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NONLINEAR INTEGRAL INEQUALITIES OF BIHARI–TYPE WITHOUT CLASS H

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Abstract. Integral inequalities of Bihari-type without restriction to the class H are discussed. The main result can be applied to generalize Pinto's results and Choi *et al*'s results. It is also applied to show boundedness of solutions of a functional differential equation.

1. Introduction

Gronwall-Bellman inequality is a very useful tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. Many results on its generalization can be found, for example in [1, 2, 3, 7, 10, 11, 12, 13, 14, 17], and, in particular, Pachpatte obtain many important results in [11, 12, 13]. Among them one of the important things is Bihari's generalization [4] for the nonlinear inequality

$$u(t) \leqslant a_0 + \int_0^t \lambda(s)w(u(s))ds, \qquad t \geqslant 0. \tag{1.1}$$

Dannan [6] considered Bihari's inequality (1.1) again with a function $a(t)$ instead of the constant a_0 while he introduced a class H consisting of all nonnegative, nondecreasing and continuous functions $w(u)$ on $[0,\infty)$ such that

$$w(u) > 0 \text{ for all } u > 0 \text{ and } w(\alpha u) \leqslant \psi(\alpha)w(u) \text{ for all } \alpha > 0 \text{ and } u \geqslant 0,$$

where ψ (called *multiplier function*) is a certain nonnegative continuous function on $[0,\infty)$. This class H allows a reduction of $a(t)$ to the case of constant a_0 by dividing $a(t)$. With this class H, Pinto [15] further investigated the inequality with function $a(t)$

$$u(t) \leqslant a(t) + f(t)\sum_{i=1}^{n} f_i(t)\int_{t_0}^{t} \lambda_i(s)w_i(u(s))ds, \ t \geqslant t_0 \geqslant 0, \tag{1.2}$$

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where all $w_i \in H$. In order to study stability of some nonlinear functional differential equations, in 1997 Choi *et al* [5] discussed the inequality

$$\begin{aligned} u(t) &\leqslant a(t) + \int_{t_0}^{t} \lambda_1(s) w_1(u(s))ds \\ &\quad + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} \lambda_3(\tau) w_2(u(\tau)) d\tau ds, \ t \geqslant t_0 \geqslant 0, \end{aligned} \tag{1.3}$$

where w_1, w_2 are both in H. Actually, when we study behaviors of solutions of a differential equation or an integral equation (e.g., see inequalities in [16] for almost periodicity of invariant manifolds), $a(t)$ may be a function but w may not satisfy the condition: $w \in H$. So it is interesting to avoid such a condition.

Motivated by this observation in this paper, we consider the inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{t_0}^{t} f_i(t,s) w_i(u(s))ds, \quad t \geqslant t_0 \geqslant 0, \tag{1.4}$$

where we do not restrict these w_i to the class H. We also show that many integral inequalities of Bihari-type such as (1.2) and (1.3) can be reduced to the form of (1.4). So our main result is applied to improve results in [5] and [15]. In particular, an error in [15] can be corrected. Our main result is also applied to estimate solutions of a functional differential equation and to prove boundedness of solutions.

2. Main results

As in [15] we say $w_1 \propto w_2$ for $w_1, w_2 : A \subset \mathbb{R} \to \mathbb{R}\backslash\{0\}$ if $\frac{w_2}{w_1}$ is nondecreasing on A. This concept helps us to compare the monotonicity of different functions. For convenience we always let t_0 represent a nonnegative constant. Consider inequality (1.4) and suppose that

(C_1) all w_i $(i = 1, \cdots, n)$ are continuous and nondecreasing functions on $[0, \infty)$ and are positive on $(0, \infty)$ such that $w_1 \propto w_2 \propto \cdots \propto w_n$;

(C_2) $a(t)$ is continuously differentiable in t and nonnegative on $[t_0, \infty)$;

(C_3) all $f_i(t,s)$ $(i = 1, \cdots, n)$ are continuous and nonnegative functions on $[t_0, \infty) \times [t_0, \infty)$.

We use the notation $W_i(u, u_i) := \int_{u_i}^{u} \frac{dz}{w_i(z)}$, for $u \geqslant u_i$, where $u_i > 0$ is a given constant. It is denoted by $W_i(u)$ simply when there is no confusion. Clearly, W_i is strictly increasing, so its inverse W_i^{-1} is well defined, continuous and increasing in its corresponding domain.

