

CONTINUITY PROPERTIES OF RIESZ POTENTIALS FOR FUNCTIONS IN $L^{p(\cdot)}$ OF VARIABLE EXPONENT

TOSHIHIDE FUTAMURA AND YOSHIHIRO MIZUTA

(communicated by B. Opic)

Abstract. Our aim in this paper is to deal with 0-Hölder continuity for Riesz potentials of functions belonging to Lebesgue's L^p space of variable exponent, in the borderline case of Sobolev's theorem. We are also concerned with exponential integrability for Riesz potentials.

1. Introduction

Let \mathbf{R}^n denote the n -dimensional Euclidean space. We consider the Riesz potential of order α for a locally integrable function f on \mathbf{R}^n , which is defined by

$$U_\alpha f(x) = \int |x - y|^{\alpha-n} f(y) dy.$$

Here $0 < \alpha < n$. Following Kováčik and Rákosník [9], we consider a positive continuous function $p(\cdot) : \mathbf{R}^n \rightarrow [1, \lambda)$, $1 < \lambda < \infty$, and a measurable function f satisfying

$$\int |f(y)|^{p(y)} dy < \infty.$$

Recently Diening [3] has established embedding results for Riesz potentials of such functions. For related results, see also Edmunds-Rákosník [4], Futamura-Mizuta-Shimomura [6] and Růžička [13]. In these discussions, the continuity of Hardy-Littlewood maximal functions is a crucial tool (see Diening [2]).

In case $p(\cdot)$ is a constant p_0 and $p_0 > n/\alpha$, well known Sobolev's theorem says that $U_\alpha f$ is continuous in \mathbf{R}^n (see e.g. [1], [10], [12]). Our first aim in this paper is to discuss the continuity for α -potentials of functions in $L^{p(\cdot)}$ spaces when $p(x) \geq n/\alpha$ for $x \in \mathbf{R}^n$ and $p(\cdot)$ satisfies a so called 0-Hölder condition, as an extension of Harjulehto-Hästö [7].

We also study exponential integrabilities of α -potentials when they are not continuous, as an extension of Trudinger's exponential integrability (see Hedberg [8] and Adams-Hedberg [1]).

Mathematics subject classification (2000): 31B15, 46E35.

Key words and phrases: Riesz potentials, Sobolev's embedding theorem, Trudinger's exponential integrability.

2. Continuity of potentials

Throughout this paper, let C denote various constants independent of the variables in question.

Let G be a bounded open set in \mathbf{R}^n and $B(x_0, r_0) \subset G$, where $B(x_0, r_0)$ denotes the open ball centered at x_0 of radius $r_0 > 0$. Consider a positive continuous function $p(\cdot)$ on G . In this and the next sections let us assume that :

$$(p1) \inf_{G \setminus B(x_0, r_0)} p(x) > p_- = n/\alpha \text{ and } p_+ = \sup_G p(x) < \infty;$$

$$(p2) p(y) \geq p_- + \frac{a \log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)} + \frac{\tilde{a}}{\log(1/|x_0 - y|)} \text{ for } y \in B(x_0, r_0),$$

where $a \geq 0$ and \tilde{a} is a real number. In our discussions we may assume that

$$(p3) p(y) \leq p_- + \frac{a \log(\log(1/|x_0 - y|))}{\log(1/|x_0 - y|)} + \frac{C}{\log(1/|x_0 - y|)} \text{ for } y \in B(x_0, r_0) .$$

Set

$$\omega_{a', a''}(r) = \frac{a' \log(\log(1/r))}{\log(1/r)} + \frac{a''}{\log(1/r)}$$

and $\omega_{a', a''}(0) = 0$. If $a' > 0$ or $a' = 0$ and $a'' > 0$, then we can find $r^* > 0$ so small that $\omega_{a', a''}$ is nondecreasing on the interval $[0, 2r^*]$ and

$$\omega_{a', a''}(s + t) \leq \omega_{a', a''}(s) + \omega_{a', a''}(t) \tag{1}$$

for $0 \leq s \leq t \leq r^*$.

Let $1/p'(x) = 1 - 1/p(x)$ and $1/p'_- = 1 - 1/p_-$.

We begin with the following result.

LEMMA 1. *If $a > (n - \alpha)/\alpha^2$, then*

$$\int_{G \cap B(x, \delta)} |x - y|^{p'(y)(\alpha - n)} dy \leq C(\log(1/\delta))^{1 - a\alpha^2/(n - \alpha)}$$

for all $x \in G$ and $\delta \in (0, 2^{-1})$.

