Mathematical
Inequalities
& Applications
Volume 8, Number 4 (2005), 655–674

DIFFERENTIAL DIFFERENCE INEQUALITIES RELATED TO HYPERBOLIC FUNCTIONAL DIFFERENTIAL SYSTEMS AND APPLICATIONS

Z. KAMONT AND K. KROPIELNICKA

(*communicated by D. Bainov*)

Abstract. Initial boundary value problems for quasilinear hyperbolic systems are transformed by discretization in space variables into systems of ordinary functional differential equations. Sufficient conditions for the convergence of the method of lines are given. An implicit difference method is proposed for the numerical solving of systems thus obtained. This leads to an implicit difference method for the original problem. A comparison technique is used. We give a complete convergence analysis for the methods and we show by an example that the new methods are considerable better than the classical schemes.

1. Introduction

For any metric spaces U and V we denote by $C(U,V)$ the class of all continuous functions from U into V. Let $M_{k\times n}$ be the space of all $k\times n$ matrices with real elements. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E = [0,a]\times[-b,b],\quad E_0 = [-b_0,0]\times[-b,b],$$
$$\partial_0 E = [0,a]\times\big([-b,b]\setminus(-b,b)\big)$$

where $a>0$, $b_0\in\mathbb{R}_+$, $\mathbb{R}_+=[0,+\infty)$, $b=(b_1,\ldots,b_n)\in\mathbb{R}^n$ and $b_i>0$ for $1\leqslant i\leqslant n$. Set $\Omega = E\times C(E_0\cup E,\mathbb{R}^k)$ and suppose that the functions

$$f:\Omega\to M_{k\times n},\quad f=\big[f_{ij}\big]_{i=1,\ldots,k,\,j=1,\ldots,n},$$
$$g:\Omega\to R^k,\quad g=(g_1,\ldots,g_k),$$
$$\varphi: E_0\cup\partial_0E\to R^k,\quad \varphi=(\varphi_1,\ldots,\varphi_k),$$

are given. We consider the problem consisting of the system of functional differential equations

$$\partial_t z_i(t,x) = \sum_{j=1}^{n} f_{ij}(t,x,z)\,\partial_{x_j} z_i(t,x) + g_i(t,x,z),\quad i=1,\ldots,k, \tag{1}$$

Mathematics subject classification (2000): 35R10, 65M20.

Key words and phrases: initial boundary value problems, differential functional inequalities, stability and convergence, nonlinear estimates of the Perron type.

Paper MIA-08-61

and the initial boundary condition

$$z(t,x) = \varphi(t,x) \ \text{ for } \ (t,x) \in E_0 \cup \partial_0 E, \tag{2}$$

where $x = (x_1, \dots, x_n)$ and $z = (z_1, \dots, z_k)$. Write

$$E_t = \big(E_0 \cup E\big) \cap \big([-b_0, t] \times \mathbb{R}^n\big), \ \ 0 \leqslant t \leqslant a.$$

We assume that f and g satisfy the following Volterra condition: if $(t,x) \in E$ and $z, \bar{z} \in C(E_0 \cup E, \mathbb{R}^k)$ are such functions that $z\big|_{E_t} = \bar{z}\big|_{E_t}$ then $f(t,x,z) = f(t,x,\bar{z})$ and $g(t,x,z) = g(t,x,\bar{z})$.

The method of lines for partial differential or functional differential equations consists in replacing derivatives with respect to spatial variables by difference expressions. Then the original problem is transformed into a system of ordinary differential or functional differential equations. It is easy to construct a differential difference system which satisfies a consistency condition with respect to the original problem on sufficiently regular solutions. The main question in these considerations is to find sufficient conditions for the stability on the numerical method of lines. The method of differential inequalities is a basic tool in the investigation of the stability.

There is a wide literature on the numerical method of lines for parabolic differential or functional differential equations and for nonlinear first order partial functional differential problems ([3], [7], [10] - [12], [14], [18]). The monographs [9], [16], [17] contain a large bibliography.

The method of lines is also treated as a tool for proving of existence theorems for differential problems corresponding to parabolic equations ([19], [20]) and for first order hyperbolic systems ([8], [15]).

Our concern is the numerical method of lines for problem (1), (2). By making use of a discretization of the spatial variable x, we associate to problem (1), (2) a net of Cauchy problems for ordinary functional differential equations. Solutions of such systems are considered as approximate solutions of (1), (2). Then we estimate the difference between the exact and approximate solutions of (1), (2) and, as a consequence, we prove that approximate solutions converge to the solution of (1), (2).

Note that theorems on the numerical method of lines for nonlinear equations ([3], [9] Chapter VI) are not applicable to (1), (2).

The second part of the paper deals with the discretization in time of differential difference systems corresponding to (1), (2). An application of one step difference methods is a natural way of the numerical solving of such problems .

In the paper we propose implicit difference schemes for the numerical solving of the above problems. They have the following properties:

(i) the numerical realization of our method is very simple,

(ii) results obtained by using implicit difference methods of the Euler type are better than those obtained by classical Runge - Kutta schemes.

We give a complete convergence analysis for the methods and we show by an example that the new methods are considerable better than the classical schemes.

The paper is organized as follows. In Section 2 we formulate a numerical method of lines for (1), (2). In the next section we present a comparison result for differential difference inequalities. It will be a generalization of corresponding results from [2] and [9]. A convergence result and an error estimate of approximate solutions are presented in Section 4. Difference problems generated by the method of lines are investigated in Section 5. Numerical examples are given in the last part of the paper.

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. For $x,\ y \in \mathbb{R}^n$, $p \in \mathbb{R}^k$, $X \in M_{k\times n}$ where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$, $p = (p_1\dots,p_k)$, and

$$X = \left[\, x_{ij} \,\right]_{i=1,\dots,k,\, j=1,\dots,k},$$

we put

$$\|x\| = \sum_{j=1}^{n} |x_j|, \quad \|p\|_0 = \max\{\, |p_i| : \quad 1 \leqslant i \leqslant k\,\},$$

and

$$\|X\| = \max\{\sum_{j=1}^{n} |x_{ij}| : \quad 1 \leqslant i \leqslant k\,\} \ \text{ and } \ x \diamond y = (x_1y_1,\dots,x_ny_n).$$

Set $\theta = (0,\dots,0) \in \mathbb{R}^n$. If $X \in M_{k\times n}$ then X^T is the transpose matrix. For $z \in C(E_0 \cup E, \mathbb{R}^k)$ we write

$$\|z\|_t = \max\{\, \|z(\tau,y)\|_0 : \quad (\tau,y) \in E_t\,\}, \quad 0 \leqslant t \leqslant a.$$

Note that differential systems with deviated variables and differential integral problems can be derived from (1) by specializing the operators f and g.

Two types of assumptions are needed in a theorem on the uniqueness of classical solutions of problem (1), (2). The first type conditions deal with the regularity of given functions. It is assumed that f and g are continuous on Ω and that they satisfy nonlinear estimates of the Perron type with respect to the functional variable. The assumptions of the second type are connected with the theory of bicharacteristics and they have the following form. Write

$$f_i = \left(f_{i1},\dots,f_{in}\right), \quad 1 \leqslant i \leqslant k.$$

It is assumed that

$$f_i(t,x,z) \diamond x \geqslant \theta, \quad 1 \leqslant i \leqslant k, \tag{3}$$

where $(t,x,z) \in \Omega$. The uniqueness criteria are consequences of comparison results for functional differential inequalities with initial boundary conditions ([2], [9]).

Existence results for (1), (2) can be deduced from [9], Chapter V. They are based on a method of bicharacteristics. Condition (3) is essential to the proof of existence results.

The existence theory of classical or generalized solutions for nonlinear differential functional systems is based on the method of bicharacteristics. The method was introduced and widely studied in nonfunctional setting by S. Cinquini and M. Cinquini Cibrario ([5], [6]). It was also adpoted by L. Cesari ([4]) and P. Bassanini ([1]) for quasilinear systems in the second canonical form.

Assumption (3) states that bicharacteristics of system (1) satisfy the following monotonicity conditions. Suppose that $v : E_0 \cup E \to \mathbb{R}^k$ is of class C^1. Let us denote by

$$g_i[v](\,\cdot\,,t,x) = \left(\, g_{i.1}[v](\,\cdot\,,t,x),\dots,g_{i.n}[v](\,\cdot\,,t,x)\,\right), \quad (t,x) \in E,$$

the $i-$th bicharacteristic of (1) corresponding to v. Then $g_i[v]$ is a solution of the Cauchy problem

$$y'(\tau) = -f_i(\tau, y(\tau), v), \quad y(t) = x.$$

Condition (3) asserts that the function $g_{ij}(\,\cdot\,, t, x)$ is non increasing if $0 \leqslant x_j \leqslant b_j$ and it is nondecreasing if $-b_j \leqslant x_j < 0$.