THEOREM 1. *Suppose* (C_1), (C_2), (C_3) *hold and* $u(t)$ *is a continuous and nonnegative function on* $[t_0, \infty)$ *satisfying* (1.4). *Then*

$$u(t) \leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^{t} \max_{t_0 \leqslant \tau \leqslant t} f_n(\tau, s) ds], \quad t_0 \leqslant t \leqslant T_1, \tag{2.1}$$

where $b_n(t)$ is determined recursively by

$$b_1(t) := a(t_0) + \int_{t_0}^{t} |a'(s)|ds,\ b_{i+1}(t) := W_i^{-1}[W_i(b_i(t)) + \int_{t_0}^{t} \max_{t_0 \leqslant \tau \leqslant t} f_i(\tau, s)ds], \quad (2.2)$$

$W_1(0) := 0$*, and* T_1 *is the largest number such that*

$$W_i(b_i(T_1)) + \int_{t_0}^{T_1} \max_{t_0 \leqslant \tau \leqslant T_1} f_i(\tau, s)ds \leqslant \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \quad i = 1, \cdots, n. \quad (2.3)$$

REMARK 1. T_1 is confined by (2.3). In particular, $T_1 = \infty$ when all w_i $(i = 1, \cdots, n)$ satisfy $\int_{u_i}^{\infty} \frac{dz}{w_i(z)} = \infty$.

REMARK 2. Different choices of u_i in W_i do not affect our results. In fact, for positive constants $v_i \neq u_i$, let $\tilde{W}_i(u) = \int_{v_i}^{u} \frac{dz}{w_i(z)}$. Then $\tilde{W}_i(u) = W_i(u) + \tilde{W}_i(u_i)$, and $\tilde{W}_i^{-1}(v) = W_i^{-1}(v - \tilde{W}_i(u_i))$. Thus, $\tilde{W}_i^{-1}[\tilde{W}_i(b_i(t)) + \int_{t_0}^{t} \tilde{f}_i(t, s)ds] = W_i^{-1}[W_i(b_i(t)) + \int_{t_0}^{t} \tilde{f}_i(t, s)ds]$, and

$$\begin{aligned}\tilde{W}_i(b_i(T_1)) + \int_{t_0}^{T_1} \tilde{f}_i(T_1, s)ds &= \tilde{W}_i(u_i) + W_i(b_i(T_1)) + \int_{t_0}^{T_1} \tilde{f}_i(T_1, s)ds \\ &\leqslant \int_{v_i}^{\infty} \frac{dz}{w_i(z)} = \int_{v_i}^{u_i} \frac{dz}{w_i(z)} + \int_{u_i}^{\infty} \frac{dz}{w_i(z)} = \tilde{W}_i(u_i) + \int_{u_i}^{\infty} \frac{dz}{w_i(z)}.\end{aligned}$$

That is, (2.1), (2.2) and (2.3) are independent of the choice of $u_i > 0$.

Proof. Obviously, $\tilde{f}_i(t, s) := \max_{t_0 \leqslant \tau \leqslant t} f_i(\tau, s)$ is nonnegative and nondecreasing in t for each fixed s, and satisfies $\tilde{f}_i(t, s) \geqslant f_i(t, s)$ for each $i = 1, \cdots, n$.

We first discuss the case that $a(t) \not\equiv 0$ for all $t \in [t_0, \infty)$. It means that $b_1(t) \not\equiv 0$ for all $t \in [t_0, \infty)$, that is, $b_1(t) > 0$ for all $t \in [t_0, \infty)$. In such a circumstance $b_1(t)$ is positive, differentiable and nondecreasing on $[t_0, \infty)$ and $b_1(t) \geqslant a(t_0) + \int_{t_0}^{t} a'(s)ds = a(t)$. Consider the auxiliary inequality

$$u(t) \leqslant b_1(t) + \sum_{i=1}^{n} \int_{t_0}^{t} \tilde{f}_i(T, s)w_i(u(s))ds, \quad t_0 \leqslant t \leqslant T, \quad (2.4)$$

where T is chosen arbitrarily such that $t_0 \leqslant T \leqslant T_1$. Having (2.4) we claim

$$u(t) \leqslant W_n^{-1}[W_n(\tilde{b}_n(T, t)) + \int_{t_0}^{t} \tilde{f}_n(T, s)ds], \quad t_0 \leqslant t \leqslant T \leqslant T_2, \quad (2.5)$$

where

$$\tilde{b}_1(T, t) = b_1(t), \quad \tilde{b}_{i+1}(T, t) = W_i^{-1}[W_i(\tilde{b}_i(T, t)) + \int_{t_0}^{t} \tilde{f}_i(T, s)ds], \quad (2.6)$$

$i = 1, \cdots, n-1$, and T_2 is the largest number such that

$$W_i(\tilde{b}_i(T, T_2)) + \int_{t_0}^{T_2} \tilde{f}_i(T, s)ds \leqslant \int_{u_i}^{\infty} \frac{dz}{w_i(z)}, \quad i = 1, \cdots, n. \quad (2.7)$$

Notice that $T_1 \leqslant T_2$. In fact, both $\tilde{b}_i(T,t)$ and $\tilde{f}_i(T,t)$ are nondecreasing in T. Thus, T_2 satisfying (2.7) gets smaller as T is chosen larger. In particular, T_2 satisfies the same (2.3) as T_1 when $T = T_1$.

