

A STUDY ON READE–WESOLOWSKI CLASS OF FUNCTIONS

R. PARVATHAM

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Abstract. A new class Reade-Wesolowski class- of functions is defined and the Fekete-Szegö problem for this class is studied. Also the action of Ruscheweyh integral operator on this class has been investigated and this gives an improvement of a result due to K. S. Padmanabhan and R. Bharati.

Let A be the class of functions f holomorphic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the normalizations $f(0) = 0 = f'(0) - 1$. Let f and g be holomorphic in U and further let g be univalent in U ; then we say that f is subordinate to g in U , written $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$. Let $P(H)$ denote the class of functions $p(z) = 1 + p_1z + p_2z^2 + \dots$ such that $p(z) \prec H(z)$ in U .

Here let us recall the definition of the Wesolowski's Class $P(H)$ [12] with $H(z) = h_{\beta', \alpha}(z) = \left(\frac{1+(1-2\alpha)z}{1-z}\right)^{\beta'}$, $z \in U$, $0 \leq \alpha \leq 1$ and $0 \leq \beta'$. This function $h_{\beta', \alpha}(z)$ is univalent in U and maps U into a convex domain $h_{\beta', \alpha}(U)$ bounded by the Wesolowski's curve E given by

$$w = \left(\frac{\alpha}{\cos \theta}\right)^{\beta'} e^{i\beta'\theta}, \quad 0 \leq \alpha \leq 1,$$

$0 \leq \beta'$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This curve has two asymptotes passing through the origin, inclined to the positive real axis at an angle $\beta'\frac{\pi}{2}$ and $-\beta'\frac{\pi}{2}$ and cut the real axis at the point $\alpha\beta'$. For the region $h_{\beta', \alpha}(U)$ bounded by E to be a convex region, β' must lie in $[0, 1]$.

In this paper we introduce a new class $\mathcal{CS}(\alpha, \beta)$ of functions $f \in A$ such that $\frac{f(z)}{g(z)} \prec h_{\beta', \alpha}(z)$ in U for some $g \in S^*$ -the well-known class of starlike univalent functions in U . This class for $\alpha = 0$ and $\beta = 1$ reduces to the class \mathcal{CS}^* of close-to-star functions of M.O.Reade [11].

At first we consider the Fekete - Szegö problem for this class $\mathcal{CS}(\alpha, \beta)$.

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THEOREM 1. Let $f(z) = z + a_2z^2 + a_3z^3 + \dots \in \mathcal{CS}(\alpha, \beta)$, $0 < \alpha \leq 1$. Then for $\beta \geq 0$ we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 + 2(1 - 2\mu)((1 - \alpha)(1 + \beta)^2 + \alpha), & \text{if } \mu \leq \frac{\beta}{2(1 + \beta)} \\ 1 + 2\beta(1 - \alpha) + \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)}(1 - \alpha\beta(1 - 2\mu)), & \\ \quad \text{if } \frac{\beta}{2(1 + \beta)} \leq \mu \leq \frac{1}{2} \\ 1 + (1 + \alpha)2\beta, & \text{if } \frac{1}{2} \leq \mu \leq \frac{2 + \beta}{2(1 + \beta)} \\ -(1 - 2\alpha) - \frac{2\alpha}{1 + \beta} + 2(1 - \alpha)((1 + \beta)^2 + \alpha)(2\mu - 1), & \\ \quad \text{if } \mu \geq \frac{2 + \beta}{2(1 + \beta)} \end{cases} \quad (1)$$

For each μ there is a function in $\mathcal{CS}(\alpha, \beta)$ such that equality holds in all cases.

To establish this result we need the following:

LEMMA 1. [2]. Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be holomorphic in U with $\operatorname{Re} p(z) > 0$ in U . Then

$$|p_n| \leq 2 \text{ for all } n \geq 1$$

and

$$|p_2 - \frac{p_1^2}{2}| \leq 2 - \frac{|p_1|^2}{2}.$$

Proof of Theorem 1. Since $f \in \mathcal{CS}(\alpha, \beta)$, we can write

$$f(z) = g(z)(q_\alpha(z))^\beta \quad (2)$$

where $g(z) = z + b_2z^2 + \dots \in S^*$ and

$$\frac{q_\alpha(z) - \alpha}{1 - \alpha} = p(z) = 1 + p_1z + p_2z^2 + \dots$$

with $p(z)$ having positive real part in U . Now equating the coefficients in (2) we get

$$\begin{aligned} a_2 &= b_2 + \beta(1 - \alpha)p_1, \\ a_3 &= b_3 + \beta(1 - \alpha)p_1 b_2 + \beta(1 - \alpha)p_2 + \frac{\beta(\beta - 1)}{2}(1 - \alpha)^2 p_1^2. \end{aligned}$$

With $x = (1 - 2\mu)$, we have

$$(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x - 1)b_2^2 + \beta(1 - \alpha)p_1 b_2 x + \beta(1 - \alpha)[p_2 + \frac{1}{2}(\beta x - 1)p_1^2]. \quad (3)$$