This property of bicharacteristics and assumptions on regularity for given function ensure the existence and uniqueness of solutions of (1), (2). It is easily seen that condition (3) may be replaced by the following assumption: there is $\bar{x} \in (-b, b)$ such that

$$f_i(t, x, z) \diamond (x - \bar{x}) \geqslant 0, \quad 1 \leqslant i \leqslant k,$$

where $(t, x, z) \in \Omega$.

We deal with the numerical method of lines and the implicit difference method for (1), (2). It is important that theorems on the uniqueness of classical solutions to (1), (2) and our convergence results have the same assumption on f and g. Condition (3) is assumed for f and for adequate coefficients of difference functional equations.

2. Differential difference problems

We define a mesh in $\mathbb{R}^n$ in the following way. Let $(h_1, \ldots, h_n) = h > \theta$ stand for steps of the mesh. For $m \in \mathbb{Z}^n$, $m = (m_1, \ldots, m_n)$, we define nodal points as follows

$$x^{(m)} = m \diamond h, \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}).$$

Let us denote by H the set of all h for which there exist $(N_1, \ldots, N_n) = N \in \mathbb{N}^n$ such that $N \diamond h = b$. We assume that $H \neq \emptyset$. Write

$$\mathbb{R}^{1+n}_{t.h} = \{\, (t, x^{(m)}) : \ t \in \mathbb{R}, \ m \in \mathbb{Z}^n \,\}$$

and

$$E_h = E \cap \mathbb{R}^{1+n}_{t.h}, \quad E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_{t.h}, \quad \partial_0 E_h = \partial_0 E \cup \mathbb{R}^{1+n}_{t.h}.$$

Elements of $E_h \cup E_{0.h}$ will be denoted by $(t, x^{(m)})$ or (t, x). Let $\mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ be the set of all functions $z : E_{0.h} \cup E_h \to \mathbb{R}^k$ such that $z(\,\cdot\,, x^{(m)}) \in C([-b_0, a], \mathbb{R}^k)$ where $-N \leqslant m \leqslant N$. In a similar way we define the set $\mathbf{F}_c(E_{0.h} \cup \partial_0 E_h, \mathbb{R}^k)$. For a function $z \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ and for a point $(t, x^{(m)}) \in E_{0.h} \cup E_h$ we write $z^{(m)}(t) = z(t, x^{(m)})$ and

$$\|z\|_{h.t} = \max \{ \|z^{(m)}(\tau)\|_0 : \quad (\tau, x^{(m)}) \in E_{0.h} \cup E_h, \quad \tau \leqslant t \}, \quad 0 \leqslant t \leqslant a.$$

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 standing on the j-th place. Given $w : E_{0.h} \cup E_h \to \mathbb{R}$, let $\delta = (\delta_1, \ldots, \delta_n)$ be the difference operator defined by

$$\delta_j w^{(m)}(t) = \frac{1}{h_j} \big[w^{(m+e_j)}(t) - w^{(m)}(t) \big] \quad \text{for} \ \ 0 \leqslant x_j^{(m_j)} < b_j, \tag{4}$$

$$\delta_j w^{(m)}(t) = \frac{1}{h_j} \big[w^{(m)}(t) - w^{(m-e_j)}(t) \big] \quad \text{for} \ \ -b_j < x_j^{(m_j)} < 0. \tag{5}$$

Write $\Omega_h = \big(E_h \setminus \partial_0 E_h\big) \times \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ and suppose that

$$f_h : \Omega_h \to M_{k\times n},\ \ f_h = \big[f_{h.ij}\big]_{i=1,\dots,k,\, j=1,\dots,n},$$
$$g_h : \Omega_h \to \mathbb{R}^k,\ \ g_h = (g_{h.1}, \dots, g_{h.k}),$$
$$\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}^k,\quad \varphi_h = (\varphi_{h.1}, \dots, \varphi_{h.k}),$$

are given functions. Let us denote by $F_h = (F_{h.1}, \dots, F_{h.k})$ the operator defined on Ω_h in the following way

$$F_{h.i}[z]^{(m)}(t) = \sum_{j=1}^{n} f_{h.ij}(t, x^{(m)}, z)\, \delta_j z_i^{(m)}(t) + g_{h.i}(t, x^{(m)}, z),\ \ 1 \leqslant i \leqslant k.$$

We will approximate classical solutions of problem (1), (2) by solutions of the system of ordinary functional differential equations

$$\frac{d}{dt} z^{(m)}(t) = F_h[z]^{(m)}(t) \tag{6}$$

with the initial boundary condition

$$z^{(m)}(t) = \varphi_h^{(m)}(t) \ \text{ on } \ E_{0.h} \cup \partial_0 E_h. \tag{7}$$

We assume that f_h and g_h satisfy the following Volterra condition: if $(t,x) \in E_h \setminus \partial_0 E_h$ and $z, \bar{z} \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ are such functions that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in (E_{0.h} \cup E_h) \cap \big([-b_0, t] \times \mathbb{R}^n\big)$ then $f_h(t,x,z) = f_h(t,x,\bar{z})$ and $g_h(t,x,z) = g_h(t,x,\bar{z})$.

3. Differential difference inequalities

For a function $z \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$, $z = (z_1, \dots, z_k)$, we write

$$D_- z^{(m)}(t) = \big(D_- z_1^{(m)}(t), \dots, D_- z_k^{(m)}(t)\big),$$

where D_- is the left hand lower Dini derivative. Put

$$\delta z^{(m)}(t) = \Big[\,\delta_j z_i^{(m)}(t)\,\Big]_{i=1,\dots,k,\, j=1,\dots,n}.$$

If $X \in M_{k\times n}$, $Y \in M_{n\times k}$ and

$$X = \Big[x_{ij}\Big]_{i=1,\dots,k,\, j=1,\dots,n},\ \ Y = \Big[y_{ij}\Big]_{i=1,\dots,n,\, j=1,\dots,k}$$

then the vector $\zeta = X \star Y$, $\zeta = (\zeta_1, \dots, \zeta_k)$, is defined by

$$\zeta_i = \sum_{j=1}^{n} x_{ij}\, y_{ji}, \quad 1 \leqslant i \leqslant k.$$

LEMMA 3.1. *Suppose that*

1) *the function* $G : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and for each* $\eta \in \mathbb{R}_+$ *the maximal solution* $\omega(\cdot,\eta)$ *of the Cauchy problem*
$$\omega'(t) = G(t,\omega(t)), \quad \omega(0) = \eta,$$
is defined on $[0,a]$,
2) *the function* $\lambda : \Omega_h \to M_{k\times n}$ *where*
$$\lambda = \Big[\lambda_{ij}\Big]_{i=1,\dots,k,\, j=1,\dots,n} \text{ and } \lambda_i = (\lambda_{i1},\dots,\lambda_{in}), \quad 1 \leqslant i \leqslant k,$$
satisfies the condition
$$\lambda_i(t,x,w) \diamond x \geqslant \theta, \quad 1 \leqslant i \leqslant k, \tag{8}$$
where $(t,x,w) \in \Omega_h$
3) *the function* $v \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$, $v = (v_1,\dots,v_k)$, *satisfies the initial boundary estimate*
$$\|v(t,x)\|_0 \leqslant \tilde{\eta} \ for \ (t,x) \in E_{0.h} \cup E_h$$
and the differential difference inequality
$$\Big\| D_- v^{(m)}(t) - \lambda(t,x,w) \star \big[\, \delta v^{(m)}(t)\,\big]^T \Big\|_0 \leqslant G(t, \|v\|_{h.t}) \tag{9}$$
holds on $E_h \setminus \partial_0 E_h$.

Under these assumptions we have
$$\|v^{(m)}(t)\|_0 \leqslant \omega(t,\tilde{\eta}) \ \ on \ \ E_h. \tag{10}$$

Proof. Write
$$\gamma(t) = \|\, v\,\|_{h.t}, \quad t \in [0,a], \tag{11}$$
and
$$J_+ = \{\, t \in (0,a] : \ \ \gamma(t) > \omega(t,\tilde{\eta})\,\}.$$
We claim that
$$D_-\gamma(t) \leqslant G(\,t,\gamma(t)\,) \ \text{ for } \ t \in J_+. \tag{12}$$
Suppose that $\tilde{t} \in J_+$. It follows from (11) that two possibilities can happen, either (i) $D_-\gamma(\tilde{t}) > 0$ or (ii) $D_-\gamma(\tilde{t}) = 0$.

