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# NECESSARY CONDITIONS FOR SOLVING INITIAL VALUE PROBLEMS WITH INFIMA OF SUPERFUNCTIONS

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*Abstract.* Goodman proved that the pointwise infimum of all superfunctions is the minimal absolutely continuous solution of

$$x' = f(t,x), \quad t \in [0,1], \ x(0) = 0,$$

in case $f$ is a $L^1$-bounded Carathéodory function. How far can Carathéodory conditions be weakened without loosing that property? First we establish necessary conditions over $f$ for Goodman's method to be valid, and then we use them as a starting point to deduce sufficient ones. In this way we obtain new existence results and we provide new insights concerning the application of Goodman's method.

## 1 Introduction

Let us consider the initial value problem

$$x'(t) = f(t,x(t)), \ t \in I = [0,1], \quad x(0) = 0, \tag{1.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$, and let us denote by $AC(I)$ the set of all real valued functions that are absolutely continuous on $I$. A (Carathéodory) solution of (1.1) is a function $x \in AC(I)$ such that $x(0) = 0$ and satisfies the differential equation for almost all (a.a.) $t \in I$. We say that a solution $x_{\min}$ is the minimal one if $x_{\min} \leqslant x$ on $I$ for any other solution $x$, and we define the maximal solution symmetrically. When both the minimal and the maximal solutions exist, we call them the extremal solutions. On the other hand, a subfunction (or lower solution) for (1.1) is a function $l \in AC(I)$ such that $l(0) \leqslant 0$ and $l'(t) \leqslant f(t,l(t))$ for a.a. $t \in I$; a superfunction (or upper solution) is defined analogously reversing the inequalities. We note that $x : I \to \mathbb{R}$ is a solution if, and only if, it is both a subfunction and a superfunction.

Subfunctions and superfunctions were first used by Peano in the proof of his classical existence result in the scalar case, see [7]. Goodman gave the above definitions in [3] to adapt Peano's technique to $L^1$-bounded Carathéodory right-hand sides, and he showed that the minimal solution is the least superfunction, and that the maximal solution is the greatest subfunction.

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It is the aim of this paper to solve as satisfactorily as possible the following problem: to find the weakest sufficient conditions over a $L^1$-bounded right-hand side $f$ so that Goodman's method can be succesfully carried out. To achieve this goal we start by answering the following question in section 2: which conditions are necessary for a bounded $f$ so that the pointwise infimum of all superfunctions of $(1.1)$ be a solution? In section 3 we show how the results of section 2 lead to a new existence result for $(1.1)$ through Goodman's approach, and in section 4 we adapt everything to subfunctions and maximal solutions. We remark that our theorem 4.1 generalizes the results established in [5, 8] for $(1.1)$ and, thanks to the new viewpoint considered here, the proofs get shorter and probably clearer.

All the results in this paper remain valid, with obvious changes, if we replace $I$ by any other compact interval $[a,b]$, and $x(0) = 0$ by any other initial condition $x(a) = x_0 \in \mathbb{R}$. We have considered $[0,1]$ and $x(0) = 0$ only for simplicity.

## 2 Properties of the infimum of superfunctions

Throughout this section we shall assume that $f$ is bounded in the following sense: there exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}$ we have $|f(t,x)| \leqslant M(t)$. This condition can probably be replaced by a more general one, but our main interest concerns the study of how $f$ depends on $(t,x)$.

We define the set of *admissible superfunctions for* $(1.1)$ as

$$\mathscr{U} := \{u \in AC(I) : u(0) \geqslant 0 \text{ and } |u'| \leqslant M+1,\ u' \geqslant f(t,u) \text{ a.e. on } I\},$$

and we note that $\mathscr{U} \neq \emptyset$ as $u(t) := \int_0^t M(r)dr$, $t \in [0,1]$, is an element of $\mathscr{U}$.

