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THE MODIFIED HYERS–ULAM–RASSIAS STABILITY OF A CUBIC TYPE FUNCTIONAL EQUATION

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(*communicated by T. M. Rassias*)

Abstract. In this paper, we obtain the solution of the following new cubic type functional equation and investigate the modified Hyers-Ulam-Rassias stability of this equation by using the fixed point alternative:

$$f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)+7f(x)+7f(-x)$$
$$=2[f(x+y)+2f(x+z)+2f(x-z)+2f(y+z)+2f(y-z)].$$

1. Introduction

In 1940, S. M. Ulam [22] proposed the following question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h: G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H: G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In next year, D. H. Hyers [8] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [16]. Since then, the stability problems of several functional equation have been extensively investigated by a number of authors (for instances, [1, 2, 3, 4, 5, 7, 9, 12, 17, 18, 19, 20, 21]).

Particularly, one of the important functional equations studied is the following functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x) = ax^2$ is a solution of this functional equation, and so one usually is said the above functional equation to be quadratic [1, 6, 11, 13].

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [21] for functions between a normed space and a Banach space. Afterwards, her result was extended by P. W. Cholewa [5] and S. Czerwik [6].

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The cubic function $f(x) = ax^3$ satisfies the functional equation

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

Hence, throughout this paper, we promise that the equation (1.1) is called a cubic functional equation and every solution of the equation (1.1) is said to be a cubic function.

The functional equation (1.1) was solved by K.-W. Jun and H.-M. Kim [10]. In fact, they proved that a function $f : X \to Y$ between real vector spaces is a solution of the functional equation (1.1) if and only if there exists a function $G : X \times X \times X \to Y$ such that $f(x) = G(x,x,x)$ for all $x \in X$, and G is symmetric for each fixed one variable and additive for fixed two variables. The function G is given by

$$G(x,y,z) = \frac{1}{24}\left[f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)\right]$$

for all $x,y,z \in X$. Moreover, they investigated the Hyers-Ulam-Rassias stability for the functional equation (1.1).

In this paper, we deal with the following new cubic type functional equation:

$$\begin{aligned} &f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)+7f(x)+7f(-x)\\ &\quad = 2[f(x+y)+2f(x+z)+2f(x-z)+2f(y+z)+2f(y-z)]. \end{aligned} \tag{1.2}$$

It is easy to see that the function $f(x) = ax^3 + b$ is a solution of the functional equation (1.2).

The main purpose of this paper is to solve the functional equation (1.2) and offer the modified Hyers-Ulam-Rassias stability result for this equation by using the fixed point alternative [14] as in [15].

2. The Required Results

We use the following theorem to demonstrate the main result for stability.

THEOREM 2.1. (The alternative of fixed point) [14]. *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \text{ for all } n \geqslant 0,$$

or

There exists a natural number n_0 such that

- *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geqslant n_0$;*
- *The sequence $(T^n x)$ is convergent to a fixed point y^* of T;*
- *y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*
- *$d(y, y^*) \leqslant \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

First, we will find out the solution of the functional equation (1.2).

LEMMA 2.2. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation (1.2) for all $x,y,z \in X$ if and only if C is cubic, where $C : X \to Y$ is a function defined by $C(x) = f(x) - f(0)$ for all $x \in X$.*

Proof. (*Necessity.*) Note that, by the assumption, we have

$$\begin{aligned} &C(x+y+2z)+C(x+y-2z)+C(2x)+C(2y)+7C(x)+7C(-x)\\ &\quad =2[C(x+y)+2C(x+z)+2C(x-z)+2C(y+z)+2C(y-z)] \end{aligned} \tag{2.1}$$

for all $x,y,z \in X$. In particular, it is clear that $C(0)=0$. Putting $x=y=0$ in (2.1), we arrive at

$$C(2z)+C(-2z)=8C(z)+8C(-z). \tag{2.2}$$

Letting $x=z$ and $y=-z$ in (2.1) gives the equation

$$7[C(z)+C(-z)]=2[C(2z)+C(-2z)]$$

and by comparing with (2.2), it follows that

$$C(z)+C(-z)=0. \tag{2.3}$$

Therefore (2.1) now becomes

$$\begin{aligned} &C(x+y+2z)+C(x+y-2z)+C(2x)+C(2y)\\ &\quad =2[C(x+y)+2C(x+z)+2C(x-z)+2C(y+z)+2C(y-z)]. \end{aligned} \tag{2.4}$$

By replacing y by x in (2.4), then we see that

$$C(2x+2z)+C(2x-2z)=8C(x+z)+8C(x-z) \tag{2.5}$$

for all $x,z \in X$.

