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A GENERALIZATION OF MACLAURIN'S INEQUALITIES AND ITS APPLICATIONS

JOSIP PEČARIĆ, JIAJIN WEN, WAN-LAN WANG AND TAO LU

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Abstract. The well-known Maclaurin's inequalities are generalized as follows: If x and y are two positive n-tuples, and y and x/y are similarly ordered, then

$$P_n^{[1]}(x)/P_n^{[1]}(y) \geqslant P_n^{[2]}(x)/P_n^{[2]}(y) \geqslant \cdots \geqslant P_n^{[k]}(x)/P_n^{[k]}(y) \geqslant \cdots \geqslant P_n^{[n]}(x)/P_n^{[n]}(y),$$

where $P_n^{[k]}(a)$ is the k-th symmetric mean of a (see[15], p. 283]). The method used in this paper is based on the computational method of descending dimension. As applications, several generalizations for the results of Izumi et al [20], Marshall and Olkin [7], Vasić et al [21], Beesack et al [22], Yang et al [5] are showed.

1. Introduction

We need the following notation and symbols:

$$x := (x_1, \ldots, x_n); x^r := (x_1^r, \ldots, x_n^r); \Re_+^n := \{x | x_i \geqslant 0, i = 1, \ldots, n\};$$
$$\Re_{++}^n := \{x | x_i > 0, i = 1, \ldots, n\}; \quad x, y \in \Re^n,$$
$$y_i \neq 0 \ (i = 1, \ldots, n), x/y := (x_1/y_1, \ldots, x_n/y_n);$$
$$1 \mp x := (1 \mp x_1, \ldots, 1 \mp x_n); \ \ x_i' := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);$$
$$E_k(x) := \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j=1}^{k} x_{i_j} \ \ (1 \leqslant k \leqslant n),$$

where $E_k(x)$ is called $k-$th elementary symmetric function of x and defined $E_0(x)=1$.

DEFINITION 1. Two n-tuples x and y are said to be similarly ordered if and only if for $i, j (1 \leqslant i, j \leqslant n)$ we have $(x_i - x_j)(y_i - y_j) \geqslant 0$; if this inequality is reversed, then x and y are said to oppositely ordered. (See [2] [15, p. 230].)

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DEFINITION 2. The k-th symmetric mean of $x \in \Re^n_{++}$ is defined by

$$P_n^{[k]}(x) := \left[\frac{E_k(x)}{\binom{n}{k}}\right]^{1/k} \quad (1 \leqslant k \leqslant n),$$

where $\binom{n}{k} = n!/[k!(n-k)!]$. Especially, $P_n^{[1]}(x) = A_n(x)$ and $P_n^{[n]}(x) = G_n(x)$ are the arithmetic mean and the geometric mean of x, respectively.

As is well known, the chain of inequalities due to Maclaurin states that

$$P_n^{[1]}(x) \geqslant P_n^{[2]}(x) \geqslant \cdots \geqslant P_n^{[k]}(x) \geqslant \cdots \geqslant P_n^{[n]}(x), \tag{1}$$

where $x \in \Re^n_{++}$. It must be noted that (1) is not only very interesting, but also useful in the theory of inequalities, and the result evoked the interest of many authors, and different proofs as well as many extensions, refinements and variants have been published. For example, in 1984 Wang and Wang [3] (also see [4]) established a Ky Fan type chain similar to (1) as follows:

$$\frac{P_n^{[1]}(x)}{P_n^{[1]}(1-x)} \geqslant \frac{P_n^{[2]}(x)}{P_n^{[2]}(1-x)} \geqslant \cdots \geqslant \frac{P_n^{[k]}(x)}{P_n^{[k]}(1-x)} \geqslant \cdots \geqslant \frac{P_n^{[n]}(x)}{P_n^{[n]}(1-x)}, \tag{2}$$

where $x \in \{x \mid x_i \in (0, 1/2], i = 1, ..., n\}$. In [18] Wang, Li and Chen also established some results similar to (1) and (2). Recently, Wen and Shi [19] obtained an amusing strengthening for (1): If $x \in \Re^n_{++}$, $n \geqslant 3$, $2 \leqslant k \leqslant n-1$, then the largest p and the smallest q , satisfying

$$[A_n(x)]^p [G_n(x)]^{1-p} \leqslant P_n^{[k]}(x) \leqslant q A_n(x) + (1-q) G_n(x), \tag{3}$$

are $p_{n,k} = (n-k)/[k(n-1)]$ and $q_{n,k} = [n/(n-1)][1-(k/n)]^{1/k}$, respectively. In [9] Wen et al discussed the separation of power means and its applications. We shall establish an extension of Maclaurin's inequalities (1) by means of the computational method of descending dimension. The method first was elaborated at the end of 20th century; and developed by the present authors of this paper; it has played an important role in establishing inequalities (see [8-11][14][19]). It seems that the following inequalities (4) has the true meaning from some mathematical and aesthetical points of view. The main result to be proved in this paper is as follows:

THEOREM 1. *Let x and $y \in \Re^n_{++}$, and let y and x/y be similarly ordered. Then*

$$\frac{P_n^{[1]}(x)}{P_n^{[1]}(y)} \geqslant \frac{P_n^{[2]}(x)}{P_n^{[2]}(y)} \geqslant \cdots \geqslant \frac{P_n^{[k]}(x)}{P_n^{[k]}(y)} \geqslant \cdots \geqslant \frac{P_n^{[n]}(x)}{P_n^{[n]}(y)}. \tag{4}$$

Equalities occur if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$. In other words, if $0 < y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n$ and $0 < x_1/y_1 \leqslant x_2/y_2 \leqslant \cdots \leqslant x_n/y_n$, inequalities (4) are equivalent to

$$\frac{E_1(x)}{E_1(y)} \geqslant \left[\frac{E_2(x)}{E_2(y)}\right]^{1/2} \geqslant \cdots \geqslant \left[\frac{E_k(x)}{E_k(y)}\right]^{1/k} \geqslant \cdots \geqslant \left[\frac{E_n(x)}{E_n(y)}\right]^{1/n}. \tag{4*}$$

Equalities occur if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$.

REMARK 1. We take $y = (1,1,\ldots,1) \in \Re^n_{++}$ in (4), since y and x/y become similarly ordered such that (4) or (4 *) reduces the original Maclaurin's inequalities (1) and hence (4) or (4 *) is a generalization of inequalities (1).