To prove (2.5) for $n = 1$, we observe that (2.4) is equivalent to $u(t) \leqslant b_1(t) + z(t)$ for $t \in [t_0, T]$ where $z(t) := \int_{t_0}^t \tilde{f}_1(T,s)w_1(u(s))ds$ is a nonnegative and differentiable function on $[t_0, T]$. Since w_1 is nondecreasing and $z(t) + b_1(t) > 0$, we have

$$\begin{aligned} \frac{z'(t) + b_1'(t)}{w_1(z(t) + b_1(t))} &\leqslant \frac{\tilde{f}_1(T,t)w_1(u(t))}{w_1(z(t) + b_1(t))} + \frac{b_1'(t)}{w_1(z(t) + b_1(t))} \\ &\leqslant \frac{\tilde{f}_1(T,t)w_1(z(t) + b_1(t))}{w_1(z(t) + b_1(t))} + \frac{b_1'(t)}{w_1(b_1(t))} \\ &\leqslant \tilde{f}_1(T,t) + \frac{b_1'(t)}{w_1(b_1(t))}. \end{aligned} \tag{2.8}$$

Integrating both side of the above inequality from t_0 to t, we obtain

$$W_1(z(t) + b_1(t)) \leqslant W_1(b_1(t)) + \int_{t_0}^t \tilde{f}_1(T,s)ds, \quad t_0 \leqslant t \leqslant T. \tag{2.9}$$

By (2.7), we see that $W_1(b_1(t)) + \int_{t_0}^t \tilde{f}_1(T,s)ds$ is in the domain of W_1^{-1} for all $t \in [t_0, T]$ for $n = 1$. Thus the monotonicity of W_1^{-1} implies

$$u(t) \leqslant b_1(t) + z(t) \leqslant W_1^{-1}[W_1(b_1(t)) + \int_{t_0}^t \tilde{f}_1(T,s)ds], \; t_0 \leqslant t \leqslant T \leqslant T_2, \tag{2.10}$$

i.e., (2.5) is true for $n = 1$.

Assume that (2.5) is true for $n = m$. Consider

$$u(t) \leqslant b_1(t) + \sum_{i=1}^{m+1} \int_{t_0}^t \tilde{f}_i(T,s)w_i(u(s))ds, \quad t_0 \leqslant t \leqslant T.$$

Let $z(t) = \sum_{i=1}^{m+1} \int_{t_0}^t \tilde{f}_i(T,s)w_i(u(s))ds$. Then $z(t)$ is differentiable, nonnegative and nondecreasing on $[t_0, T]$ and satisfies $u(t) \leqslant b_1(t) + z(t)$ for $t \in [t_0, T]$. Since w_i is nondecreasing and $z(t) + b_1(t) > 0$, we have

$$\begin{aligned} \frac{z'(t) + b_1'(t)}{w_1(z(t) + b_1(t))} &\leqslant \frac{\sum_{i=1}^{m+1} \tilde{f}_i(T,t)w_i(u(t))}{w_1(z(t) + b_1(t))} + \frac{b_1'(t)}{w_1(z(t) + b_1(t))} \\ &\leqslant \sum_{i=1}^{m+1} \tilde{f}_i(T,t)\frac{w_i(z(t) + b_1(t))}{w_1(z(t) + b_1(t))} + \frac{b_1'(t)}{w_1(b_1(t))} \\ &\leqslant \tilde{f}_1(T,t) + \sum_{i=2}^{m+1} \tilde{f}_i(T,t)\phi_i(z(t) + b_1(t)) + \frac{b_1'(t)}{w_1(b_1(t))} \\ &\leqslant \tilde{f}_1(T,t) + \sum_{i=1}^{m} \tilde{f}_{i+1}(T,t)\phi_{i+1}(z(t) + b_1(t)) + \frac{b_1'(t)}{w_1(b_1(t))}, \end{aligned}$$