Since rotations of $f(z)$ also belong to $\mathcal{CS}(\alpha, \beta)$ we can assume without loss of generality that $a_3 - \mu a_2^2$ is positive. Thus we estimate $\text{Re}(a_3 - \mu a_2^2)$ now. $g \in S^*$ implies that $zg'(z) = g(z)h(z)$ where $h(z) = 1 + h_1z + \dots$ with positive real part in U . Thus

$$b_2 = h_1, \quad b_3 = \frac{(h_2 + h_1^2)}{2}.$$

Let us suppose that $h_1 = 2\rho e^{i\varphi}$ ($0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi$) and $p_1 = 2re^{i\theta}$ ($0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$). Then an application of Lemma 1 gives

$$\begin{aligned} \text{Re}(a_3 - \mu a_2^2) &\leq (1 - \rho^2) + (1 + 2x)\rho^2 \cos 2\varphi \\ &\quad + 2\beta(1 - \alpha)[(1 - r^2) + \beta xr^2 \cos 2\theta + 2xpr \cos(\theta + \varphi)] \\ &= \psi(x) \text{ (say)}. \end{aligned} \tag{4}$$

When $\frac{\beta}{2(1+\beta)} \leq \mu \leq \frac{1}{2}$, then $0 \leq x \leq \frac{1}{1+\beta}$ and $-t^2 + t^2\beta x \cos 2\theta + 2xt$ has the maximum when $t = \frac{x}{1-\beta x \cos 2\theta}$ and hence

$$\begin{aligned} \psi(x) &\leq (1 + 2x) + 2\beta(1 - \alpha) \left(1 + \frac{x^2}{1 - \beta x} \right) \\ &= 1 + 2\beta(1 - \alpha) + \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)}(1 - \alpha\beta(1 - 2\mu)). \end{aligned} \tag{5}$$

From (4) and (5) we get the second inequality in (1). Equality occurs only if $b_2 = p_2 = 2, b_3 = 3$ and $p_1 = \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)}$ and the corresponding $f(z)$ is defined by

$$f(x) = \frac{z}{(1 - z)^2} \left[(1 - \alpha) \left\{ \lambda \frac{1 + z}{1 - z} + (1 - \lambda) \frac{1 - z}{1 + z} \right\} + \alpha \right]^\beta$$

where $\lambda = \frac{1 + (1 - 2\beta)(1 - 2\mu)}{2(1 - \beta(1 - 2\mu))}$.

Now let us consider $\mu \leq \frac{\beta}{2(1+\beta)}$ and hence $x \geq \frac{1}{1+\beta}$. With $x_0 = \frac{1}{1+\beta}$ we get

$$\begin{aligned} \psi(x) &\leq \psi(x_0) + 2(x - x_0)[(1 - \alpha)(1 + \beta)^2 + \alpha] \\ &\leq 1 + 2(1 - 2\mu)[(1 - \alpha)(1 + \beta)^2 + \alpha] \end{aligned}$$

which is the first inequality in (1). Equality occurs only if $p_1 = p_2 = b_2 = 2, b_3 = 3$ and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z}{(1 - z)^2} \left\{ (1 - \alpha) \left(\frac{1 + z}{1 - z} \right) + \alpha \right\}^\beta.$$

Let $x_1 = -\frac{1}{1+\beta}$; then for $x \leq x_1$, that is for $\mu \geq \frac{2+\beta}{2(1+\beta)}$

$\psi(x) \leq \psi(x_1) + 2 |x - x_1| [(1 - \alpha)(1 + \beta)^2 + \alpha]$. It is easy to see that $\psi(x_1) \leq (1 - \alpha)(1 + 2\beta) + \alpha$ and hence

$$\begin{aligned} \psi(x) &\leq (1 - \alpha)(1 + 2\beta) + \alpha - [(1 - \alpha)(1 + \beta)^2 + \alpha] \left[\frac{2}{1 + \beta} - (2\mu - 1) \right] \\ &\leq (2\alpha - 1) + 2(2\mu - 1)[(1 - \alpha)(1 + \beta)^2 + \alpha] - \frac{2\alpha}{1 + \beta}, \end{aligned}$$

which gives the last inequality in (1). Equality occurs only if $p_1 = 2i, p_2 = -2, b_2 = 2i, b_3 = 3$ and the corresponding function $f(z)$ is defined by

$$f(z) = \frac{z}{(1 - iz)^2} \left((1 - \alpha) \left(\frac{1 + iz}{1 - iz} \right) + \alpha \right)^\beta.$$

Finally since

$$\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0)$$

for $0 \leq \lambda \leq 1$, we get $\psi(x) \leq 1 + 2\beta(1 - \alpha)$ for $x_1 \leq x \leq 0$, that is for $\frac{1}{2} \leq \mu \leq \frac{2+\beta}{2(1+\beta)}$. Equality occurs only if $p_1 = b_2 = 0, p_2 = 2, b_3 = 1$ and the corresponding function $f(z)$ is defined by

$$f(x) = \frac{z}{1 - z^2} \left((1 - \alpha) \left(\frac{1 + z^2}{1 - z^2} \right) + \alpha \right)^\beta.$$

Thus we have established the theorem completely.