Setting $z=0$ in (2.5) leads to the identity $C(2x)=8C(x)$. Finally, we let $x=0$ and $y:=2y$ in (2.4) and then use (2.3) and the identity $C(2x)=8C(x)$ to yield

$$C(2y+z)+C(2y-z)=2C(y+z)+2C(y-z)+12C(y) \tag{2.6}$$

for all $y,z \in X$, which implies that C is cubic.

(*Sufficiency.*) Suppose that C is cubic, i.e.,

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x) \tag{2.7}$$

for all $x,y \in X$. Obviously, $C(0)=0$. Setting $x=0$ in (2.7) yields $C(-y)=-C(y)$ and by letting $y=0$ and $y=x$ in (2.7), we obtain that $C(2x)=8C(x)$ and $C(3x)=27C(x)$, respectively.

Replacing y by $2z$ in (2.7), we have

$$C(x+2z)+C(x-2z)=4C(x+z)+4C(x-z)-6C(x). \tag{2.8}$$

On the other hand, by utilizing the same computation in the proof of [10, Theorem 2.1], we obtain

$$\begin{aligned} &C(x+y+z)+C(x+y-z)+2C(x)+2C(y)\\ &\quad =2C(x+y)+C(x+z)+C(x-z)+C(y+z)+C(y-z) \end{aligned} \tag{2.9}$$

for all $x,y,z \in X$.

We substitute $z := 2z$ in (2.9) and then use (2.8) to find that

$$\begin{aligned} &C(x+y+2z)+C(x+y-2z)+8C(x)+8C(y) \\ &\quad = 2[C(x+y)+2C(x+z)+2C(x-z)+2C(y+z)+2C(y-z)]. \end{aligned} \tag{2.10}$$

By considering $C(2x) = 8C(x)$ and $C(x)+C(-x) = 0$, we note that

$$8C(x)+8C(y) = C(2x)+C(2y)+7C(x)+7C(-x).$$

Hence (2.10) can be written as

$$\begin{aligned} &C(x+y+2z)+C(x+y-2z)+C(2x)+C(2y)+7C(x)+7C(-x) \\ &\qquad = 2[C(x+y)+2C(x+z)+2C(x-z)+2C(y+z)+2C(y-z)] \end{aligned}$$

for all $x,y,z \in X$. From this, we know that a function f satisfies the functional equation (1.2) for all $x,y,z \in X$, which the proof is now complete. □

REMARK 2.3 Lemma 2.2 states that the functional equation (1.2) has a solution of the form $C(x)+B$, where C is cubic and B is a constant.

3. The modified Hyers-Ulam-Rassias stability of eq. (1.2).

In this section, let X be a real vector space and Y be a real Banach space. As a matter of convenience, for given a mapping $f : X \to Y$, we use the following abbreviation:

$$\begin{aligned} Df(x,y,z) :=\ & f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)+7f(x)+7f(-x) \\ & -2[f(x+y)+2f(x+z)+2f(x-z)+2f(y+z)+2f(y-z)] \end{aligned}$$

for all $x,y,z \in X$.

Let $\varphi : X \times X \times X \to [0,\infty)$ be a function satisfying

$$\lim_{n\to\infty} \frac{\varphi(\lambda_i^n x, \lambda_i^n y, \lambda_i^n z)}{\lambda_i^{3n}} = 0 \tag{3.1}$$

for all $x,y,z \in X$, where

$$\begin{cases} \lambda_i = 2, & \text{if} \quad i = 0 \\ \lambda_i = \dfrac{1}{2}, & \text{if} \quad i = 1. \end{cases}$$

Now, by the use of fixed point alternative, we obtain the main result as follow.