Suppose that possibility (i) holds. Then there are $(\tilde{m}_1,\dots,\tilde{m}_n) = \tilde{m}$, $-N < \tilde{m} < N$, and i, $1 \leqslant i \leqslant k$, such that $\gamma(\tilde{t}) = |v_i^{(\tilde{m})}(\tilde{t})|$. If $\gamma(\tilde{t}) = v_i^{(\tilde{m})}(\tilde{t})$ then we have
$$D_-\gamma(\tilde{t}) \leqslant D_- v_i^{(\tilde{m})}(\tilde{t}) \leqslant \sum_{j=1}^n \lambda_{ij}(\tilde{t},x^{(\tilde{m})},v)\ \delta_j v_i^{(\tilde{m})}(\tilde{t}) + G(\,\tilde{t},\gamma(\tilde{t})\,).$$
We conclude from (8) that
$$D_-\gamma(\tilde{t}) \leqslant G(\,\tilde{t},\gamma(\tilde{t})\,). \tag{13}$$
In a similar way we prove (13) if $\gamma(\tilde{t}) = -v_i^{(\tilde{m})}(\tilde{t})$. It is clear that (13) is satisfied if the case (ii) holds. Then (12) is proved.

Since $\gamma(0) \leqslant \tilde{\eta}$, it follows from (12) and from a comparison theorem for differential inequalities ([13], Theorem 1.4.2) that $\gamma(t) \leqslant \omega(t,\tilde{\eta})$ for $t \in [0,a]$ which completes the proof of (10).

4. Convergence of the numerical method of lines

Suppose that $u_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ and there are $\gamma : [0,a] \times H \to \mathbb{R}_+$ and $\alpha_0 : H \to \mathbb{R}_+$ such that

$$\| \frac{d}{dt} u_h^{(m)}(t) - F_h[u_h]^{(m)}(t) \| \leqslant \gamma(t,h) \text{ on } E_h \setminus \partial_0 E_h \tag{14}$$

and

$$\| u_h^{(m)}(t) - \varphi_h^{(m)}(t) \| \leqslant \alpha_0(h) \text{ on } E_{0.h} \cup \partial_0 E_h. \tag{15}$$

The function u_h satisfying the above relations is considered as an approximate solution of (6), (7). We prove a theorem on the estimate of the difference between the exact and approximate solutions of (6), (7).

Assumption H $[f_h, g_h]$. Suppose that

1) the function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to the second variable,
2) for each $\gamma \in C([0,a], \mathbb{R}_+)$ and $\eta \in \mathbb{R}_+$, $c \geqslant 1$, the maximal solution of the Cauchy problem

$$\omega'(t) = c\,\sigma(t, \omega(t)) + \gamma(t), \quad \omega(0) = \eta, \tag{16}$$

is defined on $[0,a]$ and the maximal solution of (16) with $\gamma(t) = 0$ for $t \in [0,a]$, $\eta = 0$, $c \geqslant 1$ is $\bar{\omega}(t) = 0$ for $t \in [0,a]$,

3) the functions $f_h : \Omega_h \to M_{k\times n}$ and $g_h : \Omega_h \to \mathbb{R}^k$ satisfy the conditions
   (i) for each $(x^{(m)}, w) \in (-b,b) \times \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ we have

$$f_h(\,\cdot\,, x^{(m)}, w) \in C([0,a], M_{k\times n}), \quad g_h(\,\cdot\,, x^{(m)}, w) \in C([0,a], \mathbb{R}^k),$$

   (ii) for each $(t,x,w) \in \Omega_h$ we have

$$f_{h.i}(t,x,w) \diamond x \geqslant \theta \text{ where } f_{h.i} = (f_{h.i1} \dots, f_{h,in}),$$

   (iii) the estimates

$$\|f_h(t,x,w) - f_h(t,x,\bar{w})\| \leqslant \sigma(t, \|w - \bar{w}\|_{h.t}),$$
$$\|g_h(t,x,w) - g_h(t,x,\bar{w})\|_0 \leqslant \sigma(t, \|w - \bar{w}\|_{h.t})$$

   are satisfied on Ω_h.

THEOREM 4.1. *Suppose that Assumption H $[f_h, g_h]$ is satisfied and*

1) $u_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ *and* $\gamma : [0,a] \times H \to \mathbb{R}_+$, $\alpha_0 : H \to \mathbb{R}_+$ *are such functions that estimates* (14), (15) *are satisfied and there is* $\tilde{c} \in \mathbb{R}_+$ *such that*

$$\| \delta_i u_h^{(m)}(t) \|_0 \leqslant \tilde{c} \;\; for \;\; (t, x^{(m)}) \in E_h \setminus \partial_0 E_h, \; 1 \leqslant i \leqslant n, \tag{17}$$

2) $\varphi_h \in \mathbf{F}_c(E_{0.h} \cup \partial_0 E_h, \mathbb{R}^k)$. *Then there is a solution* $z_h : E_{0.h} \cup E_h \to \mathbb{R}^k$ *of problem* (6), (7) *and*

$$\| z_h^{(m)}(t) - u_h^{(m)}(t) \|_0 \leqslant \omega_h(t) \text{ on } E_h, \tag{18}$$

where $\omega_h : [0,a] \to \mathbb{R}_+$ *is the solution of the Cauchy problem*

$$\omega'(t) = (1 + \tilde{c})\,\sigma(t, \omega(t)) + \gamma(t,h), \quad \omega(0) = \alpha_0(h).$$

Proof. At first we prove that the solution z_h of (6), (7) is defined on $E_{0.h} \cup E_h$. It follows that there is $\tilde{a} > 0$ such that the solution z_h is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a}), \times \mathbb{R}^n)$. Let $\tilde{z}_h \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$, $\tilde{z}_h = (\tilde{z}_{h.1}, \ldots, \tilde{z}_{h,k})$, be such a function that $\tilde{z}_h(t,x) = \varphi_h(t,x)$ for $(t,x) \in E_{0.h} \cup E_h$ and the derivative $\frac{d}{dt}\tilde{z}_h^{(m)}(t)$ exists for $(t, x^{(m)}) \in E_h \setminus \partial_0 E_h$. Let $\tilde{c}_h \in \mathbb{R}_+$ be such a constant that

$$\| \delta_j \tilde{z}_h^{(m)}(t) \|_0 \leqslant \tilde{c}_h \text{ for } (t, x^{(m)}) \in E_h \setminus \partial_0 E_h, \quad 1 \leqslant j \leqslant n. \tag{19}$$

Consider the function $\tilde{\Gamma}_h : E_h \setminus \partial_0 E_h \to \mathbb{R}^k$ defined by

$$\tilde{\Gamma}_h^{(m)}(t) = g_h(t, x^{(m)}, z_h) - g_h(t, x^{(m)}, \tilde{z}_h) + \big[f_h(t, x^{(m)}, z_h) - f_h(t, x^{(m)}, \tilde{z}_h)\big] \star \big[\, \delta \tilde{z}_h^{(m)}(t)\, \big]^T.$$

It follows from *Assumption* H $[f_h, g_h]$ and from (19) that

$$\| \tilde{\Gamma}^{(m)}(t) \|_0 \leqslant (1 + \tilde{c}_h)\, \sigma(t, \|z_h - \tilde{z}_h\|_{h.t})$$

where $(t, x^{(m)}) \in E \setminus \partial_0 E_h$. There is $\tilde{\gamma} : [0,a] \times H \to \mathbb{R}_+$ such that

$$\Big\| \frac{d}{dt} \tilde{z}_h^{(m)}(t) - F_h[\tilde{z}_h]^{(m)}(t) \Big\|_0 \leqslant \tilde{\gamma}(t,h) \text{ on } E_h \setminus \partial_0 E_h.$$

The function $z_h - \tilde{z}_h$ satisfies the relation

$$\frac{d}{dt}\big(z_h - \tilde{z}_h\big)^{(m)}(t) = f_h(t, x^{(m)}, z_h) \star \big[\, \delta(\, z_h - \tilde{z}_h\,)^{(m)}(t)\,\big]^T + F_h[\tilde{z}_h]^{(m)}(t) - \frac{d}{dt}\tilde{z}_h^{(m)}(t) + \tilde{\Gamma}_h^{(m)}(t)$$

where $(t, x^{(m)}) \in \big(E_h \setminus \partial_0 E_h\big) \cap \big([0, \tilde{a}) \times \mathbb{R}^n\big)$. We thus get the differential difference inequality

$$\Big\| \frac{d}{dt}(z_h - \tilde{z}_h)^{(m)}(t) - f_h(t, x^{(m)}, z_h) \star \big[\, \delta(z_h - \tilde{z}_h)^{(m)}\,\big]^T \Big\|_0 \leqslant (1 + \tilde{c}_h)\, \sigma(t, \|z_h - \tilde{z}_h\|_{h.t}) + \tilde{\gamma}(t,h) \text{ on } \big(E_h \setminus \partial_0 E_h\big) \cap \big([0, \tilde{a}), \times \mathbb{R}^n\big)$$

and

$$(z_h - \tilde{z}_h)^{(m)}(t) = 0 \quad \text{on } \big(E_{0.h} \cup \partial_0 E_h\big) \cap \big([-b_0, \tilde{a}) \times \mathbb{R}^n\big).$$

