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ON A NEW EXTENSION OF HILBERT'S INEQUALITY

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Abstract. This paper deals with a new extension of Hilbert's inequality with a $(p,q)-$ parameter and a best constant factor. We also consider a more extended form and the equivalent inequality.

1. Introduction

If $a_n, b_n \geqslant 0$, such that $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then the well know Hilbert's inequality is written in the following form (see Hardy et al. [1, Ch.9]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left\{\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2\right\}^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Recently, (1) had been generalized with multiple by Yang [2], and strengthened by Gao et al. [3]. In the following we provide a classical extension form of (1) with $(p,q)-$parameter as follows (see [1]): If $a_n, b_n \geqslant 0$, $p > 1, \frac{1}{p}+\frac{1}{q} = 1$, and $0 < \sum_{n=0}^{\infty} a_n^p < \infty$, $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then one has

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=0}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. And equivalent form is (see Yang et al. [4]) is the following:

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^p \sum_{n=0}^{\infty} a_n^p, \tag{3}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ is still the best possible. It is obvious that for $p = q = 2$, (2) reduces to (1). Inequality (2) is called Hardy-Hilbert's inequality, and is important in analysis and its applications (see Mitrinović et al. [5]). In recent years, by deriving the following inequality of the weight coefficient $\omega_1(r,n)$ $(n \in N_0 = N \cup \{0\}, r > 1)$, namely

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$$\omega_1(r,n) = \left(n+\frac{1}{2}\right)^{(1-\frac{1}{r})} \sum_{m=0}^{\infty} \frac{(m+1/2)^{\frac{1}{r}-1}}{m+n+1} < \frac{\pi}{\sin(\frac{\pi}{r})} - \frac{\ln 2 - \gamma}{(2n+1)^{1+\frac{1}{r}}}, \tag{4}$$

Yang [6] gave a strengthened version of (2) as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{p}}}\right] a_n^p\right\}^{\frac{1}{p}} \times \times \left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{q}}}\right] b_n^q\right\}^{\frac{1}{q}}, \tag{5}$$

where $\ln 2 - \gamma = 0.1159315^{+}$ ($\gamma = 0.57721567^{+}$ is the Euler constant). And Yang [7] also proved another strengthened version of (2), which is different of (5). By introducing a parameter λ, and the β function, Yang et al. [4] gave a generalization of (1) as follows: If $2 - \min\{p,q\} < \lambda \leqslant 2, 0 < \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q < \infty$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} < k_{\lambda}(p)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \tag{6}$$

where the constant factor $k_{\lambda}(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ is the best possible ($B(u,v)$ is the β function). For $\lambda = 1$, inequality (6) reduces to (2). Yang et al. [8] summarized the above some results.

The main objective of this paper is to estimate the weight coefficient in the form

$$\omega_{\lambda}(r,n) = \left(n+\frac{1}{2}\right)^{\lambda(1-\frac{1}{r})} \sum_{m=0}^{\infty} \frac{1}{(m+n+1)^{\lambda}}\left(m+\frac{1}{2}\right)^{\frac{\lambda}{r}-1}, \quad \begin{pmatrix} n \in N_0, r > 1, \\ 0 < \lambda \leqslant r, \end{pmatrix} \tag{7}$$

and provide a inequality related to the double series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}$ with a best constant factor, which is a new extension of (1) but different from (6). We also consider the equivalent form and certain particular results.

2. Lemmas

First, we need the following formula of the β function (see [9]):

$$B(u,v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{-1+u} dt = B(v,u) \ (u,v>0), \tag{8}$$

as well as the following estimate (see Kuang et al. [10], and Yang et al. [4]):
If $f^{(4)} \in C[0,\infty)$, $\int_0^{\infty} f(x)dx < \infty$, and $(-1)^n f^{(n)}(x) > 0$, $f^{(n)}(\infty) = 0$ $(n = 0,1,2,3,4)$, then

$$\sum_{m=0}^{\infty} f(m) < \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \tag{9}$$

LEMMA 2.1. *If* $n \in N_0 = N \cup \{0\}, r > 1, 0 < \lambda \leqslant r$, *define the function* $R_\lambda(r,n)$ *by*

$$R_\lambda(r,n) = \int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^\lambda}\left(x+\frac{1}{2}\right)^{\frac{\lambda}{r}-1} dx - \frac{4r-\lambda}{3r}\cdot\frac{1}{(n+1)^\lambda 2^{\lambda/r}} - \frac{\lambda}{6(n+1)^{\lambda+1}2^{\lambda/r}}. \tag{10}$$

Then $R_\lambda(r,n) > 0$.