THEOREM 3.1. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z)\| \leqslant \varphi(x,y,z) \tag{3.2}$$

for all $x,y,z \in X$. If there exists $L < 1$ such that the function

$$x \mapsto \psi(x) = \varphi\big(0, \frac{x}{2}, 0\big)$$

has the property

$$\psi(x) \leqslant L \cdot \lambda_i^3 \cdot \psi\Big(\frac{x}{\lambda_i}\Big) \tag{3.3}$$

for all $x \in X$*, then there exists a unique cubic function* $C : X \to Y$ *satisfying the inequality*

$$\|f(x) - C(x)\| \leqslant \frac{L^{1-i}}{1-L}\psi(x) + \|f(0)\| \tag{3.4}$$

holds for all $x \in X$*.*

Proof. Consider the set

$$\Omega := \{g : g : X \to Y,\ g(0) = 0\}$$

and introduce the generalized metric on Ω:

$$d(g,h) = d_\psi(g,h) = \inf\{K \in (0,\infty) : \|g(x) - h(x)\| \leqslant K\psi(x),\ x \in X\}.$$

It is easy to see that (Ω, d) is complete.

Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{\lambda_i^3}\, g(\lambda_i x)$$

for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) < K &\Longrightarrow \|g(x) - h(x)\| \leqslant K\psi(x),\ x \in X \\ &\Longrightarrow \Big\|\frac{1}{\lambda_i^3}\, g(\lambda_i x) - \frac{1}{\lambda_i^3}\, h(\lambda_i x)\Big\| \leqslant \frac{1}{\lambda_i^3}\, K\psi(\lambda_i x),\ x \in X \\ &\Longrightarrow \Big\|\frac{1}{\lambda_i^3}\, g(\lambda_i x) - \frac{1}{\lambda_i^3}\, h(\lambda_i x)\Big\| \leqslant LK\psi(x),\ x \in X \\ &\Longrightarrow d(Tg, Th) \leqslant LK. \end{aligned}$$

Hence, we see that

$$d(Tg, Th) \leqslant Ld(g,h)$$

for all $g, h \in \Omega$, i.e., T is a strictly self-mapping of Ω with the Lipschitz constant L.

Here we define a function $F : X \to Y$ by

$$F(x) = f(x) - f(0)$$

for all $x \in X$. Then we have $F(0) = 0$.

If we put $x = 0 = z$ in (3.2) and use (3.3), then

$$\|F(2y) - 8F(y)\| = \|[f(2y) - f(0)] - 8[f(y) - f(0)]\| \leqslant \varphi(0, y, 0), \tag{3.5}$$

which is reduced to

$$\Big\|F(y) - \frac{1}{2^3}F(2y)\Big\| \leqslant \frac{1}{2^3}\psi(2y) \leqslant L\psi(y)$$

for all $y \in X$, i.e., $d(F, TF) \leqslant L < \infty$.

If we substitute $y := \frac{y}{2}$ in (3.5) and use (3.3), then

$$\left\|F(y) - 2^3 F\left(\frac{y}{2}\right)\right\| \leqslant \psi(y)$$

for all $y \in X$, i.e., $d(F, TF) \leqslant 1 < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point C of T in Ω such that

$$C(x) = \lim_{n\to\infty} \frac{F(\lambda_i^n x)}{\lambda_i^{3n}} \tag{3.6}$$

for all $x \in X$, since $\lim_{n\to\infty} d(T^n F, C) = 0$.

To show that the function $C : X \to Y$ is cubic, let us replace x, y and z by $\lambda_i^n x$, $\lambda_i^n y$ and $\lambda_i^n z$ in (3.2), respectively and divide by λ_i^{3n}. Then it follows from (3.1) and (3.6) that

$$\|DC(x,y,z)\| = \lim_{n\to\infty} \frac{\|DF(\lambda_i^n x, \lambda_i^n y, \lambda_i^n z)\|}{\lambda_i^{3n}} \leqslant \lim_{n\to\infty} \frac{\varphi(\lambda_i^n x, \lambda_i^n y, \lambda_i^n z)}{\lambda_i^{3n}} = 0$$

for all $x, y, z \in X$, i.e., C satisfies the functional equation (1.2). Therefore Lemma 2.2 guarantees that C is cubic, since $C(0) = 0$.