It follows from Lemma 3.1 that

$$\|(z_h - \tilde{z}_h)^{(m)}(t)\| \leqslant \tilde{\omega}_h(t), \quad (t, x^{(m)}) \in E_h \cap \big(\,[0, \tilde{a}) \times \mathbb{R}^n\,\big), \tag{20}$$

where $\tilde{\omega}_h$ is the maximal solution of the Cauchy problem

$$\omega'(t) = (1 + \tilde{c}_h)\, \sigma(\, t, \omega(t)\,) + \tilde{\gamma}(t,h), \quad \omega(0) = 0.$$

The solution of (6), (7) can be extended to the boundary of the domain of the right hand sides of (6). Then we conclude from (20) that z_h is defined on $E_{0.h} \cup E_h$.

Now we prove (18). Let the function $\Gamma_h : E_h \setminus \partial_0 E_h \to \mathbb{R}^k$ be defined by

$$\Gamma_h^{(m)}(t) = \big[f_h(t, x^{(m)}, z_h) - f_h(t, x^{(m)}, u_h)\,\big] \star \big[\, \delta u_h^{(m)}(t)\,\big]^T + g_h(t, x^{(m)}, z_h) - g_h(t, x^{(m)}, u_h).$$

Then the function $z_h - u_h$ satisfies the relation

$$\frac{d}{dt}\big(z_h - u_h\big)^{(m)}(t) = f_h(t, x^{(m)}, z_h) \star \big[\, \delta(\, z_h - u_h\,)^{(m)}(t)\,\big]^T \\ + F_h[u_h]^{(m)}(t) - \frac{d}{dt}u_h^{(m)}(t) + \Gamma_h^{(m)}(t) \text{ on } E_h \setminus \partial_0 E_h.$$

It follows from *Assumption* H $[f_h, g_h]$ and from (15), (17) that the differential difference inequality

$$\Big\| \frac{d}{dt}(z_h - u_h)^{(m)}(t) - f_h(t, x^{(m)}, z_h) \star \big[\, \delta(z_h - u_h)^{(m)}\,\big]^T \Big\|_0 \\ \leqslant (1 + \tilde{c})\, \sigma(t, \|z_h - u_h\|_{h.t}) + \gamma(t, h) \text{ on } E_h \setminus \partial_0 E_h,$$

and the initial boundary estimate

$$\|(z_h - u_h)^{(m)}(t)\|_0 \leqslant \alpha_0(h), \quad (t, x^{(m)}) \in E_{0.h} \cup \partial_0 E_h,$$

are satisfied. The above conditions and Lemma 3.1 imply (18). This proves the theorem.

Now we consider a class of problems (6), (7) where f_h, g_h are superpositions of f, g and some interpolating operators. We will need the following assumptions.

Assumption H $[T_h]$. Suppose that the operator $T_h : \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k) \to C(E_0 \cup E, \mathbb{R}^k)$ satisfies the conditions

1) for z, $\bar{z} \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ we have

$$\big\| T_h[z] - T_h[\bar{z}] \big\|_t \leqslant \| z - \bar{z} \|_{h.t}, \; t \in [0, a],$$

2) for each function $z : E_0 \cup E \to \mathbb{R}^k$ which is of class C^1 there is $C \in \mathbb{R}_+$ such that

$$\| z - T_h[z_h] \|_t \leqslant C\, \|h\|, \quad t \in [0, a],$$

where z_h is the restriction of z to the set $E_{0.h} \cup E_h$.

REMARK 4.2. Condition 1) of *Assumption* H $[T_h]$ states that T_h satisfies the Volterra condition and the Lipschitz condition holds a constant $L = 1$. It follows from condition 2) that the function z is approximated by $T_h[z_h]$ and the error of this approximation is estimated by $C\|h\|$.

We will approximate solutions of (1), (2) by solutions of the system of ordinary functional differential equations

$$\frac{d}{dt} z_i^{(m)}(t) = \sum_{j=1}^{n} f_{ij}(t, x^{(m)}, T_h[z])\, \delta_j z_i^{(m)}(t) + g_i(t, x^{(m)}, T_h[z]), \quad 1 \leqslant i \leqslant k, \tag{21}$$

with the initial boundary condition

$$z^{(m)}(t) = \varphi_h^{(m)}(t) \text{ on } E_{0.h} \cup \partial_0 E_h. \tag{22}$$

We give an example of T_h which satisfies *Assumption* H $[T_h]$. Write

$$S_+ = \big\{\, s = (s_1, \ldots, s_n) : \; s_i \in \{0, 1\} \text{ for } 1 \leqslant i \leqslant n \,\big\}. \tag{23}$$

Suppose that $z \in \mathbf{F}_c(E_{0.h} \cup E_h, \mathbb{R}^k)$ and $(t,x) \in E$. There is $m \in \mathbb{Z}^n$ such that $x^{(m)} \leqslant x \leqslant x^{(m+1)}$ where $m+1 = (m_1+1, \dots, m_n+1)$ and $x^{(m)}, x^{(m+1)} \in [-b, b]$. Set

$$T_h[z](t,x) = \sum_{s \in S_+} z^{(m+s)}(t) \left(\frac{x - x^{(m)}}{h} \right)^s \left(1 - \frac{x - x^{(m)}}{h} \right)^{1-s}$$

where

$$\left(\frac{x - x^{(m)}}{h} \right)^s = \prod_{i=1}^n \left(\frac{x_i - x_i^{(m_i}}{h_i} \right)^{s_i}$$

and

$$\left(1 - \frac{x - x^{(m)}}{h} \right)^{1-s} = \prod_{i=1}^n \left(1 - \frac{x_i - x_i^{(m_i}}{h_i} \right)^{1-s_i}$$

We put $0^0 = 1$ in the above definitions.

It is easily seen that $T_h[z] \in C(E_0 \cup E, \mathbb{R}^k)$ and that *Assumption* H $[T_h]$ is satisfied. See [3], [9] for more details.

Assumption H. $[f,g]$. Suppose that

1) the function $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to the second variable,
2) for each $\gamma \in C([0,a], \mathbb{R}_+)$ and $\eta \in \mathbb{R}_+$, $c \geqslant 1$, the maximal solution of the Cauchy problem

$$\omega'(t) = c\,\sigma(t, \omega(t)) + \gamma(t), \quad \omega(0) = \eta, \tag{24}$$

   is defined on $[0,a]$ and the maximal solution of (24) with $\gamma(t) = 0$ for $t \in [0,a]$ and $c \geqslant 1$, $\eta = 0$ is $\bar{\omega}(t) = 0$ for $t \in [0,a]$,
3) $f \in C(\Omega, M_{k \times n})$, $g \in C(\Omega, \mathbb{R}^k)$ and the estimates

$$\|f(t,x,z) - f(t,x,\bar{z})\| \leqslant \sigma(t, \|z - \bar{z}\|_t),$$
$$\|g(t,x,z) - g(t,x,\bar{z})\|_0 \leqslant \sigma(t, \|z - \bar{z}\|_t)$$

   are satisfied on Ω,
4) estimates (3) are satisfied for $(t,x,z) \in \Omega$.

REMARK 4.3. If *Assumption* H $[f,h]$ is satisfied then initial boundary value problem (1), (2) admits one classical solution at the most. Indeed, if $v, \bar{v} : E_0 \cup E \to \mathbb{R}^k$ are solutions of (1), (2) then the function $\psi(t) = \|v - \bar{v}\|_t$, $t \in [0,a]$, satisfies the differential inequality

$$D_- \psi(t) \leqslant (1 + c_0)\,\sigma(t, \psi(t)), \quad t \in (0,a],$$

where $c_0 = \max\{C_v, C_{\bar{v}}\}$ and

$$C_v = \max\{\|\partial_x v(t,x)\| : (t,x) \in E\}, \quad C_{\bar{v}} = \max\{\|\partial_x \bar{v}(t,x)\| : (t,x) \in E\}.$$

Then the assertion follows from a comparison theorem for differential inequalities. See [2], [9] for more details.