Proof. First note that

$$p'(y) - p'_- = -\frac{p(y) - p_-}{(p(y) - 1)(p_- - 1)} = -\frac{p(y) - p_-}{(p_- - 1)^2} + \frac{(p(y) - p_-)^2}{(p(y) - 1)(p_- - 1)^2}.$$

Since $p_- - 1 = (n - \alpha)/\alpha$, by conditions on p , we can find $C > 0$ so that

$$p'(y) \leq p'_- - \omega_{a', -C}(|x_0 - y|) \quad (a' = a\alpha^2/(n - \alpha)^2) \tag{2}$$

for all $y \in B(x_0, r_0)$. For simplicity, set

$$\omega(r) = \omega_{a', -C}(r) = \frac{a\alpha^2}{(n - \alpha)^2} \frac{\log(\log(1/r))}{\log(1/r)} - \frac{C}{\log(1/r)}.$$

Noting that ω is nondecreasing and doubling on $[0, r_0]$ by (1), we have for $0 < \delta \leq |x_0 - x|/2$ and $x \in B(x_0, r_0/2)$

$$\begin{aligned} \int_{B(x, \delta)} |x - y|^{p'(y)(\alpha-n)} dy &\leq \sum_j \int_{B(x, 2^{-j+1}\delta) \setminus B(x, 2^{-j}\delta)} |x - y|^{p'(y)(\alpha-n)} dy \\ &\leq \sum_j (2^{-j}\delta)^{(\alpha-n)(p'_- - \omega(2^{-j}\delta))} \sigma_n(2^{-j+1}\delta)^n \\ &\leq C \sum_j (2^{-j}\delta)^{-(\alpha-n)\omega(2^{-j}\delta)} \\ &\leq C \sum_j (\log 1/(2^{-j}\delta))^{-a\alpha^2/(n-\alpha)} \\ &\leq C \int_0^\delta (\log(1/t))^{-a\alpha^2/(n-\alpha)} t^{-1} dt \\ &= C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)}, \end{aligned}$$

since $a > (n - \alpha)/\alpha^2$, where σ_n denotes the volume of the unit ball. If $y \in G \setminus B(x, |x_0 - x|/2)$, then $|x_0 - y| \leq 3|x - y|$, so that

$$\begin{aligned} \int_{B(x, \delta) \setminus B(x, |x_0 - x|/2)} |x - y|^{p'(y)(\alpha-n)} dy &\leq C \int_{G \cap B(x_0, 3\delta)} |x_0 - y|^{p'(y)(\alpha-n)} dy \\ &\leq C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)} \end{aligned}$$

when $|x_0 - x|/2 \leq \delta \leq r_0/4$. Therefore it follows that

$$\int_{B(x, \delta)} |x - y|^{p'(y)(\alpha-n)} dy \leq C(\log(1/\delta))^{1-a\alpha^2/(n-\alpha)}$$

for $0 < \delta < 1/2$ and $x \in B(x_0, r_0/2)$.

Noting from condition (p1) that $p_0 = \inf_{y \in G \setminus B(x_0, r_0/4)} p(y) > n/\alpha$, we see that

$$\int_{G \cap B(x, \delta)} |x - y|^{p'(y)(\alpha-n)} dy \leq C\delta^{(\alpha p_0 - n)/(p_0 - 1)}$$

for $x \in G \setminus B(x_0, r_0/2)$ and $\delta > 0$.

Now the proof is completed. \square

Define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot), G} = \inf\{\lambda > 0 : \int_G \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \leq 1\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$.

THEOREM 1. *Let f be a nonnegative measurable function on a bounded open set G with $\|f\|_{p(\cdot)} \leq 1$. If $a > (n - \alpha)/\alpha^2$, then $U_\alpha f$ is continuous in G . Further,*

$$|U_\alpha f(x) - U_\alpha f(z)| \leq C(\log(1/|x - z|))^{-A}$$

whenever $x, z \in G$ and $|x - z| < 1/2$, where $A = (a\alpha^2/(n - \alpha) - 1)/p'_-$.

REMARK 1. In view of Sobolev's theorem, we see that $U_\alpha f$ is continuous in $G \setminus \{x_0\}$. Harjulehto-Hästö [7] have also discussed the continuity of Sobolev functions.