The adjective "admissible" refers only to the condition that $|u'| \leqslant M+1$, which is not needed in the today standard definition of superfunction, but it wipes out some technical complications in the proofs.

In case $f$ is a Carathéodory function the minimal solution of $(1.1)$ is the least superfunction, see [3]. Thus a good candidate for being the minimal solution is

$$u_{\inf}(t) := \inf\{u(t) : u \in \mathscr{U}\}, \quad t \in I. \tag{2.1}$$

Note that if $u$ is a superfunction then $\tilde{u}(t) = \int_0^t \min\{u'(s), M(s)\}ds$, $t \in I$, defines an admissible superfunction such that $\tilde{u} \leqslant u$ on $I$. Therefore $u_{\inf}$ is also the pointwise infimum of all superfunctions.

Plainly $u_{\inf}(0) = 0$. In the next lemma we state some properties of $u_{\inf}$ which follow from our boundedness assumption and which can be proven by means of standard arguments (the reader is referred to [3] or [5, lemma 3.1]).

LEMMA 1. *There exists a nonincreasing sequence* $\{u_n\}_n \subset \mathscr{U}$ *that converges uniformly on* $I$ *to* $u_{\inf}$*; as a consequence,* $u_{\inf} \in AC(I)$ *and* $|u_{\inf}'(t)| \leqslant M(t)+1$ *for a.a.* $t \in I$.

In the following two theorems we study the behavior of $f$ over the graph of $u_{\inf}$. Such results are fundamental in this paper, as they provide a clear vision of what $f$ should satisfy so that $u_{\inf}$ be a solution.

The function $u_{\inf}$ has the following connection with $(1.1)$:

THEOREM 1. *For almost all $t \in I$ we have*

$$u_{\inf}'(t) \geqslant f(t, u_{\inf}(t))\chi_{I_1}(t) + \liminf_{y \to (u_{\inf}(t))^+} f(t,y)\chi_{I_2}(t),$$

*where $I_1 = \{t \in I : u(t) = u_{\inf}(t)$ and $u'(t) \geqslant f(t,u(t))$ for some $u \in \mathscr{U}\}$, $I_2 = I \setminus I_1$, and $\chi_{I_i}$ is the characteristic function of $I_i$, $i = 1,2$.*

*Proof.* Let $t_0 \in I_1$ be such that $u_{\inf}'(t_0)$ exists, and let $u \in \mathscr{U}$ be a superfunction corresponding to $t_0$ by definition of $I_1$. Then it is elementary matter to show that $u_{\inf}'(t_0) = u'(t_0) \geqslant f(t_0, u_{\inf}(t_0))$. Since $u_{\inf}'$ exists a.e., we conclude that

$$u_{\inf}'(t) \geqslant f(t, u_{\inf}(t)) \quad \text{for a.a. } t \in I_1 .$$

To study $u_{\inf}'$ on $I_2 = I \setminus I_1$, we take a sequence $\{u_n\}_n$ satisfying the conditions of lemma 1. Since $|u_n'|$ is uniformly $L^1$-bounded on $I$ we have that $\liminf_{n\to\infty} u_n' \in L^1(I)$; moreover for all $s,t \in I$ Fatou's lemma implies

$$u_{\inf}(t) - u_{\inf}(s) = \lim_{n\to\infty} \int_s^t u_n'(r)dr \geqslant \int_s^t \liminf_{n\to\infty} u_n'(r)dr,$$

and therefore for a.a. $t \in I$ we have

$$u_{\inf}'(t) \geqslant \liminf_{n\to\infty} u_n'(t) \geqslant \liminf_{n\to\infty} f(t, u_n(t)). \tag{2.2}$$