THEOREM 4.4. *Suppose that Assumptions* $[T_h]$ *and* $H\ [f,g]$ *are satisfied and*

1) $u : E_0 \cup E \to \mathbb{R}^k$ *is a solution of* (1), (2) *and* u *is of class* C^1,
2) *there is* $\alpha_0 : H \to \mathbb{R}_+$ *such that*

$$\|\varphi_h(t,x) - \varphi(t,x)\|_0 \leqslant \alpha_0(h) \ \text{ on } \ E_{0.h} \cup E_h$$

and $\lim_{h\to 0} \alpha_0(h) = 0$.

Then there exists a solution $z_h : E_{0.h} \cup E_h \to \mathbb{R}^k$ *of problem* (21), (7) *and there is* $\alpha : H \to \mathbb{R}_+$ *such that*

$$\|u^{(m)}(t) - z_h^{(m)}(t)\|_0 \leqslant \alpha(h) \text{ on } E_h \ \text{ and } \ \lim_{h\to 0} \alpha(h) = 0. \tag{25}$$

Proof. Write

$$u_h = u\big|_{E_{0.h}\cup E_h}, \ \ u_h = (u_{h.1}, \dots, u_{h.k}),$$

and

$$f_h(t,x,w) = f(t,x,T_h[w]), \ \ g_h(t,x,w) = g(t,x,T_h[w])$$

where $(t,x,w) \in \Omega_h$. Let the function $\Lambda_h : E_h \setminus \partial_0 E_h \to \mathbb{R}^k$, $\Lambda_h = (\Lambda_{h.1}, \dots, \Lambda_{h.k})$, be defined by

$$\Lambda_{h.i}^{(m)}(t) = \sum_{j=1}^{n} f_{ij}(t,x^{(m)},u)\,\partial_{x_j} u_i^{(m)}(t) + g_i(t,x^{(m)},u) \\ - \sum_{j=1}^{n} f_{ij}(t,x^{(m)},T_h[u_h])\,\delta_j u_{h.i}^{(m)}(t) - g_i(t,x^{(m)},T_h[u_h]), \ \ 1 \leqslant i \leqslant k.$$

We thus get

$$\frac{d}{dt} u_h^{(m)}(t) = F_h[u_h]^{(m)}(t) + \Lambda_h^{(m)}(t) \ \text{ on } \ E_h \setminus \partial_0 E_h.$$

There are $A, \tilde{c} \in \mathbb{R}_+$ and $\zeta : H \to \mathbb{R}_+$ such that

$$\|f(t,x,u)\| \leqslant A, \ \ (t,x) \in E,$$

$$\|\partial_{x_i} u^{(m)}(t) - \delta_i u_h^{(m)}(t)\|_0 \leqslant \zeta(h), \ \ (t,x^{(m)}) \in E_h \setminus \partial_0 E_h, \ \ 1 \leqslant i \leqslant n,$$

and $\lim_{h\to 0} \zeta(h) = 0$ and

$$\|\partial_{x_i} u(t,x)\|_0 \leqslant \tilde{c}, \ \ 1 \leqslant i \leqslant k, \ \ (t,x) \in E. \tag{26}$$

It follows from *Assumption* H $[f,g]$ that

$$\left\| \Lambda_h^{(m)}(t) \right\|_0 \leqslant \gamma(h), \ \ (t,x^{(m)}) \in E_h \setminus \partial_0 E_h,$$

where

$$\bar{\gamma}(h) = A\zeta(h) + (1+\tilde{c})\,\sigma(t,\|h\|).$$

Thus we see that relation (14) is satisfied with $\gamma(t,h) = \bar{\gamma}(h)$. Let $\bar{\omega}_h : [0,a] \to \mathbb{R}_+$ be the solution of the Cauchy problem

$$\omega'(t) = (1+\tilde{c})\,\sigma(t,\omega(t)) + \bar{\gamma}(h), \ \ \omega(0) = \alpha_0(h). \tag{27}$$

It follows from Theorem 4.1 that

$$\|(u_h - z_h)^{(m)}(t)\|_0 \leqslant \bar{\omega}_h(t), \quad (t, x^{(m)}) \in E_h.$$

Then we get (25) with $\alpha(h) = \bar{\omega}_h(a)$. This proves the theorem.

REMARK 4.5. If all the assumptions of Theorem 4.4 are satisfied with $\sigma(t,p) = Lp$, $(t,p) \in [0,a] \times \mathbb{R}_+$, then there are $C_0,\ C_1 \in \mathbb{R}_+$ such that

$$\|u_h - z_h\|_{h.t} \leqslant C_0 \alpha_0(h) + C_1 \|h\|, \quad t \in [0,a].$$

The above inequality may be proved by using (25) and by solving the comparison problem (27). We have assumed the Lipschitz condition for f and g in this case.

5. Difference methods generated by numerical method of lines

We are interested in discretization in time of problem (21), (22). We define a mesh on $[-b_0, a]$ in the following way. Let h_0 be the step of the mesh and $t^{(r)} = rh_0$, $r \in \mathbb{Z}$, be the nodal points. Let us denote by H' the set of all $h' = (h_0, h)$ such that $h \in H$ and there is $K_0 \in \mathbb{Z}$ with the property $K_0 h_0 = b_0$. For $h' \in H'$ we put

$$\mathbb{R}^{1+n}_{h'} = \{\, (t^{(r)}, x^{(m)}) : \ (r,m) \in \mathbb{Z}^{1+n} \,\}$$

and

$$E_{0.h'} = E_0 \cap \mathbb{R}^{1+n}_{h'}, \quad E_{h'} = E \cap \mathbb{R}^{1+n}_{h'},$$
$$\partial_0 E_{h'} = \partial_0 E \cap \mathbb{R}^{1+n}_{h'}.$$

Let $K \in \mathbb{N}$ be defined by the relations $Kh_0 \leqslant a < (K+1)h_0$. Set

$$\tilde{E}_{h'} = \{\, (t^{(r)}, x^{(m)}) \in E_{h'} \setminus \partial_0 E_{h'} : \quad 0 \leqslant r \leqslant K-1 \,\}.$$

For a function $z : E_{0.h'} \cup E_{h'} \to \mathbb{R}^k$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h'.r} = \max \{\, \|z^{(i,m)}\|_0 : \quad (t^{(i)}, x^{(m)}) \in E_{0.h'} \cup E_{h'},\ i \leqslant r \,\}, \quad 0 \leqslant r \leqslant K.$$

Suppose that *Assumption* H $[f, g]$ is satisfied with

$$\sigma(t,p) = Lp, \quad (t,p) \in [0,a] \times \mathbb{R}_+ \tag{28}$$

where $L \in \mathbb{R}_+$ and that *Assumption* H $[T_h]$ holds. Then f and g satisfy the Lipschitz condition with respect to the functional variable and the right hand sides of system (21) satisfy the Lipschitz condition with respect to the unknown function with a constant $L(h)$ and

$$\lim_{h \to 0} L(h) = +\infty. \tag{29}$$

Suppose that we apply the Euler method to solve numerically problem (21), (22). Then we get a classical difference method for (1), (2). It follows from (29) that we need additional assumptions for h_0 and h to get a stable difference scheme.

If *Assumptions* H $[f,g]$ and H $[T_h]$ are satisfied then the following inequalities

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} \left| f_{ij}(t,x,z) \right| \geqslant 0 \text{ on } \Omega \text{ for } 1 \leqslant i \leqslant k \tag{30}$$

are sufficient for the stability of the classical difference method, (see [3], [9]). The above inequalities can be considered as the Courant - Friedrichs - Levy (CFL) conditions for quasilinear hyperbolic functional differential systems.

Now we formulate a new class of difference problems corresponding to (1), (2). Consider the difference operators $(\delta_1,\ldots,\delta_n) = \delta$ defined by (4), (5) and we write

$$\delta_j w^{(r,m)} = \delta_j w^{(m)}(t^{(r)}), \quad 1 \leqslant j \leqslant n,$$

where $w \in \mathbf{F}(E_{0.h'} \cup E_{h'})$. Set

$$\delta_0 w^{(r,m)} = \frac{1}{h_0}\left(w^{(r+1,m)} - w^{(r,m)} \right).$$

For a function $z : E_{0.h'} \cup E_{h'} \to \mathbb{R}^k$ we define $\delta_0 z = (\delta_0 z_1,\ldots,\delta_0 z_k)$.