Proof of Theorem 1. First note that

$$\int_G f(y)^{p(y)} dy \leq 1 \quad (3)$$

since $\|f\|_{p(\cdot)} \leq 1$ by the assumption. Then, for $0 < \mu < 1$, we have by Young's inequality and Lemma 1

$$\begin{aligned} \int_{G \cap B(x, \delta)} |x - y|^{\alpha - n} f(y) dy &\leq \mu \int_{G \cap B(x, \delta)} \left\{ (|x - y|^{\alpha - n} / \mu)^{p'(y)} + f(y)^{p(y)} \right\} dy \\ &\leq \mu \left(\mu^{-p'_-} \int_{G \cap B(x, \delta)} |x - y|^{(\alpha - n)p'(y)} dy + 1 \right) \\ &\leq \mu \left(C \mu^{-p'_-} (\log(1/\delta))^{1 - \alpha \alpha^2 / (n - \alpha)} + 1 \right) \end{aligned}$$

whenever $x \in G$ and $0 < \delta < 1/2$. Now, considering μ such that $\mu^{p'_-} = (\log(1/\delta))^{1 - \alpha \alpha^2 / (n - \alpha)}$, we find

$$\int_{G \cap B(x, \delta)} |x - y|^{\alpha - n} f(y) dy \leq C (\log(1/\delta))^{-A}. \quad (4)$$

Hence, if $x, z \in G$ and $|x - z| < 1/4$, then we have

$$\int_{G \cap B(x, 2|x-z|)} |x - y|^{\alpha - n} f(y) dy \leq C (\log(1/|x - z|))^{-A}.$$

On the other hand we find

$$\begin{aligned} \int_{G \setminus B(x, 2|x-z|)} \left| |x - y|^{\alpha - n} - |z - y|^{\alpha - n} \right| f(y) dy \\ \leq C |x - z| \int_{G \setminus B(x, 2|x-z|)} |x - y|^{\alpha - n - 1} f(y) dy. \end{aligned}$$

This can be estimated along the same lines as above. For simplicity set $\delta = 2|x - z| < 1/2$. Then, for $\mu \geq 1$, letting

$$E = \{y \in G \setminus B(x, 2|x - z|) : |x - y|^{\alpha - n - 1} \geq \mu\},$$

we have by Young's inequality and (2)

$$\begin{aligned} \int_{G \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} |x - y|^{\alpha - n - 1} f(y) dy \\ \leq \mu \int_{G \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} \left\{ (|x - y|^{\alpha - n - 1} / \mu)^{p'(y)} + f(y)^{p(y)} \right\} dy \\ \leq C \mu \left(\int_{E \setminus B(x_0, \delta)} (|x - y|^{\alpha - n - 1} / \mu)^{p'(y)} dy + 1 \right) \\ \leq C \mu \left(\int_{E \setminus B(x_0, \delta)} (|x - y|^{\alpha - n - 1} / \mu)^{p'_- - \omega(\delta)} dy + 1 \right) \end{aligned}$$

$$\begin{aligned} &\leq C\mu \left(\mu^{-p'_- + \omega(\delta)} \int_{G \setminus B(x, \delta)} |x - y|^{(\alpha - n - 1)(p'_- - \omega(\delta))} dy + 1 \right) \\ &\leq C\mu \left(\mu^{-p'_- + \omega(\delta)} \delta^{(\alpha - n - 1)(p'_- - \omega(\delta)) + n} + 1 \right) \\ &\leq C\mu \left(\mu^{-p'_- + \omega(\delta)} \delta^{-p'_-} (\log(1/\delta))^{(\alpha - n - 1)\alpha^2 / (n - \alpha)^2} + 1 \right). \end{aligned}$$

Now, considering μ such that $\mu = \delta^{-1}(\log(1/\delta))^{-\alpha^2 / \{p'_-(n - \alpha)\}}$, we find

$$\int_{G \setminus \{B(x_0, \delta) \cup B(x, \delta)\}} |x - y|^{\alpha - n - 1} f(y) dy \leq C\delta^{-1}(\log(1/\delta))^{-\alpha^2 / \{p'_-(n - \alpha)\}}.$$

Further, we obtain by (4)

$$\begin{aligned} \int_{G \cap B(x_0, \delta) \setminus B(x, \delta)} |x - y|^{\alpha - n - 1} f(y) dy &\leq \delta^{-1} \int_{G \cap B(x_0, \delta)} |x_0 - y|^{\alpha - n} f(y) dy \\ &\leq C\delta^{-1}(\log(1/\delta))^{-A}. \end{aligned}$$