for $t_0 \leqslant t \leqslant T$, where $\phi_{i+1}(u) := \frac{w_{i+1}(u)}{w_1(u)}$, $i = 1, \cdots, m$. Integrating the above inequality from t_0 to t, we get

$$W_1(z(t) + b_1(t)) \leqslant W_1(b_1(t)) + \int_{t_0}^{t} \tilde{f}_1(T,s)ds + \sum_{i=1}^{m} \int_{t_0}^{t} \tilde{f}_{i+1}(T,s)\phi_{i+1}(z(s) + b_1(s))ds, \quad t_0 \leqslant t \leqslant T,$$

or equivalently

$$\xi(t) \leqslant c_1(t) + \sum_{i=1}^{m} \int_{t_0}^{t} \tilde{f}_{i+1}(T,s)\phi_{i+1}(W_1^{-1}(\xi(s)))ds, \quad t_0 \leqslant t \leqslant T,$$

the same as (2.5) for $n = m$, where $\xi(t) = W_1(z(t) + b_1(t))$ and $c_1(t) = W_1(b_1(t)) + \int_{t_0}^{t} \tilde{f}_1(T,s)ds$. From the assumption (C_1), each $\phi_{i+1}(W_1^{-1})$, $i = 1, \cdots, m$, is continuous and nondecreasing on $[0,\infty)$ and is positive on $(0,\infty)$ since W_1^{-1} is continuous and nondecreasing on $[0,\infty)$. Moreover, $\phi_2(W_1^{-1}) \propto \phi_3(W_1^{-1}) \propto \cdots \propto \phi_{m+1}(W_1^{-1})$. By the inductive assumption, we have

$$\xi(t) \leqslant \Phi_{m+1}^{-1}[\Phi_{m+1}(c_m(t)) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds], \quad t_0 \leqslant t \leqslant \min\{T, T_3\}, \tag{2.11}$$

where $\Phi_{i+1}(u) = \int_{\tilde{u}_{i+1}}^{u} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))}$, $u > 0, \tilde{u}_{i+1} = W_1(u_{i+1})$, Φ_{i+1}^{-1} is the inverse of Φ_{i+1}, $i = 1, \cdots, m$,

$$c_{i+1}(t) = \Phi_{i+1}^{-1}[\Phi_{i+1}(c_i(t)) + \int_{t_0}^{t} \tilde{f}_{i+1}(T,s)ds], \quad i = 1, \cdots, m-1,$$

and T_3 is the largest number such that

$$\Phi_{i+1}(c_i(T_3)) + \int_{t_0}^{T_3} \tilde{f}_{i+1}(T,s)ds \leqslant \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))}, \; i = 1, \cdots, m. \tag{2.12}$$

Note that

$$\begin{aligned}\Phi_i(u) = \int_{\tilde{u}_i}^{u} \frac{dz}{\phi_i(W_1^{-1}(z))} &= \int_{W_1(u_i)}^{u} \frac{w_1(W_1^{-1}(z))dz}{w_i(W_1^{-1}(z))} \\ &= \int_{u_i}^{W_1^{-1}(u)} \frac{dz}{w_i(z)} = W_i \circ W_1^{-1}(u), \qquad i = 2, \cdots, m+1.\end{aligned}$$

Thus, we have from (2.11) that

$$\begin{aligned} u(t) &\leqslant b_1(t) + z(t) = W_1^{-1}(\xi(t)) \\ &\leqslant W_{m+1}^{-1}[W_{m+1}(W_1^{-1}(c_m(t))) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds], \end{aligned} \tag{2.13}$$

for $t_0 \leqslant t \leqslant \min\{T, T_3\}$. Let $\tilde{c}_i(t) = W_1^{-1}(c_i(t))$. Obviously,

$$\begin{aligned}\tilde{c}_1(t) = W_1^{-1}(c_1(t)) &= W_1^{-1}[W_1(b_1(t)) + \int_{t_0}^{t} \tilde{f}_1(T,s)ds] \\ &= W_1^{-1}[W_1(\tilde{b}_1(T,t)) + \int_{t_0}^{t} \tilde{f}_1(T,s)ds] = \tilde{b}_2(T,t).\end{aligned}$$

Moreover, with the assumption that $\tilde{c}_m(t) = \tilde{b}_{m+1}(T,t)$, we have

$$\begin{aligned}\tilde{c}_{m+1}(t) &= W_1^{-1}\{\Phi_{m+1}^{-1}[\Phi_{m+1}(c_m(t)) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds]\} \\ &= W_{m+1}^{-1}[W_{m+1}(W_1^{-1}(c_m(t))) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds] \\ &= W_{m+1}^{-1}[W_{m+1}(\tilde{c}_m(t)) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds] \\ &= W_{m+1}^{-1}[W_{m+1}(\tilde{b}_{m+1}(T,t)) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds] \\ &= \tilde{b}_{m+2}(T,t).\end{aligned}$$