The case $\alpha = 0$ has been considered in [3].

From the proof of Theorem 1 in [9], we can state the following

LEMMA 2. Let $\varphi(x) = \frac{x(x^2 + (1 - \alpha)^2)}{x^2 + \alpha^2}$. For $0 < \alpha \leq \frac{1}{4}$ and $x > 0$, minimum of $\varphi(x)$ occurs at

$$x = x_0 = \sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(1 - 4\alpha)(1 - 2\alpha)(1 + 2\alpha)}}{2}} > 0;$$

thus

$$\varphi(x) \geq \varphi(x_0) = \frac{x_0(x_0^2 + (1 - \alpha^2))}{x_0^2 + \alpha^2}.$$

For $0 < \alpha \leq \frac{1}{4}$ and $x < 0$, maximum of $\varphi(x)$ occurs at

$$x = x_2 = -x_0 = -\sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(1 - 4\alpha)(1 - 2\alpha)(1 + 2\alpha)}}{2}}.$$

Thus

$$\varphi(x) \leq \varphi(x_2) = \frac{x_2(x_2^2 + (1 - \alpha)^2)}{x_0^2 + \alpha^2} = -\frac{x_0(x_0^2 + (1 - \alpha)^2)}{x_0^2 + \alpha^2}.$$

For $\frac{1}{2} \leq \alpha < 1$ and $x > 0$ minimum of $\varphi(x)$ occurs at

$$x = x_1 = \sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(1 - 4\alpha)(1 - 2\alpha)(1 + 2\alpha)}}{2}} > 0;$$

thus

$$\varphi(x) \geq \varphi(x_1) = \frac{x_1(x_1^2 + (1 - \alpha)^2)}{x_1^2 + \alpha^2}.$$

For $\frac{1}{2} \leq \alpha < 1$ and $x < 0$ maximum of $\varphi(x)$ occurs at

$$x = x_3 = -x_1 = -\sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(1 - 4\alpha)(1 - 2\alpha)(1 + 2\alpha)}}{2}} < 0;$$

thus

$$\varphi(x) \leq \varphi(x_3) = \frac{x_3(x_3^2 + (1 - \alpha)^2)}{x_3^2 + \alpha^2} = -\frac{x_1(x_1^2 + (1 - \alpha)^2)}{x_1^2 + \alpha^2}.$$

For $\frac{1}{4} \leq \alpha \leq \frac{1}{2}$, $\varphi(x)$ is an increasing function of x .

We also state without proofs the following well-known lemmas which we will be using in the sequel.

LEMMA 3. [5]. Let $g(z) \in S^*[A, B] = \{g \in A : \frac{zg'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, z \in U; -1 \leq B < A \leq 1\}$. Let $q(z) = \frac{zg'(z)}{g(z)}$ and $c \geq 0$; then $q(z) \prec \frac{1+Az}{1+Bz}$. If $q(z) + c = \rho e^{i\frac{\varphi\pi}{2}}$, then

$$\left. \begin{aligned} \frac{1-A}{1-B} + c \leq \rho \leq |q(z) + c| \leq \frac{1+A}{1+B} + c, \\ -t(A, B, c) \leq \varphi \leq t(A, B, c) \end{aligned} \right\} \text{ for } (B \neq -1)$$

where

$$\left. \begin{aligned} t(A, B, c) &= \frac{2}{\pi} \sin^{-1} \left(\frac{A - B}{1 - AB + c(1 - B^2)} \right); \\ \frac{1-A}{2} + c &< \rho < \infty \\ -1 &< \varphi < 1 \end{aligned} \right\} \text{ for } B = -1.$$

LEMMA 4. [6], [7]. Let h be a holomorphic function in \bar{U} except for at most one pole on ∂U and univalent in \bar{U} and let p be a holomorphic function in U with $p(0) = h(0)$ and $p(z) \neq p(0), z \in U$. If p is not subordinate to h then there exist points $z_0 \in U, \xi_0 \in \partial U$ and an $m \geq 1$ for which

- (1) $p\{z \in \mathbb{C} : |z| < |z_0|\} \subset h(U)$
- (2) $p(z_0) = h(\xi_0)$ and
- (3) $z_0 p'(z_0) = m \xi_0 h'(\xi_0)$.

Now we are in a position to prove our main result.

THEOREM 2.