Now if $t_0 \in I_2$ and $u_{\inf}(t_0) = u_n(t_0)$ for some $n$, the definition of $I_2$ implies that either the derivative $u_n'(t_0)$ does not exist, or $u_n'(t_0) < f(t_0, u_n(t_0))$. Since $\cup_{n\in\mathbb{N}}\{t \in I : \text{either } \nexists u_n'(t_0) \text{ or } u_n'(t_0) < f(t_0, u_n(t_0))\}$ is a null-measure set, we conclude that for a.a. $t \in I_2$ we have $u_{\inf}(t) < u_n(t)$ for all $n \in \mathbb{N}$, and then

$$u_{\inf}'(t) \geqslant \liminf_{y\to(u_{\inf}(t))^+} f(t,y) \quad \text{for a.a. } t \in I_2 ,$$

by virtue of (2.2). $\square$

The set $I_1$ may be empty in theorem 1, as it happens for $(1.1)$ with $f$ replaced by

$$f_1(t,x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Analogously, $I_2$ may be empty, and an example is furnished by $f_2 = 1 - f_1$. We note that $u_{\inf}(t) = 0$ for all $t \in [0,1]$ in the previous two examples.

REMARK 1. Bearing theorem 1 in mind, it is obvious that $u_{\inf}' \geqslant f(t, u_{\inf})$ a.e. on $I$ (and consequently $u_{\inf}$ is a superfunction) provided that the condition "$\liminf_{y\to(u_{\inf}(t))^+} f(t,y) \geqslant f(t, u_{\inf}(t))$ a.e. on $I$" is satisfied.

Next we present a necessary condition for $u_{\inf}$ being a superfunction:

THEOREM 2. *Assume that $u_{\inf}'(t) \geqslant f(t, u_{\inf}(t))$ for a.a. $t \in I$, then*

*(a) The set $J := \{t \in I : u_{\inf}'(t) > \limsup_{y\to(u_{\inf}(t))^-} f(t,y)\}$ is a countable union of sets which contain no positive measure set. Specifically, $J = \cup_{n,m\in\mathbb{N}} J_{n,m}$ where for all $n,m \in \mathbb{N}$ the set*

$$J_{n,m} := \{t \in I : u_{\inf}'(t) - 1/n > \sup\{f(t,y) : u_{\inf}(t) - 1/m < y < u_{\inf}(t)\}\}$$

*contains no positive measure set.*

*(b)* $u'_{\inf}(t) \leqslant \limsup_{y\to(u_{\inf}(t))^-} f(t,y)$ *for a.a.* $t \in I$ *provided that for all* $n,m \in \mathbb{N}$ *the set* $J_{n,m}$ *is measurable.*

*Proof.* We are going to show that $(a)$ holds. Note first that for each $t \in J$ there exists $n \in \mathbb{N}$ such that

$$u'_{\inf}(t) - \frac{1}{n} > \limsup_{y\to(u_{\inf}(t))^-} f(t,y) = \inf_{\varepsilon>0} \sup_{u_{\inf}(t)-\varepsilon<y<u_{\inf}(t)} f(t,y),$$

and therefore there exists $m \in \mathbb{N}$ such that $t \in J_{n,m}$. Conversely, for each $t \in \cup_{n,m\in\mathbb{N}} J_{n,m}$ there exist $n,m \in \mathbb{N}$ such that

$$u'_{\inf}(t) - \frac{1}{n} > \sup_{u_{\inf}(t)-1/m<y<u_{\inf}(t)} f(t,y) \geqslant \limsup_{y\to(u_{\inf}(t))^-} f(t,y),$$

and hence $t \in J$. Thus $J = \cup_{n,m} J_{n,m}$, and now it suffices to prove that for all $n,\ m \in \mathbb{N}$ the set $J_{n,m}$ contains no positive measure subset.