This proves that

$$\tilde{c}_i(t) = \tilde{b}_{i+1}(T,t), \qquad i = 1, \cdots, m.$$

Therefore, (2.12) becomes

$$\begin{aligned}W_{i+1}(\tilde{b}_{i+1}(T,T_3)) + \int_{t_0}^{T_3} \tilde{f}_{i+1}(T,s)ds &\leqslant \int_{\tilde{u}_{i+1}}^{W_1(\infty)} \frac{dz}{\phi_{i+1}(W_1^{-1}(z))} \\ &= \int_{u_{i+1}}^{\infty} \frac{dz}{w_{i+1}(z)}, \quad i = 1, \cdots, m.\end{aligned}$$

It means that $T_2 = T_3$ and $T \leqslant T_3$. From (2.13) we have

$$u(t) \leqslant W_{m+1}^{-1}[W_{m+1}(\tilde{b}_{m+1}(T,t)) + \int_{t_0}^{t} \tilde{f}_{m+1}(T,s)ds], \quad t_0 \leqslant t \leqslant T \leqslant T_2.$$

This proves (2.5) by induction.

Finally, from (1.4) we have

$$\begin{aligned}u(T) &\leqslant a(T) + \sum_{i=1}^{n} \int_{t_0}^{T} f_i(T,s) w_i(u(s))ds \\ &\leqslant b_1(T) + \sum_{i=1}^{n} \int_{t_0}^{T} \tilde{f}_i(T,s) w_i(u(s))ds,\end{aligned}$$

namely, the auxiliary inequality holds for $t = T$. By (2.5),

$$\begin{aligned}u(T) &\leqslant W_n^{-1}[W_n(\tilde{b}_n(T,T)) + \int_{t_0}^{T} \tilde{f}_n(T,s)ds] \\ &\leqslant W_n^{-1}[W_n(b_n(T)) + \int_{t_0}^{T} \tilde{f}_n(T,s)ds], \qquad t_0 \leqslant t \leqslant T_1,\end{aligned}$$

where we apply the facts that $\tilde{b}_n(T,T) = b_n(T)$ and $T_2 = T_1$, which can be easily verified and found in the sentences after (2.7) respectively. This proves (2.1) because T is arbitrarily chosen.

In case $a(t) \equiv 0$ for all $t \in [t_0, \infty)$, by definition we have that $b_1(t) \equiv 0$ for all $t \in [t_0, \infty)$. Let $b_{1,u_1}(t) := b_1(t) + u_1$ for all $t \in [t_0, \infty)$, where $u_1 > 0$ is given in $W_1(u) = \int_{u_1}^{u} \frac{dz}{w_1(z)}$. Using the same arguments as in (2.8) and (2.9) where $b_1(t)$ is replaced with the positive $b_{1,u_1}(t)$, we get

$$\int_{z(t_0)+b_{1,u_1}(t_0)}^{z(t)+b_{1,u_1}(t)} \frac{dz}{w_1(z)} \leqslant \int_{b_{1,u_1}(t_0)}^{b_{1,u_1}(t)} \frac{dz}{w_1(z)} + \int_{t_0}^{t} \tilde{f}_1(T,s)\,ds. \tag{2.14}$$

Notice that $b_{1,u_1}(t_0) = b_{1,u_1}(t) \equiv u_1$. Then the second integral of (2.14) equals 0 and we get $W_1(z(t)+u_1) \leqslant \int_{t_0}^{t} \tilde{f}_1(T,s)\,ds$, i.e.,

$$\begin{aligned} u(t) &\leqslant z(t) + b_{1,u_1}(t) \leqslant z(t) + u_1 \\ &\leqslant W_1^{-1}\Big(\int_{t_0}^{t} \tilde{f}_1(T,s)\,ds\Big), \quad t_0 \leqslant t \leqslant T \leqslant T_2, \end{aligned} \tag{2.15}$$

which is the same as (2.10) with a complementary definition that $W_1(0) := 0$. Being the first step, the estimate of (2.15) is independent of u_1. Then, as for (2.5) as the above we similarly obtain (2.1) and all $b_n(t)$ are defined by the same formula (2.2), where we note that $W_1(0) := 0$. This completes the proof. □

3. Generalization of known results

Taking $a(t) = c$, a positive constant, and $f_i(t,s) = \lambda_i(s)$ in (1.4), we get from Theorem 1 that

$$u(t) \leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^{t} \lambda_n(s)ds], \quad t_0 \leqslant t \leqslant T_1,$$

which generalizes Pinto's estimate (6) in [15]. Actually, let c_i denote the maximum of $b_i(t)$ on $[t_0, T_1]$, that is, $c_i = b_i(T_1)$. Then the above estimate where all $b_i(t)$ are replaced by c_i is just the result of Theorem 1 in [15].