REMARK 2. Since for any $x \in \Re^n_{++}(n \geqslant 2)$, there exits $y \in \Re^n_{++}$ such that y and x/y are similarly ordered (see Theorem 2), therefore combining (4) with (1), we obtain that

$$P_n^{[k]}(x) \geqslant \frac{P_n^{[k]}(y)}{P_n^{[k+1]}(y)} \cdot P_n^{[k+1]}(x) \geqslant P_n^{[k+1]}(x), \qquad (k = 1,\ldots,n-1).$$

It follows from this fact that (4) strengthens (1).

In Section 4, we shall apply Theorem 1 to generalize some well-known inequalities, e.g., the following inequality (5) and others. In order to interpret the significance of this main theorem, we will still display some geometric results of convex body.

2. Preliminaries

In this section we establish the following lemmas which will be used:

LEMMA 1. *Let $x, y, w \in \Re^n_{++}$, and let y and x/y be similarly ordered. Then the function*

$$f(r) := \left(\frac{\sum_{i=1}^n w_i x_i^r}{\sum_{i=1}^n w_i y_i^r}\right)^{1/r}, (r \in \Re)$$

is increasing with r, where

$$f(0) := \left(\frac{\prod_{i=1}^n x_i^{w_i}}{\prod_{i=1}^n y_i^{w_i}}\right)^{1/\left(\sum_{i=1}^n w_i\right)}.$$

In other words, if $r_1, r_2 \in \Re, r_1 < r_2$ then

$$f(r_1) \leqslant f(r_2). \tag{5}$$

Equality occurs if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$.

In fact, this famous result can be found in [12, pp. 48-49] [7] [15, p. 169] [18] and the references cited therein.

LEMMA 2. (The generalized Bernoulli's inequality [8]) *Let $a, t \in \Re, a > 1$.*
(I) If $t > -1$, then

$$(1+t)^a \geqslant 1 + at. \tag{6}$$

(II) If $t \geqslant -2 - c_0/a$, then

$$(1+t)|1+t|^{a-1} \geqslant 1 + at. \tag{7}$$

(III) If $t \leqslant -2 - c_0$, then

$$(1+t)|1+t|^{a-1} < 1 + at. \tag{8}$$

Here $c_0 = 2.5911211476\cdots$ is a unique positive real root of equality $ln(1+c) = 1 + (1+c)^{-1}$. Equalities in (6) and (7) occur if and only if $t = 0$.

Note that the inequality (6) is a well-known result [12, p. 65] [15, p. 5]. However, it seems that both (7) and (8) are new in literature.

LEMMA 3. *(See[15, p. 286]) Let $y \in \Re^n_{++}$ $(n \geqslant 2)$ and $1 \leqslant k \leqslant n-1$. Then*

$$[E_k(y)]^2 > E_{k-1}(y)E_{k+1}(y). \tag{9}$$

LEMMA 4. *Let $x, y \in \Re^n_{++}$ $(n \geqslant 2), x_n^{'} := (x_1, x_2, \ldots, x_{n-1})$, $y_n^{'} := (y_1, y_2, \ldots, y_{n-1})$, and*

$$0 < x_1/y_1 \leqslant x_2/y_2 \leqslant \cdots \leqslant x_{n-1}/y_{n-1} \leqslant x_n/y_n.$$

Then

$$\left[\frac{E_k(x_n')}{E_k(y_n')}\right]^{1/k} \leqslant \frac{x_n}{y_n}, (k = 1 \ldots, n-1). \tag{10}$$

In fact, it is easy to see that

$$E_k\left(x_n^{'}\right) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} \prod_{j=1}^{k} y_{i_j}\left(\frac{x_{i_j}}{y_{i_j}}\right) \leqslant \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n-1} \prod_{j=1}^{k} y_{i_j}\left(\frac{x_n}{y_n}\right) = \left(\frac{x_n}{y_n}\right)^k E_k\left(y_n^{'}\right),$$

which means that (10).

LEMMA 5. *Let $y := (\underbrace{u, \ldots, u}_{n-p}, \underbrace{c, \ldots, c}_{p}) \in \Re^n_{++}$, where $0 \leqslant p \leqslant n$, $n \geqslant 2$, and c is a positive constant. Then, for the function*

$$F_k(u) := k \cdot \frac{E_k(y)}{E_{k-1}(y)} - (k+1) \cdot \frac{E_{k+1}(y)}{E_k(y)}, 1 \leqslant k \leqslant n-1,$$

we have

$$\lim_{u \to 0} F_k(u) := \begin{cases} 0, & k \geqslant p+1, \\ c, & k \leqslant p. \end{cases} \tag{11}$$

Proof. For any integral number $r : 0 \leqslant r \leqslant n$, we have

$$E_r(y) = \sum_{i+j=r, 0 \leqslant i \leqslant n-p, 0 \leqslant j \leqslant p} \binom{n-p}{i}\binom{p}{j} u^i c^j. \tag{12}$$

From $i + j = r, 0 \leqslant i \leqslant n-p, 0 \leqslant j \leqslant p$, we get $0 \leqslant i \leqslant r, r-p \leqslant i \leqslant n-p$. Thus

$$\max\{0, r-p\} \leqslant i \leqslant \min\{r, n-p\}. \tag{13}$$

Combining (10) with (9) we have

$$E_r(y) = \sum_{i=\max\{0,\, r-p\}}^{\min\{r,\, n-p\}} \binom{n-p}{i}\binom{p}{r-i} u^i c^{r-i}. \tag{14}$$

When $r > p$, we can rewrite (14) as

$$E_r(y) = \binom{n-p}{r-p} u^{r-p} c^p + o\left(u^{r-p}\right); \tag{15}$$

when $1 \leqslant r \leqslant p$, we can rewrite (14) as

$$E_r(y) = \binom{p}{r} c^r + O(u), \tag{16}$$

where $o(u)$ is an infinitesimal of higher order than u; $O(u)$ is an infinitesimal of the same order as u (as $u \to 0$), respectively, and when $r = 0$, then $E_0(y) = 1$, and (16) also holds, but we get $O(u) \equiv 0$.