Therefore it follows that

$$\int_{G \setminus B(x, 2|x - z|)} ||x - y|^{\alpha - n} - |z - y|^{\alpha - n}| f(y) dy \leq C(\log(1/|x - z|))^{-A}.$$

Now we establish

$$\begin{aligned} &|U_\alpha f(x) - U_\alpha f(z)| \\ &\leq \int_{G \cap B(x, 2|x - z|)} |x - y|^{\alpha - n} f(y) dy + \int_{G \cap B(x, 2|x - z|)} |z - y|^{\alpha - n} f(y) dy \\ &\quad + \int_{G \setminus B(x, 2|x - z|)} ||x - y|^{\alpha - n} - |z - y|^{\alpha - n}| f(y) dy \\ &\leq C(\log(1/|x - z|))^{-A}, \end{aligned}$$

as required. \square

COROLLARY 1. *Suppose*

$$p(x) = p(x_1, \dots, x_n) \geq n/\alpha + \frac{a \log(e + \log(1/|x_n|))}{\log(e/|x_n|)}$$

for $a > (n - \alpha)/\alpha^2$. Let f be a nonnegative measurable function on $B = B(0, 1)$ with $\|f\|_{p(\cdot), B} \leq 1$. Then $U_\alpha f$ is continuous in B and it satisfies

$$|U_\alpha f(x) - U_\alpha f(z)| \leq C(\log(1/|x - z|))^{-A}$$

whenever $x, z \in B(0, 1/2)$ and $|x - z| < 1/2$, where $A = (a\alpha^2 / (n - \alpha) - 1) / p'_-$.

Proof. According to the proof of Theorem 1, it suffices to show that

$$\int_{B(x,r)} |x - y|^{(\alpha-n)p'(y)} dy \leq C(\log(1/r))^{1-a\alpha^2/(n-\alpha)} \tag{5}$$

for $0 < r < 1/2$ and $|x| < 1/2$. To show this, we may assume that

$$p'(y) \leq p'_- - \omega(|y_n|) \quad \text{for } y \in B,$$

where $\omega(r) = (a\alpha^2/(n - \alpha)^2) \log(\log(1/r))/\log(1/r) - C/\log(1/r)$ for $0 < r \leq r_0$ and $\omega(r) = \omega(r_0)$ for $r > r_0$. Then, by use of Lemma 1, we have

$$\begin{aligned} \int_{B(x,r)} |x - y|^{(\alpha-n)p'(y)} dy &\leq C \int_{\{y_n: |x_n - y_n| < r\}} |x_n - y_n|^{-1+(n-\alpha)\omega(|y_n|)} dy_n \\ &\leq C(\log(1/r))^{1-a\alpha^2/(n-\alpha)}. \end{aligned}$$

Thus (5) holds, and the proof is completed. \square

REMARK 2. Let $b > (a + 1)/n > 1$, $0 < r_0 < 1/e$ and

$$p(y) = n + \frac{a \log(\log(1/|y|))}{\log(1/|y|)}$$

for $y \in B(0, r_0)$. Consider the function

$$f(y) = |y|^{-1}(\log(1/|y|))^{-b}$$

for $y \in B(0, r_0)$ and $f = 0$ on $\mathbf{R}^n \setminus B(0, r_0)$. Then we easily see that

$$\int_{B(0,r_0)} |x - y|^{1-n} f(y) dy \geq C(\log(1/|x|))^{1-b} \quad \text{for } x \in B(0, r_0)$$

and

$$\int_{B(0,r_0)} f(y)^{p(y)} dy \leq \int_{B(0,r_0)} \{ |y|^{-1}(\log(1/|y|))^{-b} \}^n (\log(1/|y|))^a dy < \infty$$

since $-bn + a + 1 < 0$ by our assumption.

This means that the exponent A in Theorem 1 is best possible.

REMARK 3. Let $a = n - 1$ and

$$p(y) = n + \frac{(n - 1) \log(\log(1/|y|))}{\log(1/|y|)}$$

for $y \in B(0, r_0)$. Then there exists a measurable function f on \mathbf{R}^n such that $U_1 f(0) = \infty$ and $\|f\|_{p(\cdot)} < \infty$.