Assumption H $[T_{h'}]$. Suppose that the operator $T_{h'} : \mathbf{F}(E_{0.h'} \cup E_{h'}, \mathbb{R}^k) \to C(E_0 \cup E, \mathbb{R}^k)$ satisfies the conditions

1) for $z,\ \bar{z} \in \mathbf{F}(E_{0.h'} \cup E_{h'}, \mathbb{R}^k)$ we have

$$\| T_{h'}[z] - T_{h'}[\bar{z}] \|_{t^{(r)}} \leqslant \|z - \bar{z}\|_{h'.r}, \quad 0 \leqslant r \leqslant K,$$

2) for each function $z : E_0 \cup E \to \mathbb{R}^k$ which is of class C^1 there is $\tilde{C}$ such that

$$\| z - T_{h'}[z_{h'}] \|_{t^{(r)}} \leqslant \tilde{C}\, \|h'\|, \quad 0 \leqslant r \leqslant K,$$

where $z_{h'}$ is the restriction of z to the set $E_{0.h'} \cup E_{h'}$ and $\|h'\| = h_0 + \|h\|$.

REMARK 5.1. Condition 1) of *Assumption* H $[T_{h'}]$ states that the operator $T_{h'}$ satisfies the Lipschitz condition with a constant $L = 1$ and it satisfies the Volterra condition. *Assumption* 2) implies that the function z is approximated by $T_{h'}[z_{h'}]$ and the error of this approximation is estimated by $\tilde{C}\|h'\|$.

Write

$$F_{h'.i}[z]^{(r,m)} = \sum_{j=1}^{n} f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[z])\, \delta_j z_i^{(r+1,m)} + g_i(t^{(r)}, x^{(m)}, T_{h'}[z]), \quad 1 \leqslant i \leqslant k,$$

and

$$F_{h'}[z]^{(r,m)} = \left(F_{h'.1}[z]^{(r,m)}, \ldots, F_{h'.k}[z]^{(r,m)} \right).$$

We will approximate solutions of (1), (2) by solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = F_{h'}[z]^{(r,m)} \tag{31}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_{h'}^{(r,m)} \text{ on } E_{0,h'} \cup \partial_0 E_{h'} \tag{32}$$

where $\varphi_{h'} : E_{0.h'} \cup \partial_0 E_{h'} \to \mathbb{R}^k$ is a given function. Note that condition (3) implies that there exists exactly one solution $z_{h'} : E_{0,h'} \cup E_{h'} \to \mathbb{R}^k$ od (31), (32).

We give an example of $T_{h'}$ which satisfies *Assumption* H $[T_{h'}]$. Suppose that $z \in \mathbf{F}(E_{0.h'} \cup E_{h'}, \mathbb{R}^k)$ and $(t,x) \in E_0 \cup E$. Two cases will be distinguished

(I) Suppose that there is $(r,m) \in \mathbb{Z}^{1+n}$ such that $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E_{0.h'} \cup E_{h'}$ and $t^{(r)} \leqslant t \leqslant t^{(r+1)}$, $x^{(m)} \leqslant x \leqslant x^{(m+1)}$. Set

$$\begin{aligned} T_{h'}[z](t,x) = \Big(1 - \frac{t - t^{(r)}}{h_0}\Big) \sum_{s \in S_+} z^{(r,m+s)} \Big(\frac{x - x^{(m)}}{h}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s} \\ + \frac{t - t^{(r)}}{h_0} \sum_{s \in S_+} z^{(r+1,m+s)} \Big(\frac{x - x^{(m)}}{h}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s} \end{aligned} \tag{33}$$

where S_+ is defined by (23). We adopt the convention that $0^0 = 1$.

(II) If $(t,x) \in E_0 \cup E$ and $Kh_0 < t \leqslant a$ then we put $T_{h'}[z](t,x) = T_{h'}[z](Kh_0, x)$.

Then *Assumption* H $[T_{h'}]$ is satisfied. This can be found in [3], [9].

There are the following differences between the classical Euler method presented in [3], [9] and our scheme.

I. We will omit (CFL) condition (30) in a theorem on the convergence of method (31), (32).

II. Suppose that we calculate the vector $z_{h'}^{(r+1,m)}$ by using (31), (32). Then we apply the vectors

$$\delta z_{h'.j} = \big(\delta_1 z_{h'.j}, \ldots, \delta_n z_{h'.j}\big), \quad 1 \leqslant j \leqslant n,$$

considered at the point $(t^{(r+1)}, x^{(m)})$. In the classical case we use the vectors $\delta z_{h'.j}$, $1 \leqslant j \leqslant k$, at the point $(t^{(r)}, x^{(m)})$.

Numerical results obtained by (31), (32) are better than those obtained by classical methods. We show adequate examples.

THEOREM 5.2. *Suppose that Assumptions H $[f,g]$ and H $[T_{h'}]$ are satisfied and*

1) *$u : E \to \mathbb{R}^k$ is a solution of* (1), (2) *and u is of class C^1,*
2) *there is $\alpha_0 : H' \to \mathbb{R}_+$ such that*

$$\|\varphi_{h'}^{(r,m)} - \varphi^{(r,m)}\|_0 \leqslant \alpha_0(h') \ \text{ on } \ E_{0,h'} \cup \partial_0 E_{h'} \ \text{ and } \ \lim_{h' \to 0} \alpha_0(h') = 0,$$

3) *$z_{h'} : E_{0.h'} \cup E_{h'} \to \mathbb{R}^k$, $z_{h'} = (z_{h'.1} \ldots, z_{h'.k})$, is a solution of* (31), (32). *Then there is $\alpha : H' \to \mathbb{R}_+$ such that*

$$\|u_{h'}^{(r,m)} - z_{h'}^{(r,m)}\|_0 \leqslant \alpha(h') \ \text{ on } \ E_{h'} \ \text{ and } \lim_{h' \to 0} \alpha(h') = 0, \tag{34}$$

where $u_{h'}$ is the restriction of u to the set $E_{0.h'} \cup E_{h'}$.

Proof. For a function $z : E_{0.h'} \cup E_{h'} \to \mathbb{R}^k$ and for a point $(t^{(r)} x^{(m)}) \in E'_{h'}$ we put

$$P^{(r,m)}[z] = (t^{(r)}, x^{(m)}, T_{h'}[z]).$$

Let us denote by $\Gamma_{h'}$, $\Lambda_{h'} : \tilde{E}_{h'} \to \mathbb{R}^k$ where

$$\Gamma_{h'} = \big(\Gamma_{h'.1}, \ldots, \Gamma_{h'.k}\big), \quad \Lambda_{h'} = \big(\Lambda_{h'.1}, \ldots, \Lambda_{h'.k}\big),$$

the functions defined by

$$\delta_0 u_{h'}^{(r,m)} = F_{h'}[u_{h'}]^{(r,m)} - \Gamma_{h'}^{(r,m)}$$

and

$$\Lambda_{h'.i} = \sum_{j=1}^{n} \left[f_{ij}(P^{(r,m)}[z_{h'}]) - f_{ij}(P^{(r,m)}[u_{h'}]) \right] \delta_j u_{h'.i}^{(r+1,m)}$$

$$+ g_i(P^{(r,m)}, [z_{h'}]) - g_i(P^{(r,m)}[u_{h'}]), \quad 1 \leqslant i \leqslant k.$$

Then the function $v_{h'} = z_{h'} - u_{h'}$, $v_{h'} = (v_{h'.1}, \dots, v_{h'.k})$, satisfies the difference equations

$$\delta_0 v_{h'.i}^{(r,m)} = \sum_{j=1}^{n} f_{ij}(P^{(r,m)}[z_{h'}])\, \delta_j v_{h'.i}^{(r+1,m)} + \Gamma_{h.i}^{(r,m)} + \Lambda_{h'.i}^{(r,m)}, \; 1 \leqslant i \leqslant k. \tag{35}$$

Write

$$I_+[m] = \{\, i : \; 1 \leqslant i \leqslant n, \; m_i > 0 \,\} \; \text{ and } \; I_-[m] = \{1, \dots, n\} \setminus I_+[m].$$

According to (4), (5) and (35) we have

$$v_{h'.i}^{(r+1,m)} \left[1 + h_0 \sum_{j \in I_+[m]} \frac{1}{h_j} f_{ij}(P^{(r,m)}[z_{h'}])) - h_0 \sum_{j \in I_-[m]} \frac{1}{h_j} f_{ij}(P^{(r,m)}[z_{h'}]) \right]$$

$$= v_{h'.i}^{(r,m)} + \sum_{j \in I_+[m]} \frac{1}{h_j} f_{ij}(P^{(r,m)}[z_{h'}]))\, v_{h'.i}^{(r+1,m-e_j)}$$

$$-h_0 \sum_{j \in I_-[m]} \frac{1}{h_j} f_{ij}(P^{(r,m)}[z_{h'}])\, v_{h'.i}^{(r+1,m-e_j)} + h_0 \left[\Lambda_{h'.i}^{(r,m)} + \Gamma_{h'.i}^{(r,m)} \right], \; 1 \leqslant i \leqslant k. \tag{36}$$