Reasoning by contradiction, assume there exist $n,m \in \mathbb{N}$ for which the corresponding $J_{n,m}$ has a positive measure subset, denoted again by $J_{n,m}$ for simplicity. By [5, lemma 2.3] there exist $t_0 \in J_{n,m} \cap (0,1)$ and $\delta \in (0,n/m)$ such that for all $t \in (t_0, t_0+\delta)$ we have

$$\mu([t_0,t] \cap J_{n,m}) \geqslant \frac{1}{2}(t-t_0), \tag{2.3}$$

$$\int_{[t_0,t]\setminus J_{n,m}} (M(r)+1)dr \leqslant \frac{1}{4n}\mu([t_0,t] \cap J_{n,m}). \tag{2.4}$$

Let us define $u \in AC(I)$ such that $u(0)=0$ and for a.a. $t \in [0,1]$

$$u'(t) = \begin{cases} u'_{\inf}(t), & \text{if } t \leqslant t_0, \\ u'_{\inf}(t) - 1/n, & \text{if } t \in [t_0,t_0+\delta] \cap J_{n,m}, \\ M(t)+1, & \text{otherwise.} \end{cases}$$

Taking into account that $u_{\inf}$ is a superfunction, it is easily seen that $|u'(t)| \leqslant M(t)+1$ for a.a. $t \in I$ and that $u'(t) \geqslant f(t,u(t))$ for a.a. $t \in I \setminus ([t_0,t_0+\delta] \cap J_{n,m})$.

For $t \in (t_0, t_0+\delta)$ we have

$$\begin{aligned} u_{\inf}(t) - u(t) &= \int_{t_0}^{t} (u'_{\inf}(r) - u'(r))dr \\ &= \frac{1}{n}\mu([t_0,t] \cap J_{n,m}) + \int_{[t_0,t]\setminus J_{n,m}} (u'_{\inf}(r) - M(r) - 1)dr \\ &\geqslant \frac{1}{n}\mu([t_0,t] \cap J_{n,m}) - 2\int_{[t_0,t]\setminus J_{n,m}} (M(r)+1)dr \qquad \text{(by (2.4))} \\ &\geqslant \frac{1}{2n}\mu([t_0,t] \cap J_{n,m}) > 0 \ \text{(by (2.3))}, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} u_{\inf}(t) - u(t) &= \frac{1}{n}\mu([t_0,t] \cap J_{n,m}) + \int_{[t_0,t]\setminus J_{n,m}} (u'_{\inf}(r) - M(r) - 1)dr \\ &\leqslant \frac{1}{n}\mu([t_0,t] \cap J_{n,m}) \leqslant \frac{1}{n}(t-t_0) < \frac{\delta}{n} < \frac{1}{m}. \end{aligned}$$

Therefore for a.a. $t \in (t_0, t_0+\delta) \cap J_{n,m}$ we have

$$u'(t) = u'_{\inf}(t) - 1/n > \sup_{u_{\inf}(t)-1/m<y<u_{\inf}(t)} f(t,y) \geqslant f(t,u(t)),$$

and then $u'(t) \geqslant f(t,u(t))$ for a.a. $t \in I$. This is a contradiction, since $u \in \mathcal{U}$ and $u < u_{\inf}$ on $(t_0, t_0+\delta)$.

Part $(b)$ follows rightaway from $(a)$ and the extra assumption. □

REMARK 2. Combining Remark 1 with theorem 2, part $(b)$, we conclude that $u_{\inf}$ will be a solution of $(1.1)$ in case the sets $J_{n,m}$ are measurable and the following condition is fulfilled:

$$\limsup_{y\to(u_{\inf}(t))^-} f(t,y) \leqslant f(t,u_{\inf}(t)) \leqslant \liminf_{y\to(u_{\inf}(t))^+} f(t,y) \quad \text{for a.a. } t \in I.$$

Since $u_{\inf}$ is not known *a priori*, a reasonable sufficient condition to impose is that

$$\limsup_{y\to x^-} f(t,y) \leqslant f(t,x) \leqslant \liminf_{y\to x^+} f(t,y) \quad \text{for a.a. } t \in I \text{ and all } x \in \mathbb{R},$$

which was refered to as "quasisemicontinuity" in [2]. This condition seem to have been considered for the first time in connection with (1.1) in [1].