In fact, for $1/n < b \leq 1$, consider the function

$$f(y) = |y|^{-1}(\log(1/|y|))^{-1}(\log(\log(1/|y|)))^{-b}$$

for $y \in B(0, r_0)$ and $f = 0$ on $\mathbf{R}^n \setminus B(0, r_0)$. Then we have

$$\int_{B(0,r_0)} |y|^{1-n} f(y) dy = \infty.$$

Since $bn > 1$ by our assumption, we obtain

$$\begin{aligned} \int_{B(0,r_0)} f(y)^{p(y)} dy &\leq \int_{B(0,r_0)} \{ |y|^{-1} (\log(1/|y|))^{-1} (\log(\log(1/|y|)))^{-b} \}^n (\log(1/|y|))^{n-1} dy \\ &= \int_{B(0,r_0)} |y|^{-n} (\log(1/|y|))^{-1} (\log(\log(1/|y|)))^{-bn} < \infty \end{aligned}$$

In this case we can show exponential integrability (see e.g. [1]), as will be discussed soon.

3. Exponential integrability

This section concerns with $p(\cdot)$ such that

$$p(y) \leq p_- + \frac{n - \alpha \log(\log(1/|x_0 - y|))}{\alpha^2 \log(1/|x_0 - y|)}$$

for $y \in B(x_0, r_0)$. In this case, since α -potentials of $f \in L^{p(\cdot)}(G)$ may not be continuous, we discuss the exponential integrability of Trudinger type. Our discussions here can be carried out along the same lines as in Hedberg [8].

Before doing so we prepare the following lemma under conditions (p1) and (p2).

LEMMA 2. *If $0 < b < a \leq (n - \alpha)/\alpha^2$, then*

$$\int_{G \setminus B(x, \delta)} |x - y|^{(\alpha-n)p'(y)} dy \leq C (\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}$$

for $x \in G$ and $0 < \delta < 1/2$.

Proof. For $0 < b < a \leq (n - \alpha)/\alpha^2$, set

$$\omega(r) = \frac{b\alpha^2}{(n - \alpha)^2} \frac{\log(\log(1/r))}{\log(1/r)}.$$

As in (2), we can find $r_1 > 0$ such that

$$p'(y) \leq p'_- - \omega(|x_0 - y|)$$

for all $y \in B(x_0, r_1)$; in this proof we assume that $r_1 = 4r_0$.

If $x \in B(x_0, 2r_0)$, then we have

$$\begin{aligned} \int_{B(x, |x_0-x|/2) \setminus B(x, \delta)} |x - y|^{p'(y)(\alpha-n)} dy &\leq \sum_j \int_{B(x, 2^j \delta) \setminus B(x, 2^{j-1} \delta)} |x - y|^{p'(y)(\alpha-n)} dy \\ &\leq \sum_j (2^{j-1} \delta)^{(\alpha-n)(p'_- - \omega(2^{j-1} \delta))} \sigma_n(2^j \delta)^n \\ &\leq C \sum_j (2^{j-1} \delta)^{-(\alpha-n)\omega(2^{j-1} \delta)} \end{aligned}$$

$$\begin{aligned} &\leq C \sum_j (\log 1/(2^{j-1}\delta))^{-b\alpha^2/(n-\alpha)} \\ &\leq C \int_{\delta}^{r_0} (\log(1/t))^{-b\alpha^2/(n-\alpha)} t^{-1} dt \\ &= C(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}. \end{aligned}$$

If $\delta \geq |x_0 - x|/2$ and $x \in B(x_0, 2r_0)$, then

$$\begin{aligned} \int_{B(x_0, 4r_0) \setminus B(x, \delta)} |x - y|^{p'(y)(\alpha-n)} dy &\leq C \int_{B(x_0, 4r_0) \setminus B(x, \delta)} |x_0 - y|^{p'(y)(\alpha-n)} dy \\ &\leq C(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}. \end{aligned}$$

Finally, since $\inf_{G \setminus B(x_0, r_0)} p(x) > p_- = n/\alpha$, we note that

$$\int_G |x - y|^{p'(y)(\alpha-n)} dy \leq C < \infty$$

for $x \in G \setminus B(x_0, 2r_0)$.

Thus the required conclusion follows from these facts. \square

If f is a locally integrable function on G , then we consider Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\sigma_n r^n} \int_{G \cap B(x,r)} |f(y)| dy.$$

We next prove the estimate of Riesz potentials by use of maximal functions, as in Hedberg [8].