From (15), (16) and the definition of $F_k(u)$, we will discuss 4 cases as follows:

Case 1: If $k > p+1$, then $k-1 > p, k > p, k+1 > p$, and

$$F_k(u) = k \cdot \frac{\binom{n-p}{k-p} u^{k-p} c^p + o\left(u^{k-p}\right)}{\binom{n-p}{k-1-p} u^{k-1-p} c^p + o\left(u^{k-1-p}\right)} - (k+1) \cdot \frac{\binom{n-p}{k+1-p} u^{k+1-p} c^p + o\left(u^{k+1-p}\right)}{\binom{n-p}{k-p} u^{k-p} c^p + o\left(u^{k-p}\right)}.$$

It is easy to calculate that $\lim_{u\to 0} F_k(u) = 0$.

Case 2: If $k = p+1$, then $k-1 \leqslant p$, $k > p$, $k+1 > p$ and

$$F_k(u) = k \cdot \frac{\binom{n-p}{k-p} u^{k-p} c^p + o\left(u^{k-p}\right)}{\binom{p}{k-1} c^{k-1} + O(u)} - (k+1) \cdot \frac{\binom{n-p}{k+1-p} u^{k+1-p} c^p + o\left(u^{k+1-p}\right)}{\binom{n-p}{k-p} u^{k-p} c^p + o\left(u^{k-p}\right)}.$$

Similarly, we have $\lim_{u\to 0} F_k(u) = 0$.

Case 3: If $k = p$, then $k-1 \leqslant p, k \leqslant p, k+1 > p$, and

$$F_k(u) = k \cdot \frac{\binom{p}{k} c^k + O(u)}{\binom{p}{k-1} c^{k-1} + O(u)} - (k+1) \cdot \frac{\binom{n-p}{k+1-p} u^{k+1-p} c^p + o\left(u^{k+1-p}\right)}{\binom{p}{k} c^k + O(u)}.$$

It is easy to calculate that $\lim_{u\to 0} F_k(u) = p \cdot \left[c^p/(pc^{p-1})\right] = c$.

Case 4: If $1 \leqslant k \leqslant p-1$, then $k-1 \leqslant p, k \leqslant p, k+1 \leqslant p$, and

$$F_k(u) = k \cdot \frac{\binom{p}{k} c^k + O(u)}{\binom{p}{k-1} c^{k-1} + O(u)} - (k+1) \cdot \frac{\binom{p}{k+1} c^{k+1} + O(u)}{\binom{p}{k} c^k + O(u)}.$$

It is easy to calculate that

$$\lim_{u\to 0} F_k(u) = k \cdot \frac{\binom{p}{k} c^k}{\binom{p}{k-1} c^{k-1}} - (k+1) \cdot \frac{\binom{p}{k+1} c^{k+1}}{\binom{p}{k} c^k} = c.$$

Summarizing these discusses, we have completed the proof of Lemma 5.

LEMMA 6. *Let* $y \in \Re^n_{++}$ $(n \geqslant 2)$, $0 < y_i \leqslant c$ $(i = 1, \ldots, n)$,*where* c *is a positive constant. Then, for every,* k $(1 \leqslant k \leqslant n-1)$,

$$k \cdot \frac{E_k(y)}{E_{k-1}(y)} - (k+1) \cdot \frac{E_{k+1}(y)}{E_k(y)} \leqslant c. \tag{17}$$

Proof. We first consider a special case as follows:

(A) When $k = 1$, then

$$\text{the left hand side of } (17) = \sum_{i=1}^n y_i - \frac{2\sum_{1\leqslant i<j\leqslant n} y_i y_j}{\sum_{i=1}^n y_i} = \frac{\sum_{i=1}^n y_i^2}{\sum_{i=1}^n y_i} \leqslant \frac{\sum_{i=1}^n c y_i}{\sum_{i=1}^n y_i} = c.$$

Therefore, (17) holds.

(B) When $2 \leqslant k \leqslant n-1$, we denote the left-hand side of (17) by $g_k(y)$, i.e.,

$$g_k(y) := g_k\left(y_n^{'}, y_n\right) := k \cdot \frac{E_k(y)}{E_{k-1}(y)} - (k+1) \cdot \frac{E_{k+1}(y)}{E_k(y)},$$

where we used $y = (y_1, \ldots, y_{n-1}, y_n) := (y_n^{'}, y_n)$. For simplicity we denote $E_r(y_n^{'})$ by $E_r\ (0 \leqslant r \leqslant n)$, where $y_n^{'} = (y_1, \ldots, y_{n-1}) \in \Re_{++}^{n-1}$. It follows from the given relation (see [1, p. 34]) $E_k(y) = y_n E_{k-1} + E_k$ that

$$g_k(y) = k \cdot \frac{y_n E_{k-1} + E_k}{y_n E_{k-2} + E_{k-1}} - (k+1) \cdot \frac{y_n E_k + E_{k+1}}{y_n E_{k-1} + E_k}, \tag{18}$$

$$\frac{\partial g_k}{\partial y_n} = k \cdot \frac{(E_{k-1})^2 - E_k \cdot E_{k-2}}{(y_n E_{k-2} + E_{k-1})^2} - (k+1) \cdot \frac{(E_k)^2 - E_{k+1} \cdot E_{k-1}}{(y_n E_{k-1} + E_k)^2}. \tag{19}$$

By Lemma 2, we obtain that $(E_{k-1})^2 - E_k \cdot E_{k-2} > 0$ and $(E_k)^2 - E_{k+1} \cdot E_{k-1} > 0$. Now we fix y_n' arbitrarily, so we may use the following symbol:

$$H(y_n) := \frac{y_n E_{k-1} + E_k}{y_n E_{k-2} + E_{k-1}} - \left[\frac{k+1}{k} \cdot \frac{(E_k)^2 - E_{k+1} \cdot E_{k-1}}{(E_{k-1})^2 - E_k \cdot E_{k-2}}\right]^{1/2}$$

It is easy to verify that

$$\frac{\partial g_k}{\partial y_n} > 0\,(=0; <0) \iff H(y_n) > 0\,(=0; <0)\,. \tag{20}$$

From Lemma 2 we obtain that

$$\frac{dH}{dy_n} = \frac{(E_{k-1})^2 - E_k \cdot E_{k-2}}{(y_n E_{k-2} + E_{k-1})^2} > 0$$

thus $H(y_n)$ is strictly increasing with y_n.

Note that for every $y \in \Re_{++}^n : 0 < y_i \leqslant c\ (i = 1, \ldots, n)$, there exists $u : 0 < u < c$ such that $u \leqslant y_i \leqslant c\ (i = 1, \ldots, n)$.