Proof. Applying integration by parts, we obtain

$$\begin{aligned}
\int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^\lambda}(x+\frac{1}{2})^{\frac{\lambda}{r}-1}dx &= \frac{r}{\lambda}\int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^\lambda}d(x+\frac{1}{2})^{\frac{\lambda}{r}}\\
&= \frac{r}{\lambda}\cdot\frac{1}{(x+n+1)^\lambda}(x+\frac{1}{2})^{\frac{\lambda}{r}}\Big|_{-\frac{1}{2}}^{0} + r\int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^{\lambda+1}}(x+\frac{1}{2})^{\frac{\lambda}{r}}dx\\
&= \frac{r}{\lambda}\cdot\frac{1}{(n+1)^\lambda 2^{\lambda/r}} + \frac{r^2}{\lambda+r}\int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^{\lambda+1}}d(x+\frac{1}{2})^{\frac{\lambda}{r}+1}\\
&> \frac{r}{\lambda(n+1)^\lambda 2^{\lambda/r}} + \frac{r^2}{2(\lambda+r)(n+1)^{\lambda+1}2^{\lambda/r}}.
\end{aligned} \tag{11}$$

Hence by (10), since $r > 1$ and $0 < \lambda \leqslant r$, we have

$$\begin{aligned}
R_\lambda(r,n) &> \left[\frac{r}{\lambda} - \frac{4r-\lambda}{3r}\right]\frac{1}{(n+1)^\lambda 2^{\lambda/r}} + \left[\frac{r^2}{2(\lambda+r)} - \frac{\lambda}{6}\right]\frac{1}{(n+1)^{\lambda+1}2^{\lambda/r}}\\
&= \frac{(r-\lambda)(3r-\lambda)}{3r\lambda(n+1)^\lambda 2^{\lambda/r}} + \frac{3r^2-\lambda r-\lambda^2}{6(\lambda+r)(n+1)^{\lambda+1}2^{\lambda/r}} > 0.
\end{aligned} \tag{12}$$

Thus the lemma is proved.

LEMMA 2.2. *If* $n \in N_0, r > 1, 0 < \lambda \leqslant r$, *and* $\omega_\lambda(r,n)$ *is defined by* (7), *then*

$$\omega_\lambda(r,n) < B\left(\frac{\lambda}{r}, \lambda(1-\frac{1}{r})\right). \tag{13}$$

Proof. By fixed n and setting $f(x) = \frac{1}{(x+n+1)^\lambda}(x+\frac{1}{2})^{\frac{\lambda}{r}-1}, x \in (-\frac{1}{2}, \infty)$, by (9), we obtain

$$\begin{aligned}
\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^\lambda}\left(m+\frac{1}{2}\right)^{\frac{\lambda}{r}-1} &< \int_0^\infty \frac{1}{(x+n+1)^\lambda}\left(x+\frac{1}{2}\right)^{\frac{\lambda}{r}-1}dx\\
&+ \frac{1}{(n+1)^\lambda 2^{\lambda/r}} + \frac{1}{12}\left[\frac{2\lambda}{(n+1)^{\lambda+1}2^{\lambda/r}} + \frac{r-\lambda}{r}\cdot\frac{4}{(n+1)^\lambda 2^{\lambda/r}}\right]\\
&= \int_{-\frac{1}{2}}^{\infty} \frac{1}{(x+n+1)^\lambda}\left(x+\frac{1}{2}\right)^{\frac{\lambda}{r}-1}dx\\
&- \left[\int_{-\frac{1}{2}}^{0} \frac{1}{(x+n+1)^\lambda}(x+\frac{1}{2})^{\frac{\lambda}{r}-1}dx - \frac{4r-\lambda}{3r(n+1)^\lambda 2^{\lambda/r}} - \frac{\lambda}{6(n+1)^{\lambda+1}2^{\lambda/r}}\right].
\end{aligned} \tag{14}$$