LEMMA 3. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. If $0 < a \leq (n - \alpha)/\alpha^2$ and $A > (1 - a\alpha^2/(n - \alpha))/p'_-$, then*

$$U_a f(x) \leq C(\log(Mf(x) + 2))^A.$$

Proof. If $(1 - a\alpha^2/(n - \alpha))/p'_- < A$, then there exist $0 < b < a$ and $0 < p_0 < p'_-$ such that

$$(1 - b\alpha^2/(n - \alpha))/p_0 < A.$$

We can find $r_1 > 0$ such that $p'(y) > p_0$ for $y \in B(x_0, r_1)$ and

$$\int_{B(x_0, r_1) \setminus B(x, \delta)} (|x - y|^{\alpha-n}/\mu)^{p'(y)} dy \leq C\mu^{-p_0} (\log(1/\delta))^{1-b\alpha^2/(n-\alpha)}$$

for $\mu > 1$ and $x \in G$; in this proof, we may assume that $r_1 = 4r_0$. Since (3) holds by the assumption $\|f\|_{p(\cdot)} \leq 1$, we have for $\mu > 1$

$$\begin{aligned} &\int_{B(x_0, 4r_0) \setminus B(x, \delta)} |x - y|^{\alpha-n} f(y) dy \\ &\leq \mu \left(\int_{B(x_0, 4r_0) \setminus B(x, \delta)} (|x - y|^{\alpha-n}/\mu)^{p'(y)} dy + \int_{G \setminus B(x, \delta)} f(y)^{p(y)} dy \right) \\ &\leq \mu \left(C\mu^{-p_0} (\log(1/\delta))^{1-b\alpha^2/(n-\alpha)} + 1 \right). \end{aligned}$$

Now, considering μ such that $\mu^{-p_0}(\log(1/\delta))^{1-b\alpha^2/(n-\alpha)} = 1$, we have

$$\int_{B(x_0, 4r_0) \setminus B(x, \delta)} |x-y|^{\alpha-n} f(y) dy \leq C(\log(1/\delta))^\beta$$

with $\beta = \{1 - b\alpha^2/(n-\alpha)\}/p_0$. Since $\inf_{G \setminus B(x_0, r_0)} p(x) > p_- = n/\alpha$, we note that

$$\int_G |x-y|^{\alpha-n} f(y) dy \leq C$$

for $x \in G \setminus B(x_0, 2r_0)$. Consequently it follows from [1, (3.1.1)] that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x, \delta)} |x-y|^{\alpha-n} f(y) dy + \int_{G \setminus B(x, \delta)} |x-y|^{\alpha-n} f(y) dy \\ &\leq C\delta^\alpha Mf(x) + C(\log(1/\delta))^\beta. \end{aligned}$$

Here, as in the proof of Proposition 3.1.2 in [1], let

$$\delta = (Mf(x))^{-1/\alpha} (\log(Mf(x) + 2))^{\beta/\alpha}$$

when $Mf(x)$ is large enough. Then we have

$$U_\alpha f(x) \leq C(\log(Mf(x) + 2))^\beta \leq C(\log(Mf(x) + 2))^A,$$

as required. \square

By Lemma 3 and the fact that $Mf \in L^{p_-}(G)$, we establish the following exponential inequality for $f \in L^{p(\cdot)}(G)$.

THEOREM 2. For $A > (1 - a\alpha^2/(n-\alpha))/p'_- \geq 0$, there exist positive constants c_1 and c_2 such that

$$\int_G \exp(c_1 (U_\alpha f(x))^{1/A}) dx \leq c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{p(\cdot)} \leq 1$.

THEOREM 3. Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} < \infty$. If $A > (1 - a\alpha^2/(n-\alpha))/p'_- \geq 0$, then

$$\int_G \exp(c(U_\alpha f(x))^{1/A}) dx < \infty \quad \text{for all } c > 0.$$

REMARK 4. When $a = 0$, Theorems 2 and 3 hold for $A = 1/p'_- = (n-\alpha)/n$.