It follows that there is $\gamma : H' \to \mathbb{R}_+$ such that

$$\left\| \, \Gamma_{h'}^{(r,m)} \, \right\|_0 \leqslant \gamma(h') \quad \text{on } \; \tilde{E}_{h'} \quad \text{and} \quad \lim_{h \to 0} \gamma(h') = 0. \tag{37}$$

Let $\tilde{c} \in \mathbb{R}_+$ be such a constant that relations (26) are satisfied. Write

$$\varepsilon_{h'}^{(r)} = \max\{ \, \|v_{h'}^{(i,m)}\|_0 : \; (t^{(i)}, x^{(m)}) \in E_{0.h'} \cup E_{h'}, \; i \leqslant r \,\}, \; 0 \leqslant r \leqslant K.$$

It follows from *Assumption* H $[f, g]$ that

$$\left\| \, \Lambda_{h'}^{(r,m)} \, \right\|_0 \leqslant (1 + \tilde{c})\, \sigma(\, t^{(r)}, \varepsilon_{h'}^{(r)} \,) \quad \text{on } \; \tilde{E}_{h'}. \tag{38}$$

We conclude from (36) - (38) that the function $\varepsilon_{h'}$ satisfies the difference inequality

$$\varepsilon_{h'}^{(r+1)} \leqslant \varepsilon_{h'}^{(r)} + h_0(1 + \tilde{c})\, \sigma(\, t^{(r)}, \varepsilon_{h'}^{(r)} \,) + h_0 \gamma(h'), \quad 0 \leqslant r \leqslant K - 1,$$

and $\varepsilon_{h'}^{(0)} \leqslant \alpha_0(h')$.

Let us denote by $\omega_{h'} : [0, a] \to \mathbb{R}_+$ the solution of the Cauchy problem

$$\omega'(t) = (1 + \tilde{c})\, \sigma(t, \omega(t)) + \gamma(h'), \quad \omega(0) = \alpha_0(h'). \tag{39}$$

Then

$$\lim_{h'\to 0} \omega_{h'}(t) = 0 \text{ uniformly on } [0,a].$$

It is easy to see that $\omega_{h'}$ satisfies the recurrent inequality

$$\omega_{h'}^{(r+1)} \geqslant \omega_{h'}^{(r)} + h_0(1+\tilde{c})\,\sigma(\,t^{(r)}, \omega_{h'}^{(r)}\,) + h_0\gamma(h'), \quad 0 \leqslant r \leqslant K-1.$$

This gives $\varepsilon_{h'}^{(r)} \leqslant \omega_{h'}^{(r)}$ for $0 \leqslant r \leqslant K$ and condition (34) is satisfied with $\alpha(h') = \omega_{h'}(a)$. This proves the theorem.

REMARK 5.3. If all the assumptions of Theorem 5.2 are satisfied with σ given by (28) then there are $C_0\, C_1 \in \mathbb{R}_+$ such that we have the following error estimate

$$\|u_{h,}^{(r,m)} - z_{h'}^{(r,m)}\|_0 \leqslant C_0\,\alpha_0(h') + C_1\,\|h'\| \;\; on \;\; E_{h'}.$$

The above inequality is obtained from (34) and by solving of problem (39).

6. Numerical experiments

For $n=1$ we put

$$E = [0,0.5]\times[-1,1], E_0 = \{0\}\times[-1,1],\; \partial_0 E = [0,0.5]\times\big([-1,1]\setminus(-1,1)\big). \quad (40)$$

Consider the differential integral equation

$$\partial_t z(t,x) = x\Big[1 - t\sin\int_{-x}^{x} z(t,s)ds\Big]\,\partial_x z(t,x) + 12\int_{0.5(x-1)}^{0.5(x+1)} z(t,s)ds + f(t,x) \quad (41)$$

with the initial boundary condition

$$z(t,x) = 0 \;\text{ on }\; E_0 \cup \partial_0 E \quad (42)$$

where

$$f(t,x) = x^2 - 1 + 2x^2t^2\,\sin\Big[2t\big(\frac{1}{3}x^3 - x\big)\Big] + t(11 - 5x^2).$$

The solution of the above problem is known, it is $u(t,x) = t(x^2-1)$.

Write $x^{(m)} = mh$, $-N \leqslant m \leqslant N$, where $Nh = 1$. The discretization of (41), (42) with respect to x leads to the system

$$\begin{aligned} \frac{d}{dt}z^{(m)}(t) = x^{(m)}\Big[1 - t\sin\int_{-x^{(m)}}^{x^{(m)}} T_h[z](t,s)ds\Big]\,\delta z^{(m)}(t) \\ +12\int_{0.5(x^{(m)}-1)}^{0.5(x^{(m)}+1)} T_h[z](t,s)ds + f^{(m)}(t),\; -N < m < N, \end{aligned} \quad (43)$$

where

$$z^{(-N)}(t) = z^{(N)}(t) = 0 \text{ for } t \in [0,\,0.5] \;\text{ and }\; z^{(m)}(0) = 0 \;\text{ for }\; -N < m < N \quad (44)$$

and T_h is the interpolating operator defined by (33). Difference operator δ is defined according to (4), (5). Write $t^{(r)} = rh_0$, $0 \leqslant r \leqslant K$ where $Kh_0 = 0.5$.

We apply the Runge-Kutta method of order four to get the numerical solution of (43), (44). Let us denote by v_{RK} the solution which is obtained in this way.

Let v_{IM} denote the solution which is obtained by solving the implicit difference method for (41), (42). Write

$$\varepsilon_{RK}^{(r)} = \frac{1}{2N-1} \sum_{m=-(N-1)}^{N-1} \left| u^{(r,m)} - v_{RK}^{(r,m)} \right| \tag{45}$$

and

$$\varepsilon_{IM}^{(r)} = \frac{1}{2N-1} \sum_{m=-(N-1)}^{N-1} \left| u^{(r,m)} - v_{IM}^{(r,m)} \right| \tag{46}$$

where $0 \leqslant r \leqslant K$.

The numbers $\varepsilon_{RK}^{(r)}$ and $\varepsilon_{IM}^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$. We have solved problems (43), (44) and the implicit Euler method with the same steps of the mesh.

In the table we give experimental values for the functions ε_{RK} and ε_{IM} and we write “ $\times$ ” if $\varepsilon_{RK}^{(r)} > 10$.

1. Table of errors $(\varepsilon_{IM};\varepsilon_{RK})$

	$(h_0,h)=(10^{-3},2\cdot 10^{-3})$	$(h_0,h)=(5\cdot 10^{-4},10^{-3})$
$t=0.36$	$(0.0057;0.0329)$	$(0.0028;0.0167)$
$t=0.37$	$(0.0064;0.0371)$	$(0.0032;0.0188)$
$t=0.38$	$(0.0072;0.0419)$	$(0.0036;0.0213)$
$t=0.39$	$(0.0082;0.0473)$	$(0.0041;0.0240)$
$t=0.40$	$(0.0092;0.0533)$	$(0.0047;0.0276)$

2. Table of errors $(\varepsilon_{IM};\varepsilon_{RK})$

	$(h_0,h)=(5\cdot 10^{-3},\ 10^{-3})$	$(h_0,h)=(10^{-3},\ 5\cdot 10^{-4})$
$t=0.350$	$(0.0086;\times)$	$(0.0024;\times)$
$t=0.375$	$(0.0114;\times)$	$(0.0032;\times)$
$t=0.400$	$(0.0115;\times)$	$(0.0043;\times)$
$t=0.425$	$(0.0200;\times)$	$(0.0058;\times)$
$t=0.450$	$(0.0265;\times)$	$(0.0078;\times)$

If $2h_0 \leqslant h$ then the condition (CFL) is satisfied and the classical difference method for (41), (42) is stable.

In Table 1 we give the experimental values of the errors for such steps (h_0,h) that the stability condition is satisfied. We have $\varepsilon_{IM}^{(r)} < \varepsilon_{RK}^{(r)}$ for all values of r.

In Table 2 we present experimental values of the errors in the case when the stability condition is not satisfied. Then the Runge-Kutta method is not applicable and the implicit difference method is convergent.