Case 1: If $H(u) \geqslant 0$, then, for every $y_n : u \leqslant y_n \leqslant c$, $H(y_n) \geqslant H(u) \geqslant 0$. Using (20), we have $\partial g_k / \partial y_n \geqslant 0$. Thus $g_k(y)$ is strictly increasing with y_n such that $\max g_k(y) = g_k(y_n^{'}, c)$.

Case 2: If $H(c) \leqslant 0$, then, for every $y_n : u \leqslant y_n \leqslant c$, $H(y_n) \leqslant H(c) \leqslant 0$. Using (20), we have $\partial g_k / \partial y_n \leqslant 0$. Thus $g_k(y)$ is strictly decreasing with y_n such that $\max g_k(y) = g_k(y_n^{'}, u)$.

Case 3: If $H(u) < 0, H(c) > 0$, from the continuity and the strict increase of $H(y_n)$ on $[u, c]$, then there exits a unique $u_0 \in (u, c)$ such that $H(u_0) = 0$. When $u \leqslant y_n \leqslant u_0$, we can obtain that $H(y_n) \leqslant H(u_0) = 0$. Using (20), we have $\partial g_k / \partial y_n \leqslant 0$. Therefore $g_k(y)$ is strictly decreasing with y_n such that $\max g_k(y) = g_k(y_n^{'}, u)$. Similarly, when $u_0 \leqslant y_n \leqslant c$, we can obtain that $\max g_k(y) = g_k(y_n^{'}, c)$.

Summing up the above, $g_k(y)$ has maximum at $y_n = u$ or $y_n = c$. Similarly, for every $y_i : u \leqslant y_i \leqslant c, 1 \leqslant i \leqslant n-1, g_k(y)$ has maximum if and only if $y_i = u$ or

$y_i = c$. In other words, there exists $p:\ 0 \leqslant p \leqslant n$ such that every $y \in \Re^n_{++} : u \leqslant y_i \leqslant c\ (i = 1,\dots,n)$, we have

$$g_k(y) \leqslant g_k(\underbrace{u,\dots,u}_{n-p},\underbrace{c,\dots,c}_{p}) = F_k(u)$$

where $F_k(u)$ was defined by Lemma 5, and we used the symmetry of $g_k(y)$. By Lemma 4, $\lim_{u\to 0} F_k(u) \leqslant c$, thus letting $u \to 0$ in both sides $g_k(y) \leqslant F_k(u)$, we get the double-inequality

$$g_k(y) \leqslant \lim_{u\to 0} F_k(u) \leqslant c,$$

namely, (17) is valid. This completes the proof of Lemma 6.

3. Proof of Theorem 1

Now we prove our main result as follows:

If $n = 1$, the conclusion is clear. Assume that $n \geqslant 2$ below. We only prove that for every $k : 1 \leqslant k \leqslant n-1$, we have

$$\left[\frac{E_k(x)}{E_k(y)}\right]^{1/k} \geqslant \left[\frac{E_{k+1}(x)}{E_{k+1}(y)}\right]^{1/(k+1)}. \tag{21}$$

Equality holds if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$.

We give an inductive proof of (21) as follows:

(A) If $n = 2$, then $k = 1$ and (21) reduces that

$$\frac{x_2 + x_1}{y_1 + y_2} \geqslant \sqrt{\frac{x_1x_2}{y_1y_2}}. \tag{22}$$

(22) can be deduced from Lemma 1 (taking $w_1 = w_2 = 1, r_1 = 0, r_2 = 1$), and equality holds if and only if $x_1/y_1 = x_2/y_2$.

(B) Assume that Theorem 1 holds for $n-1 (n \geqslant 3)$. We prove that (21) holds for $n \geqslant 3$ as follows:

Case 1: If $k = n-1$, then (21) is equivalent to the following

$$\left[\frac{\sum_{i=1}^n \left(x_i^{-1}\prod_{j=1}^n x_j\right)}{\prod_{i=1}^n \left(y_i^{-1}\prod_{j=1}^n y_j\right)}\right]^{1/(n-1)} \geqslant \left[\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i}\right]^{1/n},$$

namely,

$$\left[\frac{\sum_{i=1}^n x_i^{-1}}{\sum_{i=1}^n y_i^{-1}}\right]^{1/(-1)} \leqslant \left[\frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i}\right]^{1/n} \tag{23}$$

Inequality (23) can be deduced from Lemma 1 (taking $w_1 = \cdots = w_n = 1, r_1 = -1, r_2 = 0$), and equality holds if and only if $x_1/y_1 = x_2/y_2 = \cdots = x_n/y_n$.

Case 2: If $1 \leqslant k \leqslant n-2$, we shall prove that inequality (21) holds. Recall that we used symbols in Lemma 4: $x_n^{'} := (x_1, x_2, \ldots, x_{n-1})$, $y_n^{'} := (y_1, y_2, \ldots, y_{n-1}) \in \Re_{++}^{n-1}$. We shall also use the following symbols:

$$T_k := \left[\frac{E_k(x_n^{'})}{E_k(y_n^{'})}\right]^{1/k}, t := \frac{x_n/y_n}{T_k}, u_k := P_{n-1}^{[k]}(y_n^{'}).$$

Since y and x/y are similarly ordered, without loss of generality, we assume that

$$0 < y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n,\ 0 < \frac{x_1}{y_1} \leqslant \frac{x_2}{y_2} \leqslant \cdots \leqslant \frac{x_n}{y_n}.$$

From Lemma 4 we have

$$t - 1 \geqslant 0. \tag{24}$$

By using the inductive hypothesis, for $k \geqslant 2$ we have

$$T_{k-1} \geqslant T_k \geqslant T_{k+1}. \tag{25}$$

It is easy to calculate that

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}, \quad \frac{\binom{n-1}{k}}{\binom{n}{k}} = \frac{n-k}{n}. \tag{26}$$

