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FUNCTIONAL MEANS WITH A PARAMETER

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Abstract. In this paper, a family of functional means with a parameter and their basic properties are introduced, some characterizations of extended mean values are obtained.

In a recent paper[1], Soon-Yeong Chung had considered the functional mean $M_f(x,y;\mu)$ and the harmonic functional mean $N_f(x,y;\mu)$ of any two positive numbers x and y with respect to a probability measure μ on $[0,1]$ for a continuous function $f(t)$ on $(0,\infty)$ by

$$M_f(x,y;\mu)=f^{-1}[\int_0^1 f(\lambda x+(1-\lambda)y)\,d\mu(\lambda)]$$

and

$$N_f(x,y;\mu)=[M_f(1/x,1/y;\mu)]^{-1}.$$

It had been shown that various means can be expressed as $M_f(x,y;\mu)$ or $N_f(x,y;\mu)$ for appropriate functions f.

Stolarsky[4], Leach and Scholander[2][3], Sun[5] had studied the so-called extended mean value which is a two-parameter mean of two positive numbers x and y as follows:

$$E_{p,q}(x,y)=(q(x^p-y^p)/p(x^q-y^q))^{1/(p-q)},\quad pq(p-q)(x-y)\neq 0;$$

$$E_{p,0}(x,y)=((x^p-y^p)/p(\log x-\log y))^{1/p},\quad p(x-y)\neq 0;$$

$$E_{p,p}(x,y)=e^{-1/p}(x^x/y^y)^{1/(x-y)},\quad p(x-y)\neq 0;$$

$$E_{0,0}(x,y)=(xy)^{1/2},\quad x\neq y;$$

$$E_{p,q}(x,x)=x,\quad x=y.$$

Clearly, $E_{2p,p}(x,y)$ and $E_{p,1}(x,y)$ are just the well known power mean and Stolarsky mean respectively.

The purpose of this paper is to consider a generalization of the functional mean $M_f(x,y;\mu)$ and it is used to find some characterizations of the extended mean values.

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1. Functional Means With A Parameter

DEFINITION. Let $f(t)$ be a continuous function on $(0,\infty)$ which is strictly monotone, let μ be a probability measure on $[0,1]$, and let p be an arbitrarily fixed nonzero number. For any two positive numbers x and y we define a functional mean $M_f(x,y;\mu;p)$ with a parameter p with respect to the probability measure μ by

$$M_f(x,y;\mu;p) = f^{-1}\Big[\int_0^1 f\big((\lambda x^p + (1-\lambda)y^p)^{1/p}\big)\,d\mu(\lambda)\Big].$$

If $p=1$, then $M_f(x,y;\mu;p) = M_f(x,y;\mu)$.

It is clear that the mean value $M_f(x,y;\mu;p)$ is uniquely determined and lies between x and y when $x \neq y$. It is also true that $M_f(x,x;\mu;p) = x$ for every $x>0$ and $M_f(x,y;\mu;p) \neq M_f(y,x;\mu;p)$ unless μ is equally distributed on $[0,1]$. When μ is the Lebesgue measure we simply write $M_f(x,y;p)$ instead of $M_f(x,y;\mu;p)$.

It is not difficult to find that the main results in [1] can be restated in like manner. In what follows, we always assume p is a fixed nonzero number.

THEOREM 1.1. *In order that*

$$M_f(x,y;\mu;p) = M_g(x,y;\mu;p)$$

for all $x,y>0$ and all probability measure μ on $[0,1]$ it is necessary and sufficient that

$$f(x) = \alpha g(x) + \beta, \quad x \in (0,\infty)$$

for some constants $\alpha (\alpha \neq 0)$ and β. We write $f \sim g$ instead of $f(x) = \alpha g(x) + \beta, x \in (0,\infty)$ for some $\alpha \neq 0$ and β.

THEOREM 1.2. *In order that*

$$M_f(kx,ky;\mu;p) = kM_f(x,y;\mu;p)$$

for every $x,y,k>0$ and every probability measure μ on $[0,1]$, it is necessary and sufficient that either $f(t) \sim t^r$ for some $r \neq 0$ or $f(t) \sim \log t$.