(i) Let $\beta \in (0, \beta_0)$ be the solution of the equation

$(\mu + \lambda)\beta\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - C'\frac{\pi}{2}$, where $\lambda > 0, \mu > 0, 0 < \alpha \leq \frac{1}{4}, c \geq 0$ and $C' \leq 2 - \frac{\mu}{2}$ where

$$C' = \frac{2\mu}{\pi} \times \tan^{-1} \left\{ \frac{(\beta x_0(x_0^2 + (1-\alpha)^2)) (\sin \frac{\pi}{2}(1-t(A, B, c)))}{2(1-\alpha)(x_0^2 + \alpha^2) (\frac{1+A}{1+B} + c) + \beta x_0(x_0^2 + (1-\alpha)^2) \cos \frac{\pi}{2}(1-t(A, B, c))} \right\}$$

with

$$x_0 = \sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(1 - 4\alpha)(1 - 2\alpha)(1 + 2\alpha)}}{2}}$$

(ii) Let $\beta \in (0, \beta_0)$ be the solution of the equation $(\mu + \lambda)\beta\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - C'\frac{\pi}{2}$ where $\lambda > 0, \mu > 0, \frac{1}{2} \leq \alpha < 1, c \geq 0$ and $C' \leq 2 - \frac{\mu}{2}$ where

$$C' = \frac{2\mu}{\pi} \times \tan^{-1} \left\{ \frac{\beta x_1(x_1^2 + (1-\alpha)^2) [\sin \frac{\pi}{2}(1-t(A, B, c))]}{2(1-\alpha)(x_1^2 + \alpha^2) (\frac{1+A}{1+B} + c) + \beta x_1(x_1^2 + (1-\alpha)^2) \cos \frac{\pi}{2}(1-t(A, B, c))} \right\}$$

with

$$x_1 = \sqrt{\frac{(1 - 2\alpha - 2\alpha^2) + \sqrt{(4\alpha - 1)(2\alpha - 1)(1 + 2\alpha)}}{2}}$$

(iii) Let $\beta \in (0, \beta_0)$ be the solution of $(\mu + \lambda)\beta\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - C'\frac{\pi}{2}$ where $\lambda > 0, \mu > 0, \frac{1}{4} \leq \alpha \leq \frac{1}{2}, c \geq 0$ and $C' \leq 2 - \frac{\mu}{2}$.

(iv) Let $\beta \in (0, \beta_0)$ be the solution of the equation $(\mu + \lambda)\beta\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - C'\frac{\pi}{2}$ where $\lambda > 0, \mu > 0, \alpha = 0, c \geq 0$ and

$$C' = (\mu + \lambda)\beta + \frac{2\mu}{\pi} \tan^{-1} \left\{ \frac{\beta(\sin \frac{\pi}{2})(1 - t(A, B, c))}{(\frac{1+A}{1+B} + c) + \beta \cos \frac{\pi}{2}(1 - t(A, B, c))} \right\}.$$

Under the above cases, if $(p(z))^{\lambda+\mu} \left(1 + \frac{1}{q(z)+c} \frac{zp'(z)}{p(z)}\right)^\mu \prec \left(\frac{1+(1-2\alpha)z}{1-z}\right)^{C'} = h_{C',\alpha}(z)$ for some $q(z) \in A$ with $q(0) = 1$ and $q(z) \prec \frac{1+Az}{1+Bz}$ for $z \in U, -1 < B < A \leq 1$ and $c \geq 0$, then $p(z) \prec \left(\frac{1+(1-2\alpha)z}{1-z}\right)^\beta = h_{\beta,\alpha}(z)$.

For $B = -1$, the above result is true with C' calculated using the corresponding estimates from Lemma 3.

Proof. Suppose on the contrary that p is not subordinate to $h_{\beta,\alpha}(z)$. Then by Lemma 4, there exist $z_0 \in U, \xi_0 \in \partial U$ and an $m \geq 1$ for which

$$p(\{z \in \mathbb{C} : |z| < |z_0|\}) \subset h_{\beta,\alpha}(U), p(z_0) = h_{\beta,\alpha}(\xi_0)$$

and

$$z_0 p'(z_0) = m \xi_0 h'_{\beta, \alpha}(\xi_0).$$

We can assume that $p(z_0) = h_{\beta, \alpha}(\xi_0) = \left(\frac{1+(1-2\alpha)\xi_0}{1-\xi_0}\right)^\beta \neq 0$ for $\xi_0 \neq -1$. Thus $\frac{1+(1-2\alpha)\xi_0}{1-\xi_0} = \alpha + ix$ for $x \in \mathbb{R}$. Consequently, $\xi_0 = \frac{xi-(1-\alpha)}{xi+(1-\alpha)}$. By a simple calculation with $q(z) + c = \rho e^{i\varphi \frac{z}{2}}$, ($z \in U$), we have

$$\begin{aligned} (p(z_0))^{\mu+\lambda} \left(\frac{z_0 p'(z_0)}{(c+q(z_0))p(z_0)} + 1 \right)^\mu &= (h_{\beta, \alpha}(\xi_0))^{\mu+\lambda} \left(\frac{m \xi_0 h'_{\beta, \alpha}(\xi_0)}{(\rho e^{i\varphi \frac{z}{2}}) h_{\beta, \alpha}(\xi_0)} + 1 \right)^\mu \\ &= (\alpha + ix)^{\beta(\mu+\lambda)} \left(1 - \frac{m\beta(x^2 + (1-\alpha)^2)(\alpha - ix) (\cos \frac{\varphi\pi}{2} - i \sin \frac{\varphi\pi}{2})}{2\rho(1-\alpha)(x^2 + \alpha^2)} \right)^\mu. \end{aligned}$$