Setting $u=(x+\frac{1}{2})/(n+\frac{1}{2})$, by (8), we find

$$\begin{aligned}&\int_{-\frac{1}{2}}^{\infty}\frac{1}{(x+n+1)^{\lambda}}\left(x+\frac{1}{2}\right)^{\frac{\lambda}{r}-1}dx\\&\quad=\left(n+\frac{1}{2}\right)^{\lambda(\frac{1}{r}-1)}\int_{0}^{\infty}\frac{u^{-1+\lambda/r}}{(1+u)^{\lambda}}du=\left(n+\frac{1}{2}\right)^{\lambda(\frac{1}{r}-1)}B\left(\frac{\lambda}{r},\lambda(1-\frac{1}{r})\right).\end{aligned}\tag{15}$$

Hence by (14), (15) and (10), one gets

$$\omega_{\lambda}(r,n)<B\left(\frac{\lambda}{r},\lambda(1-\frac{1}{r})\right)-\left(n+\frac{1}{2}\right)^{\lambda(1-\frac{1}{r})}R_{\lambda}(r,n).$$

By Lemma 2.1, we have (13) . The lemma is proved.

LEMMA 2.3. *If* $p>1,\frac{1}{p}+\frac{1}{q}=1,0<\lambda\leqslant\min\{p,q\}$ *, and* $0<\varepsilon<\lambda$*, then*

$$\begin{aligned}I:&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^{\lambda}}\left(m+\frac{1}{2}\right)^{\frac{\lambda-p-\varepsilon}{p}}\left(n+\frac{1}{2}\right)^{\frac{\lambda-q-\varepsilon}{q}}\\&>\frac{2^{\varepsilon}}{\varepsilon}B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda}{p}+\frac{\varepsilon}{q}\right)-\frac{2^{\varepsilon}q^{2}}{(\lambda-\varepsilon)(\lambda-\varepsilon+q\varepsilon)}.\end{aligned}\tag{16}$$

Proof. We have $\frac{\lambda-r-\varepsilon}{r}<0\ (r=p,q)$, and $\lambda-\varepsilon>0$. Hence we obtain by setting $u=(y+\frac{1}{2})/(x+\frac{1}{2})$,

$$\begin{aligned}I&>\int_{0}^{\infty}\left(x+\frac{1}{2}\right)^{\frac{\lambda-p-\varepsilon}{p}}\left[\int_{0}^{\infty}\frac{1}{(x+y+1)^{\lambda}}(y+\frac{1}{2})^{\frac{\lambda-q-\varepsilon}{q}}dy\right]dx\\&=\int_{0}^{\infty}\left(x+\frac{1}{2}\right)^{-1-\varepsilon}\left[\int_{\frac{1}{2x+1}}^{\infty}\frac{1}{(1+u)^{\lambda}}u^{\frac{\lambda-\varepsilon}{q}-1}du\right]dx\\&=\int_{0}^{\infty}\left(x+\frac{1}{2}\right)^{-1-\varepsilon}\left[\int_{0}^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(1+u)^{\lambda}}du-\int_{0}^{\frac{1}{2x+1}}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(1+u)^{\lambda}}du\right]dx\\&>\frac{2^{\varepsilon}}{\varepsilon}\int_{0}^{\infty}\frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(1+u)^{\lambda}}du-\int_{0}^{\infty}\left(x+\frac{1}{2}\right)^{-1-\varepsilon}[\int_{0}^{\frac{1}{2x+1}}u^{\frac{\lambda-\varepsilon}{q}-1}du]dx\\&=\frac{2^{\varepsilon}}{\varepsilon}B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda}{p}+\frac{\varepsilon}{q}\right)-\frac{2^{\varepsilon}q^{2}}{(\lambda-\varepsilon)(\lambda-\varepsilon+q\varepsilon)}.\end{aligned}\tag{17}$$

The lemma is proved.