The previous arguments reveal that quasisemicontinuity is a very fine assumption, which seems to be difficult to improve in an essential way at least on the points of the graph of $u_{\inf}$. However, it is somewhat stringent outside that graph, and in fact it can be relaxed over sufficiently nice curves of the $(t,x)$ plane, as already pointed out in [8]. We shall give a more precise formulation of this idea in section 3.

## 3 Existence of a minimal solution

As a consequence of theorems 1 and 2 we have the following existence principle for $(1.1)$:

THEOREM 3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a mapping for which the following conditions are fulfilled:*

$(i)$ *there exists $M \in L^1(0,1)$ such that for a.a. $t \in [0,1]$ and all $x \in \mathbb{R}$ we have $|f(t,x)| \leqslant M(t)$;*

$(ii)$ **either** *for a.a. $t \in I$ and all $x \in \mathbb{R}$ we have*

$$\limsup_{y\to x^-} f(t,y) \leqslant f(t,x) \leqslant \liminf_{y\to x^+} f(t,y), \tag{3.1}$$

**or** *there exist absolutely continuous* $\gamma_n : [a_n, b_n] \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, *such that for a.a.* $t \in I$ *and all* $x \in \mathbb{R} \setminus \cup_{\{n / a_n \leqslant t \leqslant b_n\}} \{\gamma_n(t)\}$ *we have (3.1), while for each* $n \in \mathbb{N}$ *and a.a.* $t \in [a_n, b_n]$ *we have either* $\gamma_n'(t) = f(t, \gamma_n(t))$, *or*

$$\gamma_n'(t) \geqslant f(t, \gamma_n(t)) \quad \textit{whenever} \quad \gamma_n'(t) \geqslant \liminf_{y \to (\gamma_n(t))^+} f(t, y), \tag{3.2}$$

*and*

$$\gamma_n'(t) \leqslant f(t, \gamma_n(t)) \quad \textit{whenever} \quad \gamma_n'(t) \leqslant \limsup_{y \to (\gamma_n(t))^-} f(t, y). \tag{3.3}$$

*Then we have the following results:*

$(a)$ $u_{\inf}'(t) = f(t, u_{\inf}(t))$ *a.a.* $t \in I \setminus J$, *where* $J$ *is a countable union of sets which contain no positive measure set. Specifically,* $J = \cup_{n,m \in \mathbb{N}} J_{n,m}$ *where for all* $n, m \in \mathbb{N}$ *the set*

$$J_{n,m} := \{t \in I : u_{\inf}'(t) - 1/n > \sup\{f(t, y) : u_{\inf}(t) - 1/m < y < u_{\inf}(t)\}\}$$

*contains no positive measure set.*

$(b)$ $u_{\inf}$ *is the minimal Carathéodory solution of* $(1.1)$ *provided that for all* $n, m \in \mathbb{N}$ *the set* $J_{n,m}$ *is measurable.*

*Proof.* We shall assume that the second alternative in $(ii)$ holds, as the proof is analogous but easier when the first alternative is satisfied.

By theorem 1 there exist $I_1 \subset I$ such that

$$u_{\inf}'(t) \geqslant f(t, u_{\inf}(t)) \chi_{I_1}(t) + \liminf_{y \to (u_{\inf}(t))^+} f(t, y) \chi_{I \setminus I_1}(t) \quad \text{for a.a. } t \in I. \tag{3.4}$$

By virtue of [6, theorem 38.2], for each $n \in \mathbb{N}$ we have $u_{\inf}'(t) = \gamma_n'(t)$ for a.a. $t \in I$ such that $u_{\inf}(t) = \gamma_n(t)$. Thus $(3.4)$ and the second inequality in $(3.1)$ imply $u_{\inf}'(t) \geqslant f(t, u_{\inf}(t))$ for a.a. $t \in I \setminus \cup_n \Gamma_n$, where $\Gamma_n := \{t \in [a_n, b_n] : u_{\inf}(t) = \gamma_n(t)$ and $\gamma_n'(t) \neq f(t, \gamma(t))\}$, $n \in \mathbb{N}$.