THEOREM 1.3. *Let f and g be continuous and strictly on $(0,\infty)$. Then a necessary and sufficient condition in order that*

$$M_f(x,y;\mu;p) \leqslant M_g(x,y;\mu;p)$$

for all x,y and μ, is that $g \circ f^{-1}$ is convex.

By Theorem 1.3 we can easily obtain a new proof of the Theorem in [2].

THEOREM 1.4. *For any function f on $(0,\infty)$ which is continuous and strictly monotone the functional mean $M_f(x,y;\mu;p)$ is continuous on $(0,\infty)\times(0,\infty)$ and increasing in the sense that*

$$\text{if } x_1 \leqslant x_2 \quad \text{and} \quad y_1 \leqslant y_2, \quad \text{then} \quad M_f(x_1,y_1;\mu;p) \leqslant M_f(x_2,y_2;\mu;p)$$

for any probability measure μ *on* $[0,1]$.

Now we define the functional harminic mean $N_f(x,y;\mu;p)$ for positive numbers x and y by

$$N_f(x,y;\mu;p) = [M_f(1/x,1/y;\mu;p)]^{-1}.$$

In particular, if μ is a Lebesgue measure we simply write $N_f(x,y;p)$ instead of $N_f(x,y;\mu;p)$.

EXAMPLE. (1) For $f(t) = 1/t^{p-q}$, where q is a nonzero number, we have

$$M_f(x,y;p) = E_{p,q}(x,y)$$

and

$$N_f(x,y;p) = [M_f(1/x,1/y;p)]^{-1} = [E_{p,q}(x,y)]^{-1} \cdot xy.$$

Hence, we obtain

$$N_f(x,y;p) \cdot M_f(x,y;p) = xy.$$

(2) Let μ be a probability measure concentrated on $\{0,1\}$ with

$$\mu(\{\lambda\}) = \begin{cases} 1/3, & \lambda = 0, \\ 2/3, & \lambda = 1. \end{cases}$$

Then

$$M_f(x,y;\mu;p) = \int_0^1 (\lambda x^p + (1-\lambda)y^p)^{1/p}\, d\mu(\lambda) = (2x+y)/3,$$

so that

$$N_f(x,y;\mu;p) = 3xy/(x+2y).$$

Hence, we obtain

$$N_f(x,y;\mu;p) \cdot M_f(x,y;\mu;p) = xy.$$

In general, we have

THEOREM 1.5. *If* $f(t)$ *is a continuous function on* $(0,\infty)$ *which is strictly monotone and is equivalent to a homogeneous function in the sense that*

$$f(kt) = \alpha(k)f(t) + \beta(k),\ t > 0,\ k > 0$$

for some real functions $\alpha(k) \neq 0$ *and* $\beta(k)$, *then*

$$N_f(x,y;\mu;p) \cdot M_f(y,x;\mu;p) = xy$$

for all $x,y > 0$ *and for every probability measure* μ.

If μ is Lebesgue measure then $M_f(x,y;p) = M_f(y,x;p)$, and it follows that

$$N_f(x,y;p) \cdot M_f(x,y;p) = xy.$$

All the proofs of the above five theorems are similar to those expressed in [1] and hence are omitted.

REMARK. The equality (2.2) in Theorem 2.1 of [1] should be corrected as follows:

$$N_f(x,y;\mu) \cdot M_f(y,x;\mu) = [G(x,y)]^2.$$

Moreover, the same correction should be made in Example (ii) before the Theorem 2.1 of [1].

2. Characterizations of extended mean values

In this section, p is always an arbitrarily fixed nonzero number, q, r, s are arbitrary nonzero numbers and distinct from p. Moreover, we may assume that the function $f(t)$ which is concerned with $M_f(x,y;p)$, is always strictly monotone, and $f''(t)$ is a continuous function on $(0,\infty)$.

LEMMA 1. *If*

$$g(x,y) := \int_0^1 f((\lambda x^p + (1-\lambda)y^p)^{1/p})\, d\lambda$$

for all positive x and y, then

$$g_{xx}(c,c) = f''(c)/3 + (p-1)f'(c)/6c$$

where c *is an arbitrarily fixed real number.*

The proof follows immediately from differentiation under the integral sign.