It suffice to show that the number mentioned above does not lie in $h_{C', \alpha}(U)$, for all real x and $m \geq 1$. Now,

$$\begin{aligned} \arg \left\{ (p(z_0))^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(c+q(z_0))p(z_0)} \right)^\mu \right\} \\ = \beta(\mu + \lambda) \arg(\alpha + ix) + \\ \mu \arg \left\{ 1 - \frac{\beta m (\cos \frac{\pi}{2} \varphi - i \sin \frac{\pi}{2} \varphi) (\alpha - ix)(x^2 + (1-\alpha)^2)}{2\rho(1-\alpha)(x^2 + \alpha^2)} \right\}; \end{aligned} \tag{6}$$

$$\begin{aligned} \arg \left\{ (p(z_0))^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(c+q(z_0))p(z_0)} \right)^\mu \right\} \\ \geq \beta(\mu + \lambda) \tan^{-1} \left(\frac{x}{\alpha} \right) \\ + \mu \tan^{-1} \left\{ \frac{[\beta m x \cos \frac{\pi\varphi}{2} + \alpha \beta m \sin \frac{\pi\varphi}{2}] (x^2 + (1-\alpha)^2)}{2\rho(1-\alpha)(x^2 + \alpha^2) + \beta m x (x^2 + (1-\alpha)^2) \sin \frac{\pi\varphi}{2}} \right\} \\ \geq \beta(\mu + \lambda) \tan^{-1} \left(\frac{x}{\alpha} \right) + \mu \tan^{-1} \left\{ \frac{(\beta x \cos \frac{\pi\varphi}{2} + \beta \alpha \sin \frac{\pi\varphi}{2}) (x^2 + (1-\alpha)^2)}{2\rho(1-\alpha)(x^2 + \alpha^2) + \beta x (x^2 + (1-\alpha)^2) \cos \frac{\pi}{2} (1-\varphi)} \right\}. \end{aligned} \tag{7}$$

Now let us apply Lemma 2 and Lemma 3 in (7).

Case 1. Let $0 < \alpha \leq \frac{1}{4}$ and $B \neq -1$. For $x > 0$, we have

$$\begin{aligned} \arg(p(z_0))^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \\ \geq 0 + \mu \tan^{-1} \left\{ \frac{\beta \sin \frac{\pi}{2} (1-t(A, B, c)) x_0 (x_0^2 + (1-\alpha)^2)}{2\rho(1-\alpha)(x_0^2 + \alpha^2) + \beta x_0 (x_0^2 + (1-\alpha)^2) \cos \frac{\pi}{2} (1-t(A, B, c))} \right. \\ \left. + \frac{\beta \alpha}{x} \frac{\cos \frac{\pi}{2} (1-t(A, B, c)) x_0 (x_0^2 + (1-\alpha)^2)}{2\rho(1-\alpha)(x_0^2 + \alpha^2) + \beta x_0 (x_0^2 + (1-\alpha)^2) \cos \frac{\pi}{2} (1-t(A, B, c))} \right\} \\ \geq \mu \tan^{-1} \left\{ \frac{\beta x_0 (x_0^2 + (1-\alpha)^2) \sin \frac{\pi}{2} (1-t(A, B, c))}{2(1-\alpha) \left(\frac{1+A}{1+B} + c \right) (x_0^2 + \alpha^2) + \beta x_0 (x_0^2 + (1-\alpha)^2) \cos \frac{\pi}{2} (1-t(A, B, c))} \right\} = C' \frac{\pi}{2} \end{aligned}$$

where

$$C' = \frac{2\mu}{\pi} \times \tan^{-1} \left\{ \frac{\beta x_0(x_0^2 + (1-\alpha)^2) \sin \frac{\pi}{2}(1-t(A, B, c))}{2(1-\alpha) \left(\frac{1+A}{1+B} + c \right) (x_0^2 + \alpha^2) + \beta x_0(x_0^2 + (1-\alpha)^2) \cos \frac{\pi}{2}(1-t(A, B, c))} \right\}$$

with x_0 as defined in Lemma 2.