Now let $n \in \mathbb{N}$ be fixed and let $t_0 \in \Gamma_n$ be such that $u_{\inf}'(t_0) = \gamma_n'(t_0)$. If $\gamma_n'(t_0) < \liminf_{y \to (\gamma_n(t_0))^+} f(t_0, y)$ then $u_{\inf}'(t_0) < \liminf_{y \to (u_{\inf}(t_0))^+} f(t_0, y)$ and then $(3.4)$ implies that either $t_0$ belongs to a null measure set or $t_0 \in I_1$, i.e., $u_{\inf}'(t_0) \geqslant f(t_0, u_{\inf}(t_0))$. If, on the other hand, we have $\gamma_n'(t_0) \geqslant \liminf_{y \to (\gamma_n(t_0))^+} f(t_0, y)$, then condition $(3.2)$ implies that either $t_0$ belongs to a null measure set or $\gamma_n'(t_0) \geqslant f(t_0, \gamma_n(t_0))$ as well, hence $u_{\inf}'(t_0) \geqslant f(t_0, u_{\inf}(t_0))$.

Therefore we have

$$u_{\inf}'(t) \geqslant f(t, u_{\inf}(t)) \quad \text{for a.a. } t \in I, \tag{3.5}$$

and then theorem 2 yields

$$u_{\inf}'(t) \leqslant \limsup_{y \to (u_{\inf}(t))^-} f(t, y) \quad \text{for a.a. } t \in I \setminus J. \tag{3.6}$$

The first inequality in $(3.1)$ implies that for a.a. $t \in I \setminus J$, $t \notin \cup_{n \in \mathbb{N}} \Gamma_n$, we have $u_{\inf}'(t) \leqslant f(t, u_{\inf}(t))$.

On the other hand, for each $n \in \mathbb{N}$ and each $t_0 \in \Gamma_n$ such that $u_{\inf}'(t_0) = \gamma_n'(t_0)$ we may have either $\gamma_n'(t_0) > \limsup_{y \to (\gamma_n(t_0))^-} f(t_0, y)$ (hence $u_{\inf}'(t_0) >$

$\limsup_{y\to(u_{\inf}(t_0))^-} f(t_0,y)$ and (3.6) implies that either $t_0$ belongs to a null set or $t_0 \in J$), or $\gamma_n'(t_0) \leqslant \limsup_{y\to(\gamma_n(t_0))^-} f(t_0,y)$ and then, by condition (3.3), either $t_0$ belongs to a null measure set or $\gamma_n'(t_0) \leqslant f(t_0,\gamma_n(t_0))$, which is equivalent to $u_{\inf}'(t_0) \leqslant f(t_0,u_{\inf}(t_0))$. Therefore $u_{\inf}'(t) \leqslant f(t,u_{\inf}(t))$ for a.a. $t \in I\setminus J$, and then (3.5) yields $u_{\inf}'(t) = f(t,u_{\inf}(t))$ for a.a. $t\in I\setminus J$.

Part $(b)$ follows from $(a)$ and the extra assumption. $\square$

REMARK 3. In order to have a clearer comparison between our results with the existing literature on the topic, we shall discuss briefly about condition $(ii)$. A family of absolutely continuous functions $\gamma_n : [a_n,b_n] \to \mathbb{R}$, $n\in\mathbb{N}$, satisfies the conditions prescribed in $(ii)$ provided that for each $n\in\mathbb{N}$ one of the following conditions holds:

(1) for a.a. $t\in[a_n,b_n]$ either $\gamma_n'(t) = f(t,\gamma_n(t))$, or there exists $\varepsilon_t > 0$ such that

$$\gamma_n'(t) \geqslant \max\{f(t,\gamma_n(t)),\ \liminf_{y\to(\gamma_n(t))^+} f(t,y),\ \limsup_{y\to(\gamma_n(t))^-} f(t,y)+\varepsilon_t\},$$

