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ON THE STABILITY OF AN ALTERNATIVE FUNCTIONAL EQUATION

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Abstract. We deal with the stability of the alternative functional equation

$$f(x+y)+f(x)+f(y)\neq 0 \implies f(x+y)=f(x)+f(y).$$

1. Introduction

The question concerning stability of functional equations was originated by S. M. Ulam (cf. [14]) and D. H. Hyers (cf. [8]). Next, this problem has been widely investigated by many authors (cf. e.g. [9]).
This paper is devoted to the stability of the alternative functional equation

$$f(x+y)+f(x)+f(y)\neq 0 \implies f(x+y)=f(x)+f(y), \tag{1}$$

which represents a class of conditional Cauchy equations with a condition dependent on the unknown function (cf. e.g. [7], [10], [11]). The notion of the stability we are going to deal with was introduced by the author in the paper [1] devoted to the stability of Mikusiski's equation and further applied to Dhombres' functional equation in [2].

Equation (1) has stemmed from

$$(f(x+y))^2=(f(x)+f(y))^2, \tag{2}$$

with a real function f, and next investigated in the form

$$|f(x+y)|=|f(x)+f(y)|, \tag{3}$$

which admits further generalizations from the real case to more general structures.

Since equation (3) has an unconditional form, one can deal with its stability in the exact Hyers–Ulam sense. Results concerning this question are contained in [3].

Moreover, there are known results concerning the stability of generalized equation (3). A natural generalization of equation (3) for normed spaces is

$$\|f(x+y)\|=\|f(x)+f(y)\|. \tag{4}$$

It is known (see [6]) that under the assumption that the norm $\|\ \|$ is strictly convex, any function f satisfying (4) has to be additive. Nevertheless, even if we consider $\mathbb{R}^2$

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with the Euclidean norm as an image space of f, equation (4) fails to be stable in the Hyers-Ulam sense (see [3]).

Another natural way to define the absolute value of $x \in \mathbb{R}$ is with the use of the order structure of $\mathbb{R}$, namely $|x| = \sup\{x, -x\}$. This motivates us to deal with the stability question in general Riesz spaces. Under some, quite natural assumptions on the Riesz space, the answer is positive (cf. [4]).

Finally, the conditional form (1) of the discussed equation allows for the application of the (δ, ε)-approach to the stability question recently introduced in [1]. The main aim of this paper is to treat equation (1) in a similar manner.

Through the paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers and complex numbers respectively.

2. Auxiliary lemmas

Let $(S, +)$ be a semigroup and let $(X, \|\cdot\|)$ be a normed space. Through this section we assume that, for some $\delta, \varepsilon \geqslant 0$ and all $x, y \in S$, a function $f : S \to X$ satisfies

$$\|f(x+y) + f(x) + f(y)\| > \delta \Longrightarrow \|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon. \tag{5}$$

For a fixed $x \in S$, we consider two complementary cases:

(i) $\left\|f(2^n x) + 2f(2^{n-1}x)\right\| > \delta$ for $n \in \mathbb{N}$;

(ii) there exists $k \in \mathbb{N}$ with $\|f(2^k x) + 2f(2^{k-1}x)\| \leqslant \delta$.

LEMMA 1. *Assume $f : S \to X$ satisfies (5). If*

$$\|f(2x) + 2f(x)\| \leqslant \delta \tag{6}$$

then

$$\|f(x)\| \leqslant \max\left\{\frac{1}{2}\varepsilon + \frac{5}{2}\delta, \frac{3}{4}\varepsilon + \frac{3}{4}\delta\right\}. \tag{7}$$

Consider $x \in S$ satisfying (6). Replacing x by $2x$ and y by x in (5) and taking into account (6) we have

$$\|f(3x) - f(x)\| \leqslant 2\delta \quad \text{or} \quad \|f(3x) + f(x)\| \leqslant \varepsilon + \delta. \tag{8}$$

Having applied (5) once again, with $3x$ and x instead of x and y respectively, yields

$$\|f(4x) + f(6x) + f(x)\| \leqslant \delta \quad \text{or} \quad \|f(4x) - f(3x) - f(x)\| \leqslant \varepsilon.$$

Now we combine the last condition and (8) to obtain

$$\begin{aligned} \|f(4x) + 2f(x)\| \leqslant 3\delta \qquad &\text{or} \quad \|f(4x) - 2f(x)\| \leqslant \varepsilon + 2\delta \\ \text{or} \quad \|f(4x)\| \leqslant \varepsilon + 2\delta \qquad &\text{or} \quad \|f(4x)\| \leqslant 2\varepsilon + \delta. \end{aligned} \tag{9}$$