3. Main results and applications

THEOREM 3.1. *If* $a_n, b_n \geqslant 0,\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leqslant \min\{p, q\},$ *such that*

$$0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{p-1-\lambda} a_n^p < \infty \text{ and } 0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{q-1-\lambda} b_n^q < \infty,$$

then we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p-1-\lambda} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q\right\}^{\frac{1}{q}}, \tag{18}$$

where the constant factor $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ *is the best possible. In particular, for* $\lambda = 1,$ *one has*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p-2} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q-2} b_n^q\right\}^{\frac{1}{q}}, \tag{19}$$

where the constant factor $\pi/\sin(\pi/p)$ *is still the best possible.*

Proof. By Hölder's inequality and (7) (for $r = p, q$), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\frac{a_m}{(m+n+1)^{\lambda/p}} \cdot \frac{(m+1/2)^{(p-\lambda)/pq}}{(n+1/2)^{(q-\lambda)/pq}}\right] \left[\frac{b_n}{(m+n+1)^{\lambda/q}} \cdot \frac{(n+1/2)^{(q-\lambda)/pq}}{(m+1/2)^{(p-\lambda)/pq}}\right] \\
&\leqslant \left\{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m^p}{(m+n+1)^{\lambda}} \cdot \frac{(m+\frac{1}{2})^{\frac{p-\lambda}{q}}}{(n+\frac{1}{2})^{\frac{q-\lambda}{q}}}\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{b_n^q}{(m+n+1)^{\lambda}} \cdot \frac{(n+\frac{1}{2})^{\frac{q-\lambda}{p}}}{(m+\frac{1}{2})^{\frac{p-\lambda}{p}}}\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=0}^{\infty} \omega_\lambda(q, m) \left(m+\frac{1}{2}\right)^{p-1-\lambda} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \omega_\lambda(p, n) \left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{20}$$

Hence by (13), since $\omega_\lambda(r, n) < B(\lambda/p, \lambda/q)$ $(r = p, q),$ we have (18).

For $0 < \varepsilon < \lambda,$ setting $\widetilde{a}_m$ and $\widetilde{b}_n$ by

$$\widetilde{a}_m = \left(m + \frac{1}{2}\right)^{\frac{\lambda-p-\varepsilon}{p}}, \widetilde{b}_n = \left(n + \frac{1}{2}\right)^{\frac{\lambda-q-\varepsilon}{q}}, \text{ for } m, n \in N_0,$$

we have

$$\begin{aligned}
&\left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{p-1-\lambda} \widetilde{a}_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2}\right)^{q-1-\lambda} \widetilde{b}_n^q\right\}^{\frac{1}{q}} = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-1-\varepsilon} \\
&= 2^{1+\varepsilon} + \sum_{n=1}^{\infty} \left(n + \frac{1}{2}\right)^{-1-\varepsilon} < 2^{1+\varepsilon} + \int_0^{\infty} \left(x + \frac{1}{2}\right)^{-1-\varepsilon} dx = 2^{1+\varepsilon} + \frac{2^{\varepsilon}}{\varepsilon}.
\end{aligned} \tag{21}$$

If there exists a positive parameter $\lambda(\leqslant \min\{p,q\})$, such that the constant factor in (18) is not the best possible, then, there exists a positive number $K < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$, such that (18) is valid if one replaces $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ by K. In particular, we have

$$\varepsilon I = \varepsilon\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(m+n+1)^{\lambda}} < \varepsilon k\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}\widetilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-1-\lambda}\widetilde{b}_n^q\right\}^{\frac{1}{q}}, \tag{22}$$

By (16) and (21), we find

$$2^{\varepsilon}B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda}{p}+\frac{\varepsilon}{q}\right)-\varepsilon\frac{2^{\varepsilon}q^2}{(\lambda-\varepsilon)(\lambda-\varepsilon+q\varepsilon)} < K(\varepsilon 2^{1+\varepsilon}+2^{\varepsilon}).$$

Setting $\varepsilon\to 0^+$ in the above inequality, we conclude that $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\leqslant K$. This is a contradiction. Hence the constant factor $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ in (18) is the best possible. The theorem is proved.

REMARK 3.2. (*i*) For $p=q=2$ and $\lambda=1$, inequality (18) reduces to (1). It follows that (18) is a further extension of (1), but different from (6).

(*ii*) Since (19) and (2) are different, it is obvious that (18) is not an extension of (2).

(*iii*) Both (19) and (2) are extensions of (1) with (p,q)-parameter and the same best constant factor.