From the above definitions and the facts we get

$$\begin{aligned}
E_k(x) &= x_n E_{k-1}\left(x_n'\right) + E_k\left(x_n'\right) \quad \text{(by [1, p. 34])}\\
&= x_n E_{k-1}\left(y_n'\right)\left(T_{k-1}\right)^{k-1} + E_k\left(y_n'\right)\left(T_k\right)^k \quad \text{(by the definition of } T_k)\\
&= x_n \binom{n-1}{k-1}\left(u_{k-1}\right)^{k-1}\left(T_{k-1}\right)^{k-1} + \binom{n-1}{k}\left(u_k\right)^k\left(T_k\right)^k \text{ (by the definition of } u_k)\\
&= \frac{1}{n}\binom{n}{k}\left[k x_n\left(u_{k-1}\right)^{k-1}\left(T_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k\left(T_k\right)^k\right] \text{ (by the equalities (26))}\\
&\geqslant \frac{1}{n}\binom{n}{k}\left[k x_n\left(u_{k-1}\right)^{k-1}\left(T_k\right)^{k-1} + (n-k)\left(u_k\right)^k\left(T_k\right)^k\right]; \text{(by the inequalities (25))}\\
&= \frac{1}{n}\binom{n}{k}\left(T_k\right)^k\left[k y_n t\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k\right] \text{ (by the definition of } t).
\end{aligned} \tag{27}$$

If $k = 1$, for every $T_0 \neq 0$, the equality in (27) holds, so we have (27) for $1 \leqslant k \leqslant n-2$. Similarly, from the inductive assumption, i.e., (25), for every $k : 1 \leqslant k \leqslant n-2$, we have $T_k \geqslant T_{k+1}$ and

$$\begin{aligned}
&E_{k+1}(x)\\
&= \frac{1}{n}\binom{n}{k+1}\left[(k+1)x_n\left(u_k\right)^k\left(T_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}\left(T_{k+1}\right)^{k+1}\right]\\
&\leqslant \frac{1}{n}\binom{n}{k+1}\left[(k+1)x_n\left(u_k\right)^k\left(T_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}\left(T_k\right)^{k+1}\right] \text{ (by inequalities (25))}\\
&= \frac{1}{n}\binom{n}{k+1}\left(T_k\right)^{k+1}\left[(k+1)y_n t\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}\right] \text{ (by the definition of } t).
\end{aligned} \tag{28}$$

Using an argument similar to (27) and replacing x by y, we can obtain that

$$E_k(y) = \frac{1}{n}\binom{n}{k}\left[ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k\right], \tag{29}$$

$$E_{k+1}(y) = \frac{1}{n}\binom{n}{k+1}\left[(k+1)\,y_n\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}\right]. \tag{30}$$

Combining (27), (28) and (29) with (30) we have

$$\begin{aligned}\left[\frac{E_k(x)}{E_k(y)}\right]^{1/k} &\geqslant T_k\left[\frac{ky_nt\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k}{ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k}\right]^{1/k} \\ &= T_k\left[1 + \frac{ky_n\left(u_{k-1}\right)^{k-1}}{ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k}\cdot(t-1)\right]^{1/k}\end{aligned} \tag{31}$$

$$\begin{aligned}\left[\frac{E_{k+1}(x)}{E_{k+1}(y)}\right]^{1/(k+1)} &\leqslant T_k\left[\frac{(k+1)\,y_nt\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}}{(k+1)\,y_n\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}}\right]^{1/(k+1)} \\ &= T_k\left[1+\frac{(k+1)\,y_n\left(u_k\right)^k}{(k+1)\,y_n\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}}\cdot(t-1)\right]^{1/(k+1)}.\end{aligned} \tag{32}$$

For the argument of (21), from (31) and (32) it is enough to prove that

$$\begin{aligned}&\left[1 + \frac{ky_n\left(u_{k-1}\right)^{k-1}}{ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k}\cdot(t-1)\right]^{(k+1)/k} \\ &\qquad\geqslant 1 + \frac{(k+1)\,y_n\left(u_k\right)^k}{(k+1)\,y_n\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}}\cdot(t-1).\end{aligned} \tag{33}$$

From Bernoulli's inequality (6) and (24) we get that

$$\text{the left-hand side of (33)} \geqslant 1 + \frac{(k+1)\,y_n\left(u_{k-1}\right)^{k-1}}{ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k}\cdot(t-1). \tag{34}$$

It follows from (33) and (34) that we need to prove that

$$\begin{aligned}&\frac{(k+1)y_n\left(u_{k-1}\right)^{k-1}}{ky_n\left(u_{k-1}\right)^{k-1} + (n-k)\left(u_k\right)^k} \geqslant \frac{(k+1)y_n\left(u_k\right)^k}{(k+1)y_n\left(u_k\right)^k + (n-k-1)\left(u_{k+1}\right)^{k+1}} \\ &\iff \quad ky_n + (n-k)\frac{\left(u_k\right)^k}{\left(u_{k-1}\right)^{k-1}} \leqslant (k+1)\,y_n + (n-k-1)\frac{\left(u_{k+1}\right)^{k+1}}{\left(u_k\right)^k} \\ &\iff \quad (n-k)\,\frac{\left(u_k\right)^k}{\left(u_{k-1}\right)^{k-1}} - (n-k-1)\frac{\left(u_{k+1}\right)^{k+1}}{\left(u_k\right)^k} \leqslant y_n.\end{aligned} \tag{35}$$

Since for every $k : 1 \leqslant k \leqslant n-2$ we have

$$\frac{(u_k)^k}{(u_{k-1})^{k-1}} = \frac{\binom{n-1}{k-1}}{\binom{n-1}{k}} \cdot \frac{E_k(y_n')}{E_{k-1}(y_n')} = \frac{k}{n-k} \cdot \frac{E_k(y_n')}{E_{k-1}(y_n')};$$
$$\frac{(u_{k+1})^{k+1}}{(u_k)^k} = \frac{\binom{n-1}{k}}{\binom{n-1}{k+1}} \cdot \frac{E_{k+1}(y_n')}{E_k(y_n')} = \frac{k+1}{n-k-1} \cdot \frac{E_{k+1}(y_n')}{E_k(y_n')},$$

herefore (35) is equivalent to that

$$k \cdot \frac{E_k(y_n')}{E_{k-1}(y_n')} - (k+1) \cdot \frac{E_{k+1}(y_n')}{E_k(y_n')} \leqslant y_n. \tag{36}$$

As pointed out that $y_n' \in \Re_{++}^{n-1}, 0 < y_i \leqslant y_n \, (i = 1, \dots, n-1)$, $n-1 \geqslant 2$, $1 \leqslant k \leqslant n-2$, thus (36) holds from Lemma 6. In other words, the desired inequality (21) holds.