4. Sobolev's inequality

In this section we are concerned with $p(\cdot)$ satisfying :

$$(p4) \quad 1 < p_- = \inf_G p(x) \leq p(x) < p_+ = \sup_G p(x) = n/\alpha ;$$

$$(p5) \quad |p(x) - p(y)| \leq \frac{\tilde{a}}{\log(1/|x - y|)} \quad \text{whenever } |x - y| < 1/2, \text{ for some } \tilde{a} > 0.$$

As an example, we may consider the function of the form

$$p(y) = p_0 - \omega(|x_0 - y|), \quad \omega(r) = \frac{\tilde{a}}{\log(1/r)},$$

for $y \in B(x_0, r_0)$ with r_0 chosen sufficiently small; set $p(y) = p_+ - \omega(r_0)$ outside $B(x_0, r_0)$. Note here that

$$\omega(s + t) \leq \omega(s) + \omega(t)$$

for $0 < s < r_0$ and $0 < t < r_0$.

$$\text{Let } 1/p^\sharp(x) = 1/p(x) - \alpha/n.$$

LEMMA 4. *If $\mu > 1$ and $0 < \delta < 1/2$, then*

$$\int_{G \setminus B(x, \delta)} (|x - y|^{\alpha-n}/\mu)^{p'(y)} dy \leq C \left(\mu^{-p'(x)} \frac{\delta^{-q(x)/(p(x)-1)}}{q(x)} + 1 \right)$$

for $x \in G$, where $q(x) = n - \alpha p(x) > 0$.

Proof. First find $C > 0$ such that

$$|p'(y) - p'(x)| \leq \frac{C}{\log(1/|x - y|)}$$

whenever $|x - y| < 1/2$. Then we have for $\mu > 1$

$$\begin{aligned} & \int_{B(x, \mu^{1/(\alpha-n)}) \setminus B(x, \delta)} (|x - y|^{\alpha-n}/\mu)^{p'(y)} dy \\ & \leq C \mu^{-p'(x)} \int_{B(x, \mu^{1/(\alpha-n)}) \setminus B(x, \delta)} |x - y|^{(\alpha-n)p'(x)} dy \\ & \leq C \mu^{-p'(x)} \frac{\delta^{p'(x)(\alpha-n/p(x))}}{-p'(x)(\alpha - n/p(x))}, \end{aligned}$$

which yields the required inequality. \square

LEMMA 5. *Let f be a nonnegative measurable function on G with $\|f\|_{p(\cdot)} \leq 1$. Then*

$$\int_G |x - y|^{\alpha-n} f(y) dy \leq C \tilde{q}(x)^{-\alpha(p(x)-1)/n} Mf(x)^{p(x)/p^\sharp(x)}.$$

for $x \in G$, where $q(x) = n - \alpha p(x) > 0$ and $\tilde{q}(x) = \min\{q(x), 1\}$.

Proof. First consider the case

$$Mf(x)\tilde{q}^{1/p'(x)} > 2^{\alpha+q(x)/p(x)}. \tag{6}$$

Since $\|f\|_{p(\cdot)} \leq 1$, we have for $\mu > 1$

$$\begin{aligned} \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy &\leq \mu \left(\int_{G \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + \int_{G \setminus B(x,\delta)} f(y)^{p(y)} dy \right) \\ &\leq C\mu \left(\mu^{-p'(x)} \frac{\delta^{-q(x)/(p(x)-1)}}{q(x)} + 1 \right) \end{aligned}$$

because of Lemma 4. Now if we set

$$\mu^{-p'(x)} \frac{\delta^{-q(x)/(p(x)-1)}}{\tilde{q}(x)} = 1,$$

then

$$\int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \leq C \frac{\delta^{-q(x)/p(x)}}{\tilde{q}(x)^{1/p'(x)}}.$$

It follows from [1, (3.1.1)] that

$$\begin{aligned} \int_G |x-y|^{\alpha-n} f(y) dy &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy + \int_{G \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\ &\leq C\delta^\alpha Mf(x) + C \frac{\delta^{-q(x)/p(x)}}{\tilde{q}(x)^{1/p'(x)}}. \end{aligned}$$

Letting $Mf(x)\tilde{q}(x)^{1/p'(x)} = \delta^{-\alpha-q(x)/p(x)}$ by (6) as in the proof of Proposition 3.1.2 in [1], we find

$$\int_G |x-y|^{\alpha-n} f(y) dy \leq CMf(x)^{p(x)/p^\sharp(x)} \frac{1}{\tilde{q}(x)^{\alpha(p(x)-1)/n}}.$$