On the other hand, from the connection between β and C' we get

$$\arg \left\{ p(z_0)^{\mu+\lambda} \left\{ 1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right\}^\mu \right\} \leq (\mu + \lambda)\beta \frac{\pi}{2} + \mu \frac{\pi}{2} = 2\pi - C' \frac{\pi}{2}.$$

Thus we have for $x > 0$,

$$C' \frac{\pi}{2} \leq \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \leq 2\pi - \frac{C' \pi}{2}. \tag{8}$$

For $x < 0$, applying the same method we can show that

$$-(2\pi - C' \frac{\pi}{2}) \leq \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \leq -C' \frac{\pi}{2} \tag{9}$$

From (8) and (9) we get

$$(2\pi - C' \frac{\pi}{2}) \geq \left| \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \right| \geq C' \frac{\pi}{2} \quad \text{for } x \neq 0.$$

When $x = 0$, $\xi_0 = -1$; also we have from (6)

$$\begin{aligned} p(z_0)^{\lambda+\mu} \left[1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right]^\mu &= h_{\beta, \alpha}^{\lambda+\mu} \left[1 + \frac{m \xi_0 h'_{\beta, \alpha}(\xi_0)}{(q(z) + c)h_{\beta, \alpha}(\xi_0)} \right]^\mu \\ &= \alpha^{\beta(\lambda+\mu)} \left[1 - e^{-i\frac{\varrho\pi}{2}} \frac{m\beta(1-\alpha)}{\rho 2\alpha} \right]^\mu \\ &< \alpha^{\beta(\lambda+\mu)} \\ &= \alpha^{4-\mu-C'} < \alpha^{C'} \text{ if } C' \leq \frac{4-\mu}{2}; \\ &\notin h_{C', \alpha}(U). \end{aligned}$$

Thus for all $x \in \mathbb{R}$, we have $p(z_0)^{\lambda+\mu} \left[1 + \frac{z_0 p'(z_0)}{(c + q(z))p(z_0)} \right]^\mu$ does not belong to the region bounded by the Wesolowski curve E , with $\beta' = C'$, a contradiction to the hypothesis $p(z_0)^{\lambda+\mu} \left[1 + \frac{z_0 p'(z_0)}{(c + q(z))p(z)} \right]^\mu < \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^{C'}$.

Hence $p(z)$ must be subordinate to $h_{\beta, \alpha}(z) = \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^\beta$ under the stated conditions of the theorem.

Case 2. Let $\frac{1}{2} \leq \alpha < 1$ and $B \neq -1$. A similar argument as in Case 1 gives the result in this case and hence the details are omitted.

Case 3. Let $\frac{1}{4} \leq \alpha \leq \frac{1}{2}$ and $B \neq -1$. Then $\varphi(x)$ in Lemma 2 is an increasing function of x . Hence for $x \neq 0$,

$$-\beta(\mu + \lambda)\frac{\pi}{2} - \mu\frac{\pi}{2} \leq \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \leq \beta(\mu + \lambda)\frac{\pi}{2} + \mu\frac{\pi}{2}.$$

The rest of the proof follows from the relation between C' and β (as in case (i)).

Case 4. Let $\alpha = 0$ and $B \neq -1$. Then from (6) for $x > 0$, left hand side of (6) $\geq C'\frac{\pi}{2}$, where

$$\frac{C'\pi}{2} = (\mu + \lambda)\frac{\beta\pi}{2} + \mu \tan^{-1} \left\{ \frac{\beta \sin \frac{\pi}{2}(1-t(A, B, c))}{\left(\frac{1+A}{1+B} + c\right) + \beta \cos \frac{\pi}{2}(1-t(A, B, c))} \right\}$$

On the other hand, from the connection between C' and β we get

$$\begin{aligned} \arg \left\{ p(z_0)^{\lambda+\mu} \left[1 + \frac{z_0 p'(z_0)}{(c + q(z))p(z_0)} \right]^\mu \right\} &\leq (\lambda + \mu)\beta\frac{\pi}{2} + \mu\frac{\pi}{2} \\ &= 2\pi - C'\frac{\pi}{2}. \end{aligned}$$

Thus for $x > 0$

$$C'\frac{\pi}{2} \leq \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \leq 2\pi - C'\frac{\pi}{2}.$$

For $x < 0$, a similar argument gives

$$-(2\pi - C'\frac{\pi}{2}) \leq \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z) + c)p(z_0)} \right)^\mu \right\} \leq -C'\frac{\pi}{2}.$$

Thus for all $x \neq 0$,

$$(2\pi - C'\frac{\pi}{2}) \geq \left| \arg \left\{ p(z_0)^{\mu+\lambda} \left(1 + \frac{z_0 p'(z_0)}{(q(z)c)p(z_0)} \right)^\mu \right\} \right| \geq C'\frac{\pi}{2}.$$

When $x = 0$ left hand side of (6) $= 0 \notin h_{C',0}(\mu)$.

Hence for all $x \in \mathbb{R}$, in case $\alpha = 0$, we get a contradiction.

The case $B = -1$ can be treated similarly with corresponding estimates from Lemma 3.

Thus we have established the Theorem completely.

This theorem when $q(z) = 1$ has been treated in [9].