From the above process of argument, we know that the equality in (21) holds if and only if

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_{n-1}}{y_{n-1}} = T_k = T_{k+1},$$
$$(\text{for every } k : 1 \leqslant k \leqslant n-1), t = 1, \text{ i.e., } T_k = x_n/y_n$$
$$\Longleftrightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}.$$

The proof of Theorem 1 is complete.

REMARK 3. Thorem 1 remains valid if $x, y \in \Re_{++}^n$ is replaced by $x \in \Re_{+}^n, y \in \Re_{++}^n$.

REMARK 4. The fundamental idea of proving Theorem 1 is as follows: The problem for proving analytic inequalities of higher dimension reduces to the problem of lower dimension (even one dimension) first and then the problem can be treated by means of routine calculus. This method is called the computional method of descending dimension. This method can treat not only the different analytic inequalities of higher dimension, but also the optimizing problems of inequalities. The present authors have used it many a time. (See [8-11][13-14][19].)

4. Some applications

First we establish a chain of inequalities similar to (2).

THEOREM 2. *Let* $x \in \Re_{+}^n$. *Then*

$$\begin{aligned} \frac{A(x)}{A_n(1+x)} = \frac{P_n^{[1]}(x)}{P_n^{[1]}(1+x)} &\geqslant \frac{P_n^{[2]}(x)}{P_n^{[2]}(1+x)} \geqslant \cdots \\ &\geqslant \frac{P_n^{[k]}(x)}{P_n^{[k]}(1+x)} \geqslant \cdots \geqslant \frac{P_n^{[n]}(x)}{P_n^{[n]}(1+x)} = \frac{G_n(x)}{G_n(1+x)}. \end{aligned} \tag{37}$$

Equalities occur if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. For any $i,j : 1 \leqslant i,j \leqslant n$, we have

$$[(1+x_i)-(1+x_j)]\left(\frac{x_i}{1+x_i}-\frac{x_j}{1+x_j}\right)=\frac{(x_i-x_j)^2}{(1+x_i)(1+x_j)}\geqslant 0.$$

Since Theorem 1 holds, therefore (34) also holds. And equality condition is that $x_1/(1+x_1)=\cdots=x_n/(1+x_n)$, i.e., $x_1=x_2=\cdots=x_n$. This completes our proof.

THEOREM 3. (The generalization of inequality (5)) *Let $x,\ y\in\Re^n_{++}$, and let $x,\ y$ be similarly ordered. Then function in $r(r\in\Re)$*

$$f_k(r):=f_k(r,x,y):=\left[\frac{E_k(x^r)}{E_k(y^r)}\right]^{1/r},(1\leqslant k\leqslant n)$$

is increasing with r, where $f_k(0):=\lim_{r\to 0}f_k(r)=[G_n(x)/G_n(y)]^k$, where $G_n(\cdots)$ is the geometric mean (see Definition 2). In other words, if $r_1<r_2$, then

$$f_k(r_1)\leqslant f_k(r_2). \tag{38}$$

Equality occurs if and only if $x_1/y_1=x_2/y_2=\cdots=x_n/y_n$ or $k=n$.

Proof. By using induction we first prove that if $r>1$ then

$$\frac{E_k(x^r)}{E_k(y^r)}\geqslant\left[\frac{E_k(x)}{E_k(y)}\right]^r,\ (1\leqslant k\leqslant n). \tag{39}$$

Without loss of generality, assume that

$$0<y_1\leqslant y_2\leqslant\cdots\leqslant y_n,\ 0<\frac{x_1}{y_1}\leqslant\frac{x_2}{y_2}\leqslant\cdots\leqslant\frac{x_n}{y_n}.$$

(A) When $n=1,2$ or $k=1$, using inequality (5) we deduce (39). And (39) reduces to an equality when $k=n$, thus (39) is still valid.

(B) Now suppose (38) has been proved for integers less than n, then (39) is valid for $n-1$ $(n\geqslant 3)$, $2\leqslant k<n$ and $1\leqslant k\leqslant n-1$. We shall prove that (38) holds for $2\leqslant k<n\,(n\geqslant 3)$. By using the following relations

$$E_k(x^r)=y_n^rE_{k-1}(y_n'^r)\left(\frac{x_n}{y_n}\right)^r\frac{E_{k-1}(x_n'^r)}{E_{k-1}(y_n'^r)}+E_k(y_n'^r)\frac{E_k(x_n'^r)}{E_k(y_n'^r)},$$
$$E_k(y^r)=y_n^rE_{k-1}(y_n'^r)+E_k(y_n'^r),$$
$$\frac{E_{k-1}(x_n'^r)}{E_{k-1}(y_n'^r)}\geqslant\left[\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')}\right]^r,\frac{E_k(x_n'^r)}{E_k(y_n'^r)}\geqslant\left[\frac{E_k(x_n')}{E_k(y_n')}\right]^r,$$

(by the inductive assumption and the power mean inequality)

$$\frac{p_1a_1+p_2a_2}{p_1+p_2}\geqslant\left(\frac{p_1a_1+p_2a_2}{p_1+p_2}\right)^r\ (r>1),$$

taking

$$p_1 = y_n^r E_{k-1}(y_n'^r), \quad p_2 = E_k(y_n'^r),$$
$$a_1 = \frac{x_n}{y_n}\left[\frac{E_{k-1}(x_n'^r)}{E_{k-1}(y_n'^r)}\right]^{1/r} \quad \left(\geqslant \frac{x_n}{y_n}\cdot\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')}\right),$$
$$a_2 = \left[\frac{E_k(x_n'^r)}{E_k(y_n'^r)}\right]^{1/r} \quad \left(\geqslant \frac{E_k(x_n')}{E_k(y_n')}\right),$$

we have

$$\begin{aligned}\frac{E_k\left(x^r\right)}{E_k(y^r)} &= \frac{\left[y_n^r E_{k-1}(y_n'^r)\cdot\left(\frac{x_n}{y_n}\right)^r\cdot\frac{E_{k-1}(x_n'^r)}{E_{k-1}(y_n'^r)} + E_k(y_n'^r)\cdot\frac{E_k(x_n'^r)}{E_k(y_n'^r)}\right]}{\left[y_n^r E_{k-1}(y_n'^r) + E_k(y_n'^r)\right]} \\ &\geqslant \frac{\left[y_n^r E_{k-1}(y_n'^r)\cdot\frac{x_n}{y_n}\cdot\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')} + E_k(y_n'^r)\cdot\frac{E_k(x_n')}{E_k(y_n')}\right]}{\left[y_n'^r E_{k-1}(y_n'^r) + E_k(y_n'^r)\right]^r} \\ &= \left\{\frac{x_n}{y_n}\cdot\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')} + \frac{E_k(x_n')/E_k(y_n') - (x_n/y_n)[E_{k-1}(x_n')/E_{k-1}(y_n')]}{y_n^r E_{k-1}(y_n'^r)/E_k(y_n'^r)+1}\right\}^r.\end{aligned} \tag{40}$$