Consider the inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_{t_0}^{t} g_i(t,s) \int_{t_0}^{s} h_i(s,\tau) w_i(u(\tau)) d\tau ds, \quad t \geqslant t_0 \geqslant 0, \tag{3.1}$$

which looks more complicated than (1.4).

COROLLARY 1. *Suppose that (C_1) and (C_2) hold and that the functions $g_i(t,s), h_i(t,s)$ are both nonnegative and continuous on $[t_0,\infty) \times [t_0,\infty)$, $i = 1, \cdots, n$. If $u(t)$ is a continuous and nonnegative function such that (3.1) holds on $[t_0,\infty)$, then*

$$u(t) \leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^{t} \max_{t_0 \leqslant \tau \leqslant t} g_n(\tau, v) \int_{t_0}^{v} h_n(v,\tau) ds dv], \quad t_0 \leqslant t \leqslant T_B,$$

where b_n *and its related functions are defined as in Theorem* 1 *by replacing* $f_i(t,s)$ *with* $\int_s^t \max_{t_0\leqslant\tau\leqslant t} g_i(\tau,v)h_i(v,s)dv$.

Proof. Because g_i, h_i and w_i are continuous, we have

$$\int_{t_0}^t g_i(t,s)\int_{t_0}^s h_i(s,\tau)w_i(u(\tau))d\tau ds = \int_{t_0}^t w_i(u(\tau))\int_\tau^t g_i(t,s)h_i(s,\tau)dsd\tau$$
$$= \int_{t_0}^t w_i(u(s))\int_s^t g_i(t,\tau)h_i(\tau,s)d\tau ds \leqslant \int_{t_0}^t f_i(t,s)w_i(u(s))ds,$$

where $f_i(t,s) := \int_s^t \max_{t_0\leqslant\tau\leqslant t} g_i(\tau,v)h_i(v,s)dv$. Then (3.1) is reduced to

$$u(t) \leqslant a(t) + \sum_{i=1}^n \int_{t_0}^t f_i(t,s)w_i(u(s))ds, \quad t \geqslant t_0,$$

which is just the form of (1.4). Note that for fixed s the function $f_i(t,s)$ is increasing in t. So $\tilde{f_i}(t,s) := \max_{t_0\leqslant\tau\leqslant t} f_i(\tau,s) = f_i(t,s)$. By Theorem 1,

$$\begin{aligned} u(t) &\leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^t f_n(t,s)ds] \\ &\leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^t \int_s^t \max_{t_0\leqslant\tau\leqslant t} g_n(\tau,v)h_n(v,s)dvds] \\ &\leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^t \max_{t_0\leqslant\tau\leqslant t} g_n(\tau,v)\int_{t_0}^v h_n(v,s)dsdv], \quad t_0 \leqslant t \leqslant T_B. \quad \square \end{aligned}$$

Similarly, (1.3) can also be changed into the form of (1.4). Hence, Theorem 1 also answers in the cases discussed in Theorems in [5].

COROLLARY 2. *Suppose that* (C_1) *and* (C_3) *hold and that* $a(t)$ *is continuous, nondecreasing and positive on* $[t_0,\infty)$. *Let each function* $w_i(u)$ *be in the class* H *with a multiplier* ψ_i, $i = 1,\cdots,n$. *If* $u(t)$ *is a continuous and nonnegative function satisfying* (1.4) *on* $[t_0,\infty)$, *then*

$$u(t) \leqslant a(t)W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^t \frac{1}{a(s)}\max_{t_0\leqslant\tau\leqslant t} f_n(\tau,s)\psi_n(a(s))ds], \quad t_0 \leqslant t \leqslant T_1,$$

where b_n *and other notations are defined in Theorem* 1 *by replacing* $a(t_0), a'(s), f_i(t,s)$ *with* $1, 0, f_i(t,s)\psi_i(a(s))/a(s)$ *respectively.*