(2) for a.a. $t\in[a_n,b_n]$ either $\gamma_n'(t) = f(t,\gamma_n(t))$, or there exists $\varepsilon_t > 0$ such that

$$\gamma_n'(t) \leqslant \min\{f(t,\gamma_n(t)),\ \liminf_{y\to(\gamma_n(t))^+} f(t,y)-\varepsilon_t,\ \limsup_{y\to(\gamma_n(t))^-} f(t,y)\}.$$

In turn (1) is implied by

$(1')$ there exists $\varepsilon > 0$ and $\delta > 0$ such that

$$\gamma_n'(t) \geqslant \varepsilon + f(t,x) \quad \text{for a.a. } t\in[a_n,b_n] \text{ and all } x\in[\gamma_n(t)-\delta,\gamma_n(t)+\delta],$$

and there is analogous condition stronger than (2). This shows that theorem 3 improves the information given in [8, theorem 3.1] concerning minimal solutions.

REMARK 4. Part (a) of theorem 3 ensures that $u_{\inf}$ is a sort of "weak" Carathéodory solution. The problem with that concept of a solution is that it is extremely weak, as countable unions of sets having no positive measure subset may be rather big. Indeed, the very real line can be expressed as such an union: see, for instance, the construction of non measurable sets described in pages 69 and 70 in [4].

It would have been more interesting to have obtained absolutely continuous "solutions" which solve the differential equation on $I\setminus K$, for some $K$ containing no positive measure set. Such "solutions" $x$ immediately become Carathéodory solutions in case $t\in I\mapsto f(t,x(t))$ is measurable. Unfortunately, so far we have not been able to obtain this type of solutions just by supposing that the conditions of theorem 3 hold, neither to construct any counterexample.

Anyway, we emphasize the fact that only measurability of the sets $J_{n,m}$ is sufficient to turn $u_{\inf}$ into a Carathéodory solution, and thus only invoking the axiom of choice can one construct a function $f$ satisfying the conditions of theorem 3 and such that $u_{\inf}$ is not a Carathéodory solution.

The following lemma is a mild extension of [5, lemma 2.1], and it gives an easily verifiable sufficient condition for the measurability of $J_{n,m}$. The proof is included for completeness.

LEMMA 2. *Let $N\subset I$ be a null-measure set and let $f : I\times\mathbb{R}\to\mathbb{R}$ be such that $f(\cdot,q)$ is measurable for each $q\in\mathbb{Q}$.*

*If, moreover, for all* $t \in I \setminus N$ *and all* $x \in \mathbb{R}$ *we have*

$$\max\left\{\liminf_{y\to x^-} f(t,y), \liminf_{y\to x^+} f(t,y)\right\} \geqslant f(t,x),$$

*then the mapping* $t \in I \mapsto \sup\{f(t,y) \,:\, x_1(t) < y < x_2(t)\}$ *is measurable for each pair* $x_1, x_2 \in \mathscr{C}(I)$ *such that* $x_1(t) < x_2(t)$ *for all* $t \in I$.

*Proof.* We denote by $\mathscr{S}$ the following set of step functions: $v : [0,1) \longrightarrow \mathbb{R}$ belongs to $\mathscr{S}$ if $v$ assumes only rational values, $x_1(t) < v(t) < x_2(t)$ on $[0,1)$ and there exists $j \in \mathbb{N}$ such that $v$ is constant on every interval

$$\left[0, \frac{1}{j}\right), \left[\frac{1}{j}, \frac{2}{j}\right), \dots, \left[\frac{j-1}{j}, 1\right).$$

As $x_1$, $x_2$ are continuous and $x_1 < x_2$ on $[0,1]$, one can prove reasoning by contradiction that there is $j \in \mathbb{N}$ such that for all $k \in \{0,1,\dots,j-1\}$ we have

$$\max\{x_1(t) \,:\, t \in [k/j,(k+1)/j]\} < \min\{x_2(t) \,:\, t \in [k/j,(k+1)/j]\},$$

which implies that $\mathscr{S}$ is not empty. Note, moreover, that for each $q \in (x_1(t), x_2(t)) \cap \mathbb{Q}$ there exists $v \in \mathscr{S}$ such that $v(t) = q$.