LEMMA 2. *If*

$$M_f(x,y;p) = E_{r,s}(x,y)$$

holds for all positive x *and* y, *then*

$$f''(t) + (2p-(r+s)+1)f'(t)/t = 0$$

holds for $t \in (0,\infty)$.

Proof. By assumption we have

$$\int_0^1 f((\lambda x^p + (1-\lambda)y^p)^{1/p})\, d\lambda = f((s(x^r-y^r)/r(x^s-y^s))^{1/(r-s)}).$$

Set $g(x,y) = \int_0^1 f((\lambda x^p + (1-\lambda)y^p)^{1/p})\, d\lambda$. It follows that

$$g(x,y) = f((s(x^r-y^r)/r(x^s-y^s))^{1/(r-s)}).$$

Differentiating both sides on x twice and setting $x=c, y=c$ in the resulting equality by applying L'Hôpital's rule yields

$$g_{xx}(c,c) = f''(c)/4 + (r+s-3)f'(c)/12c.$$

By Lemma 1 we obtain

$$f''(c)/3 + (p-1)f'(c)/6c = f''(c)/4 + (r+s-3)f'(c)/12c,$$

and therefore

$$f''(c) + (2p-(r+s)+1)f'(c)/c = 0.$$

Since c is an arbitrarily fixed positive number, we can replace c by a positive real variable t in the above equality. Hence we have

$$f''(t) + (2p-(r+s)+1)f'(t)/t = 0$$

on $(0,\infty)$. □

THEOREM 2.1. *Let* $A(\neq 0)$ *and* B *be arbitrary real constants.*

(1) $M_f(x,y;p) = E_{p,q}(x,y)$ *holds for all positive* x *and* y *if and only if* $f(t) = A/t^{p-q}+B$.

(2) $M_f(x,y;p) = E_{p,p}(x,y)$ *holds for all positive* x *and* y *if and only if* $f(t) = A\log t+B$.

(3) $M_f(x,y;p) = E_{p,0}(x,y)$ *holds for all positive* x *and* y *if and only if* $f(t) = A/t^{p}+B$.

(4) $M_f(x,y;p) = E_{0,0}(x,y)$ *holds for all positive* x *and* y *if and only if* $f(t) = A/t^{2p}+B$.

Proof.

(1) Suppose $M_f(x,y;p) = E_{p,q}(x,y)$ holds for all positive x and y. Then by Lemma 2 we have

$$f''(t)+(p-q+1)f'(t)/t = 0$$

on $(0,\infty)$. This implies

$$f(t) = A/t^{p-q}+B$$

on $(0,\infty)$.

On the contrary, suppose $f(t) = A/t^{p-q}+B$ and $t = (\lambda x^p+(1-\lambda)y^p)^{1/p}$. Then $f^{-1}(t) = (A/(t-B))^{1/(p-q)}$, and so

$$\begin{aligned} M_f(x,y;p) &= f^{-1}[\textstyle\int_0^1 f((\lambda x^p+(1-\lambda)y^p)^{1/p})\,d\lambda] \\ &= f^{-1}[A\textstyle\int_0^1(\lambda x^p+(1-\lambda)y^p)^{q/p-1}\,d\lambda+B] \\ &= (\textstyle\int_0^1(\lambda x^p+(1-\lambda)y^p)^{q/p-1}\,d\lambda)^{1/(q-p)} \\ &= (q(x^p-y^p)/p(x^q-y^q))^{1/(p-q)} \\ &= E_{p,q}(x,y) \end{aligned}$$

for all positive x,y.

(2) Suppose $M_f(x,y;p) = E_{p,p}(x,y)$ holds for all positive x and y. Then by Lemma 2 we have

$$f''(t)+f'(t)/t = 0$$

on $(0,\infty)$. This implies

$$f(t) = A\log t+B$$

on $(0,\infty)$.

On the contrary, suppose $f(t) = A\log t+B$ and $t = (\lambda x^p+(1-\lambda)y^p)^{1/p}$. Then $f^{-1}(t) = exp((t-B)/A)$, and so

$$\begin{aligned} M_f(x,y;p) &= f^{-1}[A\textstyle\int_0^1 \log(\lambda x^p+(1-\lambda)y^p)^{1/p}\,d\lambda+B] \\ &= exp((1/p)\textstyle\int_0^1 \log(\lambda x^p+(1-\lambda)y^p)\,d\lambda) \\ &= exp((x^p\log x-y^p\log y)/(x^p-y^p)-(1/p)) \\ &= (x^{x^p}/y^{y^p})^{1/(x^p-y^p)}e^{-1/p} \\ &= E_{p,p}(x,y) \end{aligned}$$

for all positive x,y.