Proof. We can not apply Theorem 1 immediately because (C_2) does not hold. However, w_i is in the class H. Thus from (1.4) we have

$$\begin{aligned} z(t) &\leqslant 1 + \sum_{i=1}^n \int_{t_0}^t \frac{f_i(t,s)}{a(t)} w_i(a(s)z(s))ds \\ &\leqslant 1 + \sum_{i=1}^n \int_{t_0}^t \frac{f_i(t,s)}{a(s)} \psi_i(a(s))w_i(z(s))ds, \end{aligned} \tag{3.2}$$

where $z(t) := u(t)/a(t)$. Clearly inequality (3.2) is in the form of (1.4) and the term corresponding to $a(t)$ becomes 1, a constant function, which of course satisfies (C_2). Applying Theorem 1 to (3.2) we get

$$z(t) \leqslant W_n^{-1}[W_n(b_n(t)) + \int_{t_0}^{t} \frac{1}{a(s)} \max_{t_0 \leqslant \tau \leqslant t} f_n(\tau, s)\psi_n(a(s))ds], \quad t_0 \leqslant t \leqslant T_1.$$

Together with the relation $u(t) = a(t)z(t)$, it gives the estimate of u in the corollary. □

As (1.3) can be changed into the form of (1.4), Corollary 2 also answers in the case of Corollary 2.4 in [5]. Moreover, by taking $f_i(t,s)$ to be $\lambda_i(s)$, $f_i(t)\lambda_i(s)$ and $f(t)f_i(t)\lambda_i(s)$ respectively, our Corollary 2 also implies Pinto's Theorem 3, Corollaries 1 and 2 in [15]. In particular, as in Theorem 3 in [15], we get from our Corollary 2 that

$$u(t) \leqslant h(t)W_n^{-1}[W_n(c_{n-1}) + \int_a^t \frac{\lambda_n(s)}{h(s)} r_n(h(s))ds],$$

which corrects an error in [15]. In fact, the first line on page 395 of [15] should be

$$z(t) \leqslant 1 + \sum_{i=1}^{n} \int_a^t \frac{\lambda_i(s)}{h(s)} r_i(h(s))\omega_i(z(s))ds.$$

Corresponding to Theorem 4 in [15], another inequality

$$\begin{aligned} u(t) \ &\leqslant \ a(t) + \int_{t_0}^{t} f_1(t,s)w_1(u(s))ds \\ &\quad + \int_{t_0}^{t} f_2(t,s)w_2(\int_{t_0}^{s} f_3(s,\tau)w_3(u(\tau))d\tau)ds, \quad t \geqslant t_0 \geqslant 0, \end{aligned} \tag{3.3}$$

is also interesting. It is just the case of Theorem 4 in [15] where $a(t) = c$ (a positive constant) and $f_i(t,s) = \lambda_i(s),\ i = 1,2,3$. The following result gives an estimate to inequality (3.3).

COROLLARY 3. *Suppose that functions* a, f_i *and* w_i*,* $i = 1,2,3$*, satisfy* (C_1)*,* (C_2) *and* (C_3)*. If the continuous and nonnegative function* $u(t)$ *satisfies* (3.3) *on* $[t_0,\infty)$*, then*

$$u(t) \leqslant W_3^{-1}[W_3(b_3(t)) + \int_{t_0}^{t} \max_{t_0 \leqslant \tau \leqslant t} f_3(\tau,s)ds], \quad t_0 \leqslant t \leqslant T_1,$$

where the notations b_3 *and* T_1 *are defined as in Theorem* 1.

Proof. Take T arbitrarily such that $t_0 \leqslant T \leqslant T_1$. Let

$$\begin{aligned} b_1(t) &= a(t_0) + \int_{t_0}^{t} |a'(s)|ds, \\ z_1(t) &= \int_{t_0}^{t} \tilde{f}_1(T,s)w_1(u(s))ds + \int_{t_0}^{t} \tilde{f}_2(T,s)w_2(z_2(s))ds, \\ z_2(t) &= \int_{t_0}^{t} \tilde{f}_3(T,s)w_3(u(s))ds, \end{aligned}$$

where $\tilde{f}_i(t,s) := \max_{t_0 \leqslant \tau \leqslant t} f_i(\tau,s)$. Thus, $b_1(t), z_1(t), z_2(t)$ are all nonnegative, differentiable and nondecreasing on $[t_0,T]$, and $u(t) \leqslant b_1(t)+z_1(t)+z_2(t)$ for $t \in [0,T]$. Note that the function $z(t) := b_1(t)+z_1(t)+z_2(t)$ satisfies

$$\begin{aligned} z'(t) &= |a'(t)| + \tilde{f}_1(T,t)w_1(u(t)) + \tilde{f}_2(T,t)w_2(z_2(t)) + \tilde{f}_3(T,t)w_3(u(t)) \\ &\leqslant |a'(t)| + \tilde{f}_1(T,t)w_1(z(t)) + \tilde{f}_2(T,t)w_2(z(t)) + \tilde{f}_3(T,t)w_3(z(t)) \end{aligned}$$

because $w_i\ (i=1,2,3)$ are nondecreasing. Integrating the above inequality, we get

$$z(t) \leqslant b_1(t) + \sum_{i=1}^{3} \int_{t_0}^{t} \tilde{f}_i(T,s)w_i(z(s))ds, \quad t_0 \leqslant t \leqslant T,$$

which is in the form of (1.4). Applying Theorem 1 we obtain the result of this corollary. □