By using Theorem 1 and Lemma 4, for the difference in (40), we have

$$\frac{E_k(x_n')}{E_k(y_n')} - \frac{x_n}{y_n}\cdot\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')} \leqslant \left[\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')}\right]^{k/(k-1)} - \frac{x_n}{y_n}\cdot\frac{E_{k-1}(x_n')}{E_{k-1}(y_n')} \leqslant 0. \tag{41}$$

Based on (40) and (41), for (39) it suffices to prove that

$$\frac{y_n^r E_{k-1}(y_n'^r)}{E_k(y_n'^r)} + 1 \geqslant \frac{y_n E_{k-1}(y_n')}{E_k(y_n')} + 1,$$

or,

$$\frac{y_n^{r-1} E_{k-1}(y_n'^r)}{E_{k-1}(y_n')} \geqslant \frac{E_k(y_n'^r)}{E_k(y_n')}. \tag{42}$$

Because y_n' and $y_n'^r/y_n'$ are similarly ordered, from Theorem 1 we get

$$\frac{E_{k-1}(y_n'^r)}{E_{k-1}(y_n')} \geqslant \left[\frac{E_k(y_n'^r)}{E_k(y_n')}\right]^{(k-1)/k}. \tag{43}$$

To prove (42), from (43) it is enough to prove that

$$y_n^{r-1}\left[\frac{E_k(y_n'^r)}{E_k(y_n')}\right]^{(k-1)/k} \geqslant \frac{E_k(y_n'^r)}{E_k(y_n')},$$

or,

$$y_n^{-r}\left[E_k(y_n'^r)\right]^{1/k} \leqslant y_n^{-1}\left[E_k(y_n')\right]^{1/k}. \tag{44}$$

In fact, by $(y_i/y_n)^r \leqslant y_i/y_n \; (i=1,\dots,n-1)$ we have

$$y_n^{-r}\left[E_k(y_n'^r)\right]^{1/k} = \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n-1}\prod_{j=1}^{k}(y_{i_j}/y_n)^r\right]^{1/k}$$

$$\leqslant \left[\sum_{1\leqslant i_1<\cdots<i_k\leqslant n-1}\prod_{j=1}^{k}(y_{i_j}/y_n)\right]^{1/k} = y_n^{-1}\left[E_k(y_n')\right]^{1/k}.$$

This is just our desired (44). From induction principle, the inequality (39) holds.

We prove that (38) holds as follows: When $0<r_1<r_2$, then $r_2/r_1>1$, From (39) and y^{r_1} and x^{r_1}/y^{r_1} are similarly ordered we obtain that

$$\frac{E\left((x^{r_1})^r\right)}{E_k((y^{r_1})^r)} \geqslant \left[\frac{E_k(x^{r_1})}{E_k(y^{r_1})}\right]^r, \quad r=\frac{r_2}{r_1}>1,$$

or,

$$f_k(r_2)=\left\{\left[\frac{E_k\left((x^{r_1})^{r_2/r_1}\right)}{E_k\left((y^{r_1})^{r_2/r_1}\right)}\right]^{1/(r_2/r_1)}\right\}^{1/r_1} \geqslant \left[\frac{E_k(x^{r_1})}{E_k(y^{r_1})}\right]^{1/r_1} = f_k(r_1).$$

When $r_1<r_2<0$, then $0<-r_2<-r_1$. From y^{-1} and x^{-1}/y^{-1} are similarly ordered, we have

$$f_k(r_2)=\left\{\left[\frac{E_k\left((x^{-1})^{-r_2}\right)}{E_k\left((y^{-1})^{-r_2}\right)}\right]^{1/(-r_2)}\right\}^{-1} \geqslant \left\{\left[\frac{E_k\left((x^{-1})^{-r_1}\right)}{E_k\left((y^{-1})^{-r_1}\right)}\right]^{1/(-r_1)}\right\}^{-1} = f_k(r_1).$$

When $r_1\leqslant 0\leqslant r_2$, from the continuity of $f_k(r)$ on $\Re$ we get

$$f_k(r_1)\leqslant f_k(0)\leqslant f_k(r_2).$$

From the above argument, we deduce that (38) is valid, and the equality condition is $x_1/y_1=x_2/y_2=\cdots=x_n/y_n$ or $k=n$. The proof of (38) has been proved.

REMARK 5. It is easy to see that when $k=1$, (38) is equivalent to inequality (5). It follows from this that (38) is a true generalization of inequality (5).

Recall that Pečarić et al established the following interesting result in [17]:

If $x\in\Re^n_{++}$, then

$$A_n(x)=\frac{E_1(x)}{\binom{n}{1}} \geqslant \frac{E_2(x^{1/2})}{\binom{n}{2}} \geqslant\cdots\geqslant \frac{E_k(x^{1/k})}{\binom{n}{k}} \geqslant\cdots\geqslant \frac{E_n(x^{1/n})}{\binom{n}{n}} = G_n(x). \quad (45)$$

THEOREM 4. *Let $x,y\in\Re^n_{++}$, and let y and x/y be similarly ordered. Then*

$$\frac{A_n(x)}{A_n(y)}=\frac{E_1(x)}{E_1(y)} \geqslant \frac{E_2(x^{1/2})}{E_2(y^{1/2})} \geqslant\cdots\geqslant \frac{E_k(x^{1/k})}{E_k(y^{1/k})} \geqslant\cdots\geqslant \frac{E_n(x^{1/n})}{E_n(y^{1/n})} = \frac{G_n(x)}{G_n(y)}. \quad (46)$$

Equalities occur if and only if $x_1/y_1=x_2/y_2=\cdots=x_n/y_n$.