Next consider the case

$$Mf(x)\tilde{q}^{1/p'(x)} \leq 2^{\alpha+q(x)/p(x)}.$$

Then we have

$$\begin{aligned} \int_G |x-y|^{\alpha-n} f(y) dy &\leq CMf(x) \\ &= C \left(Mf(x)\tilde{q}^{1/p'(x)} \right) \tilde{q}^{-1/p'(x)} \\ &\leq C \left(Mf(x)\tilde{q}^{1/p'(x)} \right)^{p(x)/p^\sharp(x)} \tilde{q}^{-1/p'(x)} \\ &= CMf(x)^{p(x)/p^\sharp(x)} \frac{1}{\tilde{q}(x)^{\alpha(p(x)-1)/n}}, \end{aligned}$$

as required. \square

In view of Lemma 5 we see that

$$\left(\tilde{q}(x)^{\alpha(p(x)-1)/n} U_\alpha f(x) \right)^{p^\sharp(x)/p(x)} \leq CMf(x)$$

for all nonnegative measurable functions f on G with $\|f\|_{p(\cdot)} \leq 1$. Since M is bounded from $L^{p(\cdot)}$ to itself according to the result by Diening [2], we have the following result.

THEOREM 4. *There exist positive constants c_1 and c_2 such that*

$$\int_G \left(c_1 \tilde{q}(x)^{\alpha(p(x)-1)/n} U_\alpha f(x) \right)^{p^\sharp(x)} dx \leq c_2$$

for all nonnegative measurable functions f on G with $\|f\|_{p(\cdot)} \leq 1$.

When $\alpha = 1$, we refer the reader to the paper by Edmunds-Rákosník [4]; compare also with the paper by Diening [3] concerning Sobolev's embeddings.

REMARK 5. For $0 < \varepsilon < 1$, set $p(x) = n - \varepsilon$ and $1/q = 1/p(x) - 1/n$. Then we see from Lemma 5 that

$$(1/q)^{n/(n-1)} \|U_1 f\|_q \leq C \|f\|_{n-\varepsilon}$$

(see also [11]). Hence we have the following fact by Fusco-Lions-Sbordone [5]:

If f is a nonnegative measurable function on G such that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^\delta \int_G f(y)^{n-\varepsilon} dy = 0$$

for some $0 < \delta < 1$, then

$$\int_G \exp(c(U_1 f(x))^{1/A}) dx < \infty \quad \text{for all } c > 0,$$

where $A = (n - 1 + \delta)/n$.

REFERENCES

- [1] D. R. ADAMS AND L. I. HEDBERG, *Function spaces and potential theory*, Springer, 1996.
- [2] L. DIENING, *Maximal functions on generalized Lebesgue spaces $L^{p(\cdot)}$* , Math. Inequal. Appl., **7**, 2 (2004), 245–253.
- [3] L. DIENING, *Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$* , Math. Nachr., **268**, (2004), 31–43.
- [4] D. E. EDMUNDS AND J. RÁKOSNÍK, *Sobolev embedding with variable exponent*, II, Math. Nachr. **246-247** (2002), 53–67.
- [5] N. FUSCO, P. L. LIONS AND C. SBORDONE, *Sobolev embedding theorems in borderline cases*, Proc. Amer. Math. Soc. **124** (1996), 561–565.
- [6] T. FUTAMURA, Y. MIZUTA AND T. SHIMOMURA, *Sobolev embeddings for Riesz potential space of variable exponent*, preprint.
- [7] P. HARJULEHTO AND P. HÄSTÖ, *A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces*, Rev. Mat. Complut., **17**, (2004), 129–146.
- [8] L. I. HEDBERG, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.
- [9] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$* , Czechoslovak Math. J. **41** (1991), 592–618.
- [10] Y. MIZUTA, *Potential theory in Euclidean spaces*, Gakkōtoshō, Tokyo, 1996.
- [11] Y. MIZUTA AND T. SHIMOMURA, *Exponential integrability for Riesz potentials of functions in Orlicz classes*, Hiroshima Math. J. **28** (1998), 355–371.
- [12] Y. MIZUTA AND T. SHIMOMURA, *Continuity and differentiability for weighted Sobolev spaces*, Proc. Amer. Math. Soc. **130** (2002), 2985–2994.

- [13] M. RŮŽIČKA, *Electrorheological fluids : modeling and Mathematical theory*, Lecture Notes in Math. **1748**, Springer, 2000.

(Received April 5, 2004)

Toshihide Futamura
Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530

Japan
e-mail: futamura@daido-it.ac.jp

Yoshihiro Mizuta
The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521
Japan

e-mail: mizuta@mis.hiroshima-u.ac.jp