Now we consider the differential equation with deviated variables. Suppose that E, E_0, $\partial_0 E$ are the sets defined by (40). We take into considerations the differential equations

$$\begin{aligned}\partial_t z(t,x) = \sin x\big[1 - x\cos\big(z(t,0.5(x+1)) - z(t,0.5(x-1))\big)\big]\partial_x z(t,x)\\ + 2\big[\,z(t,0.5x) + z(t,-0.5x)\,\big] + f(t,x)\end{aligned} \tag{47}$$

with the initial boundary condition (42) where

$$f(t,x) = x^2 - 1 - t(x^2 - 4) - 2tx\big[1 - x\,\cos(tx)\big]\,\sin x.$$

The function $u(t,x) = t(x^2 - 1)$ is the solution of the above problem.

Consider the numerical method of lines

$$\begin{aligned}\frac{d}{dt}z^{(m)}(t) = \sin x^{(m)}\big[1 - x^{(m)}\cos\big(T_h[z](t,0.5(x^{(m)}+1)) - T_h[z](t,x^{(m)}-1)\big)\big)\delta z^{(m)}(t)\\ +2\big[T_h[a](t,0.5x^{(m)}) + t_h[z](t,-0.5^{(m)}\big] + f^{(m)}(t),\ -N < m < N,\end{aligned} \tag{48}$$

with initial boundary condition (44) and δ is defined according to (4), (5). Let us denote by v_{RK} the numerical solution the above problem which is obtained by using the Runge Kutta method of order four.

We have solved also problem (47), (42) by using the implicit difference method presented in Section 4. Let v_{IM} be the solution. Suppose that the function ε_{RK} and ε_{IM} are defined by (45), (46). The experimental values of ε_{RK} and ε_{IM} are listed in the tables. We write " $\times$ " if $\varepsilon_{RK}^{(r)} > 10$.

3. Table of errors $(\varepsilon_{IM}; \varepsilon_{RK})$

	$(h_0,h) = (10^{-3},\ 2\cdot 10^{-3})$	$(h_0,h) = (2\cdot 10^{-4},\ 4\cdot 10^{-4})$
$t = 0.300$	$(3.187\cdot 10^{-5}\ ;\ 0.0010)$	$(6.379\cdot 10^{-6}\ ;\ 0.0002)$
$t = 0.325$	$(3.537\cdot 10^{-5}\ ;\ 0.0012)$	$(7.080\cdot 10^{-6}\ ;\ 0.0002)$
$t = 0.350$	$(3.902\cdot 10^{-5}\ ;\ 0.0013)$	$(7.810\cdot 10^{-6}\ ;\ 0.0002)$
$t = 0.375$	$(4.281\cdot 10^{-5}\ ;\ 0.0015)$	$(8.570\cdot 10^{-6}\ ;\ 0.0003)$
$t = 0.400$	$(4.676\cdot 10^{-5}\ ;\ 0.0017)$	$(9.359\cdot 10^{-6}\ ;\ 0.0003)$

4. Table of errors $(\varepsilon_{IM}; \varepsilon_{RK})$

	$(h_0,h) = (10^{-3},\ 5\cdot 10^{-4})$	$(h_0,h) = (2.5\cdot 10^{-4},\ 2\cdot 10^{-4})$
$t = 0.350$	$(0.00023; \times)$	$(6.194\cdot 10^{-5}; \times)$
$t = 0.375$	$(0.00026; \times)$	$(6.800\cdot 10^{-5}; \times)$
$t = 0.400$	$(0.00028; \times)$	$(7.446\cdot 10^{-5}; \times)$
$t = 0.425$	$(0.00030; \times)$	$(8.136\cdot 10^{-5}; \times)$
$t = 0.450$	$(0.00033; \times)$	$(8.875\cdot 10^{-5}; \times)$

The classical difference method for (47), (42) is stable for $2h_0 \leqslant h$. In Table 3 we give the experimental values of the errors for such steps (h_0, h) that the stability condition is satisfied. We have $\varepsilon_{IM}^{(r)} < \varepsilon_{RK}^{(r)}$ for all values of r.

In Table 4 we present experimental values of the errors in the case when the condition (CFL) is not satisfied. Then the Runge-Kutta method is not applicable and the implicit difference method is convergent.

Note that we have a little better results for the differential equation with deviated variables then for the differential integral problem. This is due to the fact that we use interpolating values $T_{h'}[z](t,\cdot)$ on the intervals $[x^{(-m)}, x^{(m)}]$ and $[0.5(x^{(m)} - 1), 0,5(x^{(m)}+1)]$ in the first example we calculate the function $T_{h'}[z](t,\cdot)$ at the points $x^{(-m)}$, $x^{(m)}$ and $0.5(x^{(m)}-1)$, $0,5(x^{(m)}+1)$ in the second example.

Implicit difference methods, described in Section 5, have the potential for applications in the numerical solving of differential integral equations or equations with deviated variables.

REFERENCES

[1] P. BASSANINI, *Iterative methods for quasilinear hyperbolic systems*, Boll. Un. Mat. Ital., **6**, 1 - B (1982), 225–250.

[2] P. BRANDI, Z. KAMONT AND A. SALVADORI, *Differential and difference inequalities related to mixed problems for first order partial differential functional equations*, Atti Sem. Mat. Fis. Univ. Modena, **39**, (1991), 255–276.

[3] P. BRANDI, Z. KAMONT AND A. SALVADORI, *Approximate solutions of mixed problems for first order partial differential functional equations*, Atti Sem. Mat. Fis. Univ. Modena, **39**, (1991), 277–302.

[4] L. CESARI, *Un problema ai limiti per sistemi di equazion iperbolici quasi lineari nell forma canonica di Schauder*, Rend. Accad. Naz. Lincei, **57**, (1975), 303–307.

[5] S. CINQUINI, *Sopra i sistemi iperbolici equazioni a derivate parziali (nonlineari) in piú variabili indipendenti*, Ann. Mat. Pura ed Appl., **120**, (1979), 201–214.

[6] M. CINQUINI CIBRARIO, *Sopra una classe di sistemi di equazioni nonlineari a derivate parziali in piú variabili indipendenti*, Ann. Mat. Pura ed Appl., **140**, (1985), 223–253.

[7] I. GYÖRY, *The method of lines for the solutions of some nonlinear partial differential equations*, Comput. Math. Appl., **15**, (1988), 635–648.

[8] D. JARUSZEWSKA-WALCZAK, Z. KAMONT, *Existence of solutions of first order partial differential functional equations via the method of lines*, Serdica Bulg. Math. Publ., **16**, (1990), 104–114.

[9] Z. KAMONT, *Hyperbolic Functional Differential Inequalities and Applications*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1999.

[10] Z. KAMONT, S. ZACHAREK, *The lines method for parabolic differential-functional equations with initial boundary conditions of the Dirichlet type*, Atti Sem. Mat. Fis. Univ. Modena, **35**, (1987), 249–262.

[11] J. P. KAUTHEN, *The method of lines for parabolic partial integrodifferential equations*, Journ. Integral. Equat., **4**, (1992), 69–81.

[12] S. KOZIEŁ, *Infinite systems of quasilinear differential difference inequalities and applications*, Appl. Anal., **82**, (2003), 311–327.

[13] V. LAKSHMIKANTHAM, S. LEELA, *Differential and Integral Inequalities*, Acad. Press, New York, 1969.

[14] H. LESZCZYŃSKI, *On the method of lines for a nonlinear heat equation with functional dependence*, Ann. Polon. Math., **69**, (1998), 61–74.

[15] S. ŁOJASIEWICZ, *Sur probléme de Cauchy pour les systémes d' équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépedanates*, Ann. Polon. Math.**3**, (1956), 87–117.

[16] W. E. SCHIESSER, *The Numerical Method of Lines: Integration of Partial Differential Equations*, Acad. Press, San Diego, 1991.

[17] A. VANDE WOUWER, PH. SAUCEZ AND W. E. SCHIESSER, *Adaptive Method of Lines, Capman & Hall / CRC*, Boca Raton, 2001.

[18] A. VOIGT, *The method of lines for nonlinear parabolic differential equations with mixed derivatives*, Numer. Math., **32**, (1979), 197–207.

[19] W. WALTER, *Existenzsätze in Grossen für nichtlineare parabolische Differentialgleichungen aufrung der Linienmethode*, Math. Zeit., **107**, (1968), 173–188.

[20] W. WALTER, *Differential and Integral Inequalities*, Springer-Varlag, Berlin, 1970.

(Received January 4, 2005)

Z. Kamont
Institute of Mathematics
University of Gdańsk
Wit Stwosz Street 57
80-952 Gdańsk
Poland
e-mail: zkamont@math.univ.gda.pl

K. Kropielnicka
Gdańsk University of Technology
Department of Mathematics
Narutowicz Street 11/12
80 - 952 Gdańsk
Poland