By (20), for $\lambda=1$, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{m+n+1}\leqslant\left\{\sum_{m=0}^{\infty}\omega_1(q,m)\left(m+\frac{1}{2}\right)^{p-2}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\omega_1(p,n)\left(n+\frac{1}{2}\right)^{q-2}b_n^q\right\}^{\frac{1}{q}}.$$

Hence, by (4), we have a strengthened version of (19) as follows:

COROLLARY 3.1. *If* $a_n,b_n\geqslant 0,\ p>1,\frac{1}{p}+\frac{1}{q}=1,\gamma=0.57721567^+$ *is the Euler constant, and* $0<\sum_{n=0}^{\infty}(n+\frac{1}{2})^{p-2}a_n^p<\infty$, $0<\sum_{n=0}^{\infty}(n+\frac{1}{2})^{q-2}b_n^q<\infty$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_mb_n}{m+n+1}<\left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{p}}}\right]\left(n+\frac{1}{2}\right)^{p-2}a_n^p\right\}^{\frac{1}{p}}\times$$
$$\times\left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})}-\frac{\ln 2-\gamma}{(2n+1)^{1+\frac{1}{q}}}\right]\left(n+\frac{1}{2}\right)^{q-2}b_n^q\right\}^{\frac{1}{q}}, \tag{23}$$

THEOREM 3.2. *If* $a_n\geqslant 0, p>1,\frac{1}{p}+\frac{1}{q}=1, 0<\lambda\leqslant\min\{p,q\}$, *satisfy* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}a_n^p<\infty$, *then*

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{\lambda(p-1)-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p<\left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}a_n^p, \tag{24}$$

where the constant factor $\left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p$ *is the best possible; Inequality* (24) *is equivalent to* (18). *In particular, for* $\lambda = 1$, *we obtain*

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-2}\left(\sum_{m=0}^{\infty}\frac{a_m}{m+n+1}\right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^p \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-2} a_n^p, \tag{25}$$

where the constant factor $[\frac{\pi}{\sin(\pi/p)}]^p$ *is still the best possible.*

Proof. Since $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{p-1-\lambda}a_n^p < \infty$, there exists $k_0 \in N_0$, such that for any $k \geqslant k_0$, to imply $0 < \sum_{n=0}^{k}(n+\frac{1}{2})^{p-1-\lambda}a_n^p < \infty$. We set

$$b_n(k) := \left(n+\frac{1}{2}\right)^{\lambda(p-1)-1}\left[\sum_{m=0}^{k}\frac{a_m}{(m+n+1)^{\lambda}}\right]^{p-1}, k \geqslant k_0,$$

and use (18) to obtain

$$\begin{aligned}
0 &< \sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q(k)\\
&= \sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{\lambda(p-1)-1}\left[\sum_{m=0}^{k}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p = \sum_{n=0}^{k}\sum_{m=0}^{k}\frac{a_m b_n(k)}{(m+n+1)^{\lambda}} \\
&< B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\sum_{n=0}^{k}(n+\frac{1}{2})^{p-1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q(k)\right\}^{\frac{1}{q}}.
\end{aligned} \tag{26}$$

Hence we find

$$\left[\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q(k)\right]^{\frac{1}{p}} < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\sum_{n=0}^{k}\left(n+\frac{1}{2}\right)^{p-1-\lambda} a_n^p. \tag{27}$$

It follows that $0 < \sum_{n=0}^{\infty}(n+\frac{1}{2})^{q-1-\lambda}b_n^q(\infty) < \infty$. Hence (26) is valid as $k \to \infty$ by (18). Thus (27) is also valid. Thus inequality (24) holds.

We have proved that (18) implies (24). We need to show that (24) implies (18). By Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &= \sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)^{(\lambda+1-q)/q}\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]\left[\left(n+\frac{1}{2}\right)^{(q-1-\lambda)/q} b_n\right]\\
&\leqslant \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{\lambda(p-1)-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(m+n+1)^{\lambda}}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-1-\lambda} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{28}$$

By (24), we have (18). It follows that (18) and (24) are equivalent.

If the constant factor $\left[B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p$ in (24) is not the best possible, then by (28), one lead to a contradiction. The theorem is proved.

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