Proof. From Theorem 1, for every $k : 1 \leqslant k \leqslant n-1$, we have

$$\frac{E_k(x)}{E_k(y)} \geqslant \left[\frac{E_{k+1}(x)}{E_{k+1}(y)}\right]^{k/(k+1)}. \tag{47}$$

And for $k : 1 \leqslant k \leqslant n-1$, $y^{1/k}$ and $x^{1/k}/y^{1/k}$ are similar ordered, combining this with (47) we get

$$\frac{E_k\left(x^{1/k}\right)}{E_k(y^{1/k})} \geqslant \left[\frac{E_{k+1}(x^{1/k})}{E_{k+1}(y^{1/k})}\right]^{k/(k+1)}. \tag{48}$$

Recall the definition of the function $f_k(r,x,y)$ in Theorem 3, we can rewrite (48) in the form

$$\frac{E_k(x^{1/k})}{E_k(y^{1/k})} \geqslant f_{k+1}\left(\frac{k+1}{k}, x^{1/(k+1)}, y^{1/(k+1)}\right). \tag{49}$$

Because $y^{1/(k+1)}$ and $x^{1/(k+1)}/y^{1/(k+1)}$ are similar ordered, from $(k+1)/k > 1$ and Theorem 3, we obtain that

$$f_{k+1}\left(\frac{k+1}{k}, x^{1/(k+1)}, y^{1/(k+1)}\right) \geqslant f_{k+1}\left(1, x^{1/(k+1)}, y^{1/(k+1)}\right) = \frac{E_{k+1}(x^{1/(k+1)})}{E_{k+1}(y^{1/(k+1)})}. \tag{50}$$

Combining (50) with (49), so (46) is proved, and the equality condition is also true. The proof of Theorem 4 is complete.

THEOREM 5. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ positive definite Hermitian matrices, and let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$, where the components λ_i and μ_i are eignevalues of A and B, respectively. If λ and μ are similarly ordered, then*

$$\frac{\operatorname{tr}(A)}{\operatorname{tr}(B)} \geqslant \left[\frac{E_2[A]}{E_2[B]}\right]^{1/2} \geqslant \cdots \geqslant \left[\frac{E_k[A]}{E_k[B]}\right]^{1/k} \geqslant \cdots \geqslant \left[\frac{\det A}{\det B}\right]^{1/n}. \tag{51}$$

Equalities hold iff $\lambda_1/\mu_1 = \lambda_2/\mu_2 = \cdots = \lambda_n/\mu_n$, where $\operatorname{tr}(\diamond)$ and $\det(\diamond)$ are the trace and determinant of matrix $\diamond$, and $E_k[\diamond]$ denotes the sum of all principal k-rowed minors.

In fact, since for matrix A, we have $E_k[A] = E_k(\lambda)(1 < k < n)$, $\operatorname{tr}(A) = E_1(\lambda), \det A = E_n(\lambda)$; for B, we also have the same result, it follows from Theorem 1 that the chain (48) of inequalities holds. And equality condition is also valid.

Let $\sigma_N(P) := \{P_1, \dots, P_N\}$ $(N > n)$ be a set in the k-dimensional Euclidean space E^n. For any $k+1$ points in $\sigma_N(P)$, we can construct a k-dimensional simplex with these points as vertices. Denote by $N_k(P)(k = 1, \dots, n)$ the sum of the squares of all k-dimensional contents of these k-dimensional simplexes. In [5][13][23] the following interesting results have been obtained:

$$\frac{[N_k(P)]^l}{[N_l(P)]^k} \geqslant \frac{\left[(n-l)! \cdot (l!)^3\right]^k}{\left[(n-k)! \cdot (k!)^3\right]^l} (n!N)^{l-k} \ (1 \leqslant k < l \leqslant n). \tag{52}$$

The following result is an extension of (52):

THEOREM 6. *Let* $\sigma_N(P) := \{P_1, \ldots, P_N\}$ *and* $\sigma_N(Q) := \{Q_1, \ldots, Q_N\}$ *be two sets in* E^n $(N > n \geqslant 2)$, *and let* $a_i > 0$ *and* $b_i > 0$ *be the semi-axises of the so-called " dense ellipsoid ". If* $a^2 = (a_1^2, \ldots, a_n^2)$ *and* $b^2 = (b_1^2, \ldots, b_n^2)$, *and* a^2 *and* a^2/b^2 *are similarly ordered, then*

$$\left[\frac{N_k(P)}{N_k(Q)}\right]^l \geqslant \left[\frac{N_l(P)}{N_l(Q)}\right]^k, (1 \leqslant k < l \leqslant n). \tag{53}$$

Equality holds iff $a_1^2/b_1^2 = \cdots = a_n^2/b_n^2$, *i.e., two dense ellipsoids are similar.*

Proof. By inequality (3.1) of [5] we observe that

$$E_k(a^2) = \frac{(k!)^2}{N} \cdot N_k(P), \quad E_k(b^2) = \frac{(k!)^2}{N} \cdot N_k(Q).$$

Thus

$$\frac{E_k(a^2)}{E_k(b^2)} = \frac{N_k(P)}{N_k(Q)}. \tag{54}$$

From (4^*) we have

$$\left[\frac{E_k(a^2)}{E_k(b^2)}\right]^{1/k} \geqslant \left[\frac{E_l(a^2)}{E_l(b^2)}\right]^{1/l}, \quad (1 \leqslant k < l \leqslant n). \tag{55}$$

Combining (55) with (54), (53) can be deduced, and equality condition is also obtained from Theorem 1.

REMARK 6 If $a_i > 0, b_i > 0$ are replaced by $a_i \geqslant 0, b_i \geqslant 0$, then we have

$$[N_k(P)]^l [N_l(Q)]^k \geqslant [N_k(Q)]^l [N_l(P)]^k.$$

But " *a^2 and a^2/b^2 are similarly ordered* " reads " $(b_i^2 - b_j^2)(a_i^2 b_j^2 - a_j^2 b_i^2) \geqslant 0, (i,j = 1,2,\ldots,n)$."

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Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr

Jiajin Wen
Department of Computer Science and Mathematics
Chengdu University
Sichuan 610106
P. R. China
e-mail: wenjiajin623@163.com

Wan-lan Wang
Department of Computer Science and Mathematics
Chengdu University
Sichuan 610106
P. R. China
e-mail: wanlanwang@163.com

Tao Lu
School of Mathematics
Sichhuan University
Sichuan 610024
P. R. China
e-mail: lutao@public.cd.sc.cn