(3) Suppose $M_f(x,y;p) = E_{p,0}(x,y)$ holds for all positive x and y. Then by Lemma 2 we have

$$f''(t) + (p+1)f'(t)/t = 0$$

on $(0,\infty)$. This implies

$$f(t) = A/t^p + B$$

on $(0,\infty)$.

On the contrary, suppose $f(t) = A/t^p + B$ and $t = (\lambda x^p + (1-\lambda)y^p)^{1/p}$. Then $f^{-1}(t) = (A/(t-B))^{1/p}$, and so

$$\begin{aligned} M_f(x,y;p) &= f^{-1}[A\int_0^1 (\lambda x^p + (1-\lambda)y^p)^{-1}\,d\lambda + B] \\ &= (\int_0^1 (\lambda x^p + (1-\lambda)y^p)^{-1}\,d\lambda)^{-1/p} \\ &= ((x^p - y^p)/p(\log x - \log y))^{1/p} \\ &= E_{p,0}(x,y) \end{aligned}$$

for all positive x, y.

(4) Suppose $M_f(x,y;p) = E_{0,0}(x,y)$ holds for all positive x and y. Then by Lemma 2 we have

$$f''(t) + (2p+1)f'(t)/t = 0$$

on $(0,\infty)$. This implies

$$f(t) = A/t^{2p} + B$$

on $(0,\infty)$.

On the contrary, suppose $f(t) = A/t^{2p} + B$ and $t = (\lambda x^p + (1-\lambda)y^p)^{1/p}$. Then $f^{-1}(t) = (A/(t-B))^{1/2p}$, and so

$$\begin{aligned} M_f(x,y;p) &= f^{-1}[A\int_0^1 (\lambda x^p + (1-\lambda)y^p)^{-2}\,d\lambda + B] \\ &= (\int_0^1 (\lambda x^p + (1-\lambda)y^p)^{-2}\,d\lambda)^{-1/2p} \\ &= (xy)^{1/2} \\ &= E_{0,0}(x,y) \end{aligned}$$

for all positive x, y. □

THEOREM 2.2. *Let $p \neq r$ and $\{r,s\} \neq \{0,0\}$. Then there exists no $f(t)$ such that $M_f(x,y;p) = E_{r,s}(x,y)$ holds for all positive numbers x and y.*

Proof. Suppose there exists some $f(t)$ such that $M_f(x,y;p) = E_{r,s}(x,y)$ holds for all positive x and y. Then by Lemma 2 we have

$$f''(t) + (2p - (r+s) + 1)f'(t)/t = 0$$

on $(0,\infty)$. This implies

$$f(t) = A/t^{2p-r-s} + B$$

on $(0,\infty)$, where A and B are arbitrary real constants with $A \neq 0$.

However, if $f(t) = A/t^{2p-r-s} + B$ and $t = (\lambda x^p + (1-\lambda)y^p)^{1/p}$ then $f^{-1}(t) = (A/(t-B))^{1/(2p-r-s)}$ and it follows that

$$\begin{aligned} M_f(x,y;p) &= f^{-1}[A \int_0^1 (\lambda x^p + (1-\lambda)y^p)^{(r+s)/p-2}\, d\lambda + B] \\ &= (\int_0^1 (\lambda x^p + (1-\lambda)y^p)^{(r+s)/p-2}\, d\lambda)^{1/(r+s-2p)} \\ &= ((r+s-p)(x^p - y^p)/p(x^{r+s-p} - y^{r+s-p}))^{1/(2p-r-s)} \\ &= E_{p,r+s-p}(x,y) \end{aligned}$$

for all positive x, y.

This leads to a contradiction and hence the theorem is proved. □

REMARK. It is not difficult to prove that $M_f(x,y;p) = E_{p,r+s-p}(x,y)$ holds for all positive numbers x and y if and only if $f(t) = A/t^{2p-r-s} + B$, where A and B are arbitrary real constants with $A \neq 0$.

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