According to the fixed point alternative, since C is the *unique* fixed point of T in the set $\Delta = \{g \in \Omega : d(F, g) < \infty\}$, C is the unique function such that

$$\|F(x) - C(x)\| \leqslant K\psi(x)$$

for all $x \in X$ and some $K > 0$. Again, using the fixed point alternative, we have

$$d(F, C) \leqslant \frac{1}{1-L} d(F, TF),$$

and so we obtain the inequality

$$d(F, C) \leqslant \frac{L^{1-i}}{1-L},$$

which yields the inequality (3.4). This completes the proof of the theorem. □

From Theorem 3.1, we obtain the following corollary concerning the Hyers-Ulam-Rassias stability [16] of the functional equation (1.2).

COROLLARY 3.2. *Let X and Y be a normed space and a Banach space, respectively. Let $p \geqslant 0$ be given with $p \neq 3$. Assume that $\delta \geqslant 0$ and $\varepsilon \geqslant 0$ are fixed. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y,z)\| \leqslant \delta + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \tag{3.7}$$

for all $x, y, z \in X$. Moreover, assume that $\delta = 0$ in (3.7) for the case $p > 3$. Then there exists a unique cubic function $C : X \to Y$ satisfying the inequality

$$\|f(x) - C(x)\| \leqslant \frac{\delta}{2^{3-p} - 1} + \frac{\varepsilon}{8 - 2^p\|x\|^p + \|f(0)\|} \tag{3.8}$$

holds for all $x \in X$, *where* $p < 3$,
or
the inequality

$$\|f(x) - C(x)\| \leqslant \frac{\varepsilon}{2^p - 8}\|x\|^p + \|f(0)\| \tag{3.9}$$

holds for all $x \in X$, *where* $p > 3$.

Proof. Let

$$\varphi(x,y,z) := \delta + \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then it follows that

$$\frac{\varphi(\lambda_i^n x, \lambda_i^n y, \lambda_i^n z)}{\lambda_i^{3n}} = \frac{\delta}{\lambda_i^{3n}} + (\lambda_i^n)^{p-3}\varepsilon(\|x\|^p + \|y\|^p + \|z\|^p) \longrightarrow 0$$

as $n \to \infty$, where

$$\begin{cases} p < 3, & \text{if} \quad i = 0 \\ p > 3, & \text{if} \quad i = 1, \end{cases}$$

i.e., (3.1) is true.

Since the inequality

$$\frac{1}{\lambda_i^3}\psi(\lambda_i x) = \frac{\delta}{\lambda_i^3} + \frac{\lambda_i^{p-3}}{2^p}\varepsilon\|x\|^p \leqslant \lambda_i^{p-3}\psi(x)$$

holds for all $x \in X$, where

$$\begin{cases} p < 3, & \text{if} \quad i = 0 \\ p > 3, & \text{if} \quad i = 1, \end{cases}$$

we see that the inequality (3.3) holds with either $L = 2^{p-3}$ or $L = \frac{1}{2^{p-3}}$. Now the inequality (3.4) yields the inequalities (3.8) and (3.9), which complete the proof of the corollary.

The following corollary is the Hyers-Ulam stability [8] of the functional equation (1.2).

COROLLARY 3.3. *Let* X *and* Y *be a normed space and a Banach space, respectively. Assume that* $\theta \geqslant 0$ *is fixed. Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x,y,z)\| \leqslant \theta \tag{3.10}$$

for all $x, y, z \in X$. *Then there exists a unique cubic function* $C : X \to Y$ *satisfying the inequality*

$$\|f(x) - C(x)\| \leqslant \frac{1}{21}\theta + \|f(0)\| \tag{3.11}$$

holds for all $x \in X$

Proof. Considering $\delta := 0$, $p := 0$ and $\varepsilon := \frac{\theta}{3}$ in the corollary 3.2, we arrive at the conclusion of the corollary. $\square$

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