4. Applications

Consider the differential equation

$$\dot{x}(t) = \frac{1}{t^2} + \exp(-t)\sqrt{|x(t)|+1} + t\exp(-t)\mathfrak{T}x(t), \tag{4.1}$$

where $x : [0,\infty) \to \mathbf{R}$ is a differentiable function and $\mathfrak{T}$ is a continuous operator on $C(\mathbb{R},\mathbb{R})$ such that $|\mathfrak{T}x| \leqslant c_0|x|$ for a constant $c_0 > 0$. In particular, when we take $\mathfrak{T}x(t) = \int_{t_0}^{t} H(t,s,x(s))ds$ or $\mathfrak{T}x(t) = x(t-\tau)$, equation (4.1) becomes an integro-differential equation or retarded functional differential equation. General theory can be found, for example, in [8, 9]. From (4.1),

$$|x(t)| \leqslant b_1(t) + \int_{t_0}^{t} f(s)w_1(|x(s)|)ds + \int_{t_0}^{t} g(s)w_2(|x(s)|)ds, \quad t \geqslant t_0 > 0, \tag{4.2}$$

where $b_1(t) = |x(t_0)| + \frac{1}{t_0} - \frac{1}{t}$, $w_1(u) = \sqrt{u+1}$, $w_2(u) = c_0 u$, $f(t) = \exp(-t)$, $g(t) = t\exp(-t)$. Clearly, $\frac{w_2(u)}{w_1(u)} = c_0\frac{u}{\sqrt{u+1}}$ is nondecreasing for $u > 0$, that is, $w_1 \propto w_2$. Then for $u_1, u_2 > 0$

$$\begin{aligned} W_1(u) &= \int_{u_1}^{u} \frac{dz}{\sqrt{z+1}} = 2(\sqrt{u+1} - \sqrt{u_1+1}), \quad W_1^{-1}(u) = (\frac{u}{2} + \sqrt{u_1+1})^2 - 1, \\ W_2(u) &= \int_{u_2}^{u} \frac{dz}{c_0 z} = \frac{1}{c_0}\ln\frac{u}{u_2}, \qquad W_2^{-1}(u) = u_2\exp(c_0 u), \\ b_2(t) &= [\sqrt{|x(t_0)| + \frac{1}{t_0} - \frac{1}{t} + 1} + \frac{1}{2}(\exp(-t_0) - \exp(-t))]^2 - 1. \end{aligned}$$

Note that $T_1 = \infty$ because $\int_{u_1}^{\infty} \frac{dz}{w_1(z)} = \int_{u_1}^{\infty} \frac{dz}{\sqrt{z+1}} = \infty$ and $\int_{u_2}^{\infty} \frac{dz}{w_2(z)} = \int_{u_2}^{\infty} \frac{dz}{c_0 z} = \infty$. Then

$$
\begin{aligned}
|x(t)| &\leqslant W_2^{-1}[W_2(b_2(t)) + \int_{t_0}^{t} g(s)ds] \\
&\leqslant \{[\sqrt{|x(t_0)| + \frac{1}{t_0} - \frac{1}{t} + 1} + \frac{1}{2}(\exp(-t_0) - \exp(-t))]^2 - 1\} \times \\
&\qquad \times \exp[c_0(t_0+1)\exp(-t_0) - c_0(t+1)\exp(-t)], \quad \forall t \geqslant t_0.
\end{aligned}
$$

In particular,

$$
\lim_{t\to\infty} \sup |x(t)| \leqslant \{[\sqrt{|x(t_0)| + \frac{1}{t_0} + 1} + \frac{1}{2}\exp(-t_0)]^2 - 1\} \exp[c_0(t_0+1)\exp(-t_0)].
$$

This implies that every solution of (4.1) is bounded.

It is worth mentioning that theorems and corollaries in [15] and [5] do not work in this example because the inequality corresponding to (4.2) involves the non-constant function $b_1(t)$ and w_1 is not in the class H.

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