Since $\mathscr{S}$ is a countable family and any composition $f(\cdot, v(\cdot))$ with $v \in \mathscr{S}$ is measurable on $[0,1)$, it suffices to prove that

$$\sigma(t) := \sup_{y\in(x_1(t),x_2(t))} f(t,y) = \sup_{v\in\mathscr{S}} f(t,v(t)) =: \sigma_0(t)$$

a.e. on $[0,1)$ to deduce that $\sigma$ is measurable.

Clearly, $\sigma(t) \geqslant \sigma_0(t)$ on $[0,1)$. To prove that $\sigma(t) \leqslant \sigma_0(t)$ on $[0,1)\setminus N$, we fix $t \in [0,1)\setminus N$ and we take a sequence $\{y_n\}_n \subset (x_1(t), x_2(t))$ such that

$$\lim_{n\to\infty} f(t,y_n) = \sigma(t). \tag{3.7}$$

Our assumptions guarantee that for each $n$ we have

$$\liminf_{y\to y_n^-} f(t,y) \geqslant f(t,y_n) \quad \left(\text{or} \quad \liminf_{y\to y_n^+} f(t,y) \geqslant f(t,y_n)\right),$$

thus there exists $q_n \in (x_1(t), y_n) \cap \mathbb{Q}$ (or $q_n \in (y_n, x_2(t)) \cap \mathbb{Q}$) such that $f(t,q_n) \geqslant f(t,y_n) - 1/n$. Since there exists $v_n \in \mathscr{S}$ such that $v_n(t) = q_n$ we have, for all $n$, that

$$\sigma_0(t) = \sup_{v\in\mathscr{S}} f(t,v(t)) \geqslant f(t,v_n(t)) \geqslant f(t,y_n) - \frac{1}{n}$$

and, using (3.7), we conclude that

$$\sigma_0(t) \geqslant \lim_{n\to\infty}\left[f(t,y_n) - \frac{1}{n}\right] = \sigma(t).$$

□

## 4 Extremal solutions

The reader can verify that analogous arguments with the set of admissible subfunctions $\mathscr{L} := \{l \in AC(I) \,:\, l(0) \leqslant 0 \text{ and } |l'| \leqslant M+1,\, l' \leqslant f(t,l) \text{ a.e. on } I\}$, and

$$l_{\sup}(t) = \sup\{l(t) \,:\, l \in \mathscr{L}\} \quad \text{for all } t \in I, \tag{4.1}$$

lead to the corresponding obvious versions concerning maximal solutions for all the previous results. We shall only state here the following consequence of theorem 3, lemma 2, and their analogous for $l_{\sup}$, which improves the main results for scalar problems established in [1, 2, 5, 8]:

THEOREM 4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a mapping which satisfies $(i)$, $(ii)$, and*

$(iii)$ *$f(\cdot, q)$ is measurable for each $q \in \mathbb{Q}$ and, moreover, for a.a. $t \in I$ and all $x \in \mathbb{R}$ we have*

$$\min\left\{\limsup_{y \to x^-} f(t,y), \limsup_{y \to x^+} f(t,y)\right\} \leqslant f(t,x) \leqslant \max\left\{\liminf_{y \to x^-} f(t,y), \liminf_{y \to x^+} f(t,y)\right\}.$$

*Then $(2.1)$ is the minimal solution of $(1.1)$ and $(4.1)$ is the maximal one.*

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