Now we have:

COROLLARY 1. Let $f \in A$. If

$$\left(\left(\frac{f'(z)}{g'(z)} \right)^\mu \left(\frac{f(z)}{g(z)} \right)^\lambda \right) \prec \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^{C'}$$

for some $g \in M(\mu, A, B)$ - the generalized Mocanu class of functions -

$$= \left\{ g \in A : (1-\mu) \frac{zg'(z)}{g(z)} + \mu \left(1 + \frac{zg''(z)}{g'(z)} \right) \prec \frac{1+Az}{1+Bz} \text{ for } z \in U, -1 \leq B < A \leq 1 \right\},$$

then $\frac{f(z)}{g(z)} \prec \left(\frac{1+(1-2\alpha)z}{1-z} \right)^\beta$, where β and C' are connected as in Theorem 2 with $c = 0$.

For $\alpha = 0$ the above result has been obtained in [3].

Proof. Let $p(z) = \frac{f(z)}{g(z)}$ and $q(z) = \frac{zg'(z)}{g(z)}$. Then by an application of differential subordinations, we get immediately $g \in S^*(A, B)$. Now applying Theorem 2 with $c = 0$ we get this result immediately.

First we define the new class $\mathcal{CS}(\alpha, \beta, A, B)$.

DEFINITION 1. Let $\mathcal{CS}(\alpha, \beta, A, B)$ denote the class of functions $f \in A$ such that $\frac{f(z)}{g(z)} \prec h_{\beta, \alpha}(z) = \left(\frac{1+(1-2\alpha)z}{1-z} \right)^\beta$ for $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ and $g \in S^*(A, B)$. This is a generalized class of the class \mathcal{CS}^* of close-to-star functions of M. O. Reade [11].

We now define another new class $\mathcal{CS}_\mu(\alpha, \beta, A, B)$ - Reade-Wesolowski class of functions.

DEFINITION 2. Let $\mathcal{CS}_\mu(\alpha, \beta, A, B)$ be the class of functions $f \in A$ such that $\frac{f^\mu(z)f^{1-\mu}(z)}{g^\mu(z)g^{1-\mu}(z)} \prec h_{\beta, \alpha}(z)$ where $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, \mu \geq 0$ and $g \in M(\mu, A, B), -1 \leq B < A \leq 1$ - the generalized Mocanu class of functions.

For $\alpha = 0, \beta = 1, A = 1, B = -1$ this class reduces to the class of μ -close-to-convex functions of R. Bharati [1].

Further if $\mu = 0$ then this class reduces to the class of \mathcal{CS}^* of M. O. Reade.

It is immediate from the Corollary 1 with $\lambda = 1 - \mu$ that if $f \in \mathcal{CS}_\mu(\alpha, C', A, B)$ with $g \in M(\mu, A, B) \subset S^*(A, B)$, then $\frac{f(z)}{g(z)} \prec \left(\frac{1+(1-\alpha)z}{1-z} \right)^\beta$ with $g \in S^*(A, B)$; that is $f \in \mathcal{CS}(\alpha, \beta, A, B)$ where β and C' are connected as in Theorem 2 with $c = 0$.

Let $F^{\frac{1}{\mu}}(z) = \frac{(c+\frac{1}{\mu})}{z^c} \int_0^z t^{c-1} f^{\frac{1}{\mu}}(t) dt$ be the Ruscheweyh integral operator of f where $\mu > 0$. Then we get $\frac{1}{\mu} z^c F^{\frac{1}{\mu}-1}(z)F(z) + cz^{c-1}F^{\frac{1}{\mu}}(z) = \left(c + \frac{1}{\mu} \right) z^{c-1} f^{\frac{1}{\mu}}(z)$; or $\frac{1}{\mu} zF'(z)F^{\frac{1}{\mu}-1}(z) + cF^{\frac{1}{\mu}}(z) = \left(c + \frac{1}{\mu} \right) f^{\frac{1}{\mu}}(z)$. This on differentiation again with respect to z gives

$$\begin{aligned} \frac{1}{\mu} (F'(z) + zF''(z))F^{\frac{1}{\mu}-1} + \frac{1}{\mu} zF'(z) \left(\frac{1}{\mu} - 1 \right) F^{\frac{1}{\mu}-2}(z)F'(z) + \frac{c}{\mu} F^{\frac{1}{\mu}-1}(z)F'(z) \\ = \left(c + \frac{1}{\mu} \right) \frac{1}{\mu} f^{\frac{1}{\mu}-1}(z)f'(z). \end{aligned}$$

Thus

$$F^{\frac{1}{\mu}-1}F'(z) \left\{ 1 + z \frac{F''(z)}{F'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zF'(z)}{F(z)} + c \right\} = \left(c + \frac{1}{\mu} \right) f'(z) f^{\frac{1}{\mu}-1}(z). \quad (10)$$

Similarly we get

$$G^{\frac{1}{\mu}-t}G'(z) \left\{ 1 + z \frac{G''(z)}{G'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zG'(z)}{G(z)} + c \right\} = \left(c + \frac{1}{\mu} \right) g'(z) g^{\frac{1}{\mu}-1}(z) \quad (11)$$

where G is the Ruscheweyh integral operator of g .

Using the differential subordination technique it is easy to show that for $c \geq 0$,

$$G(z) \in M(\mu, A, B) \text{ whenever } g(z) \in M(\mu, A, B). \text{ Let } p(z) = \frac{F^{\frac{1}{\mu}-1}(z)F'(z)}{G^{\frac{1}{\mu}-1}(z)G'(z)}.$$

Taking logarithmic derivatives on both sides we get

$$\begin{aligned} z \frac{p'(z)}{p(z)} &= \frac{zF''(z)}{F'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zF'(z)}{F(z)} - \left(\frac{zG''(z)}{G'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zG'(z)}{G(z)} \right) \\ &= 1 + c + \frac{zF''(z)}{F'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zF'(z)}{F(z)} \\ &\quad - \left(1 + c + \frac{zG''(z)}{G'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zG'(z)}{G(z)} \right) \end{aligned}$$

$$\text{Let } q(z) = 1 + z \frac{zG''(z)}{G'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zG'(z)}{G(z)}.$$

Then

$$\frac{zp'(z)}{p(z)} + q(z) + c = 1 + c + \frac{zF''(z)}{F'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zF'(z)}{F(z)};$$

from (10) and (11) we get

$$\begin{aligned} \frac{zp'(z)}{p(z)(q(z) + c)} + 1 &= \frac{1 + \frac{zF''(z)}{F'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zF'(z)}{F(z)} + c}{1 + z \frac{G''(z)}{G'(z)} + \left(\frac{1}{\mu} - 1 \right) \frac{zG'(z)}{G(z)} + c} \\ &= \frac{f'(z)f^{\frac{1}{\mu}-1}(z)}{F^{\frac{1}{\mu}-1}F'(z)}; \frac{G^{\frac{1}{\mu}-1}(z)G'(z)}{g'(z)g^{\frac{1}{\mu}-1}(z)} \\ &= \frac{f'(z)f^{\frac{1}{\mu}-1}(z)}{g'(z)g^{\frac{1}{\mu}-1}(z)} \frac{1}{p(z)}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{zp'(z)}{q(z) + c} + p(z) &= \frac{f'(z)f^{\frac{1}{\mu}-1}(z)}{g'(z)g^{\frac{1}{\mu}-1}(z)} \\ p^\mu(z) \left(\frac{zp'(z)}{(q(z) + c)p(z)} + 1 \right)^\mu &= \frac{(f')^\mu f^{1-\mu}(z)}{(g')^\mu g^{1-\mu}(z)} \prec \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^{C'} = h_{C',\alpha}(z) \end{aligned}$$

since $f \in \mathcal{CS}_\mu(\alpha, C', A, B)$ with respect to $g \in M(\mu, A, B)$. Then by Theorem 2 with $\lambda = 0$, we get

$$p(z) \prec \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^\beta = h_{\beta,\alpha}(z)$$

where C' and β are connected as in the Theorem 2 with $\lambda = 0$.

Thus we get

THEOREM 3. If $f \in \mathcal{CS}_\mu(\alpha, C', A, B)$ then

$$\frac{F^{\frac{1}{\mu}-1}(z)F'(z)}{G^{\frac{1}{\mu}-1}(z)G'(z)} \prec h_{\beta, \alpha}(z) = \left(\frac{1 + (1 - 2\alpha)z}{1 - z} \right)^{\beta}$$

or $F \in \mathcal{CS}_\mu(\alpha, \beta\mu, A, B)$ where $\beta \in (0, \beta_0)$ be the solution of the equation $\beta\mu\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - \frac{\pi}{2}C'$ for various cases of Theorem 2.

REMARK. Let $\beta \in (0, \frac{\beta_0}{2})$ where

$$\beta_0\mu\frac{\pi}{2} = (2 - \frac{\mu}{2})\pi - C'\frac{\pi}{2} = (4 - \mu - C')\frac{\pi}{2}.$$

or $\beta_0 = \frac{4-\mu-C'}{\mu}$ and $\beta_0\mu \leq 2$. Then for $2 - C' \leq \mu \leq \min_i\{\frac{2}{\beta_0}, 4 - C'\}$, $F \in \mathcal{CS}(\alpha, \beta\mu, A, B)$ whenever $f \in \mathcal{CS}(\alpha, C', A, B)$. In particular, if $C' = 1$ and $f \in \mathcal{CS}_\mu(\alpha, 1, A, B)$ then $F \in \mathcal{CS}_\mu(\alpha, \beta\mu, A, B)$ for $\beta \in (0, \frac{\beta_0}{2})$ $\beta_0 = \frac{3-\mu}{\mu}$ and $1 \leq \mu \leq \min(\frac{2}{\beta_0}, 3)$. Further if $\alpha = 0, A = 1, B = -1$ and $1 \leq \mu \leq \min(3, \frac{2}{\beta_0})$, this result is an improvement of a result by K.S.Padmanabhan and R.Bharati [8] regarding the closure of the class of μ -close-to-convex functions under the Ruscheweyh integral operator.

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R. Parvatham
The Ramanujan Institute for Advanced Study in Mathematics
University of Madras
Chennai-600 005
India
e-mail: parvatham@hotmail.com