Finally, substituting $2x$ for x and y in (5) and taking into account (6), we have

$$\|f(4x) - 4f(x)\| \leqslant 3\delta \quad \text{or} \quad \|f(4x) + 4f(x)\| \leqslant \varepsilon + 2\delta,$$

which along with (9) results in (7). $\square$

LEMMA 2. *Assume $f : S \to X$ satisfies (5). If k is the smallest nonnegative integer with $\|f(2^{k+1}x) + 2f(2^k x)\| \leqslant \delta$ then*

$$\|f(x)\| \leqslant \max\left\{\varepsilon, \|f(2^k x)\|\right\}. \tag{10}$$

Let us consider an arbitrary $x \in S$ satisfying (ii) and $k > 0$ (the case $k = 0$ is trivial). Using (5), with x and y replaced sequentially by x, $2x$, ..., $2^{k-1}x$, and combining the obtained inequalities, one can obtain

$$\left\|\frac{f(2^k x)}{2^k} - f(x)\right\| \leqslant \left(1 - \frac{1}{2^k}\right)\varepsilon,$$

which easily results in (10). $\square$

As a consequence of both the above lemmas we have

COROLLARY 1. *Assume $f : S \to X$ satisfies (5). If $x \in S$ satisfies* (ii) *then*

$$\|f(x)\| \leqslant \max\left\{\varepsilon, \frac{1}{2}\varepsilon + \frac{5}{2}\delta, \frac{3}{4}\varepsilon + \frac{3}{4}\delta\right\}.$$

LEMMA 3. *Assume $f : S \to X$ satisfies (5). Let an additive function $a : S \to X$ and a constant $K \geqslant 0$ be such that*

$$\|f(x) - a(x)\| \leqslant K \quad \textit{for } x \in S,$$

and

$$\|f(x) - a(x)\| \leqslant \varepsilon \quad \textit{if } x \textit{ satisfies } \text{(i)} \textit{ or } a(x) \neq 0. \tag{11}$$

Then

$$\|f(x) - a(x)\| \leqslant \max\{\varepsilon, \delta\} \quad \textit{for } x \in S.$$

It is enough to consider $x \in S$ satisfying (ii), such that $a(x) = 0$ (denote this subset of S by $\hat{S}$) and $K > \max\{\varepsilon, \delta\}$.

At first we will prove that for an arbitrary constant $A \geqslant 0$ the following implication holds

$$\begin{aligned} &\|f(x)\| \leqslant A \quad \text{for } x \in \hat{S} \\ &\implies \|f(x)\| \leqslant \max\left\{\varepsilon, \frac{1}{2}\delta + \frac{1}{2}\varepsilon, \frac{1}{2}\delta + \frac{1}{2}A\right\} \quad \text{for } x \in \hat{S}. \end{aligned} \tag{12}$$

Assume for the proof that $\|f(x)\| \leqslant A$ for $x \in \hat{S}$, fix an arbitrary $x \in \hat{S}$ and consider the smallest nonnegative integer k with

$$\|f(2^{k+1}x) + 2f(2^k x)\| \leqslant \delta. \tag{13}$$

If $2^{k+1}x \in \hat{S}$ then $\|f(2^{k+1}x)\| \leqslant A$. In the opposite case $\|f(2^{k+1}x)\| \leqslant \varepsilon$, on account of (11). Thus, (13) results in

$$\|f(2^k x)\| \leqslant \frac{1}{2}\delta + \frac{1}{2}\max\{\varepsilon, A\},$$

which, along with lemma (2), completes the proof of (12).

Using (12) one can show by induction that

$$\|f(x)\| \leqslant \max\left\{\varepsilon, \left(1-\frac{1}{2^n}\right)\delta + \frac{1}{2^n}K\right\} \quad \text{for } x \in \hat{S}, \quad n \in \mathbb{N}.$$

Letting $n \to +\infty$ we finish the proof. □

3. Main results

Now we will prove the main result of this paper concerning stability of the alternative equation (1).

THEOREM 1. *Let $(S,+)$ be an abelian semigroup and let $(X,\|\cdot\|)$ be a Banach space. If, for some $\delta,\varepsilon \geqslant 0$, a function $f : S \to X$ satisfies*

$$\|f(x+y)+f(x)+f(y)\| > \delta \Longrightarrow \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon \tag{14}$$

for $x,y \in S$, then there exists a unique additive function $a : S \to X$ such that

$$\|f(x)-a(x)\| \leqslant \max\{\varepsilon,\delta\} \quad \textit{for } x \in S. \tag{15}$$

The proof runs in a few steps.

Step 1. We will prove that there exists a constant $K \geqslant 0$ such that

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \leqslant \left(1-\frac{1}{2^n}\right)K \quad \text{for } x \in S,\ n \in \mathbb{N}. \tag{16}$$

Fix an arbitrary $x \in S$ and define $K := \max\{\delta+4\varepsilon, 2\varepsilon+11\delta, 3\varepsilon+4\delta\}$. Using (14) with y replaced by x in the case where $\|f(2x)+2f(x)\| > \delta$, or Corollary (1) in the opposite case, we have $\|f(2x)-2f(x)\| \leqslant K$. Making use of this, one can easily prove (16) by induction.

By (16) one can show that $\left(\frac{f(2^n x)}{2^n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for an arbitrary $x \in S$. Thus, the map $a : S \to X$ given by

$$a(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for } x \in S$$

is well defined.

Step 2. We will prove the additivity of a. Given $x,y \in S$ with $a(x+y)+a(x)+a(y) \neq 0$, we observe that $\|f(2^n(x+y))+f(2^n x)+f(2^n y)\| > \delta$ for a sufficiently large $n \in \mathbb{N}$. Thus, on account of the definition of a and (14), we obtain $a(x+y) = a(x)+a(y)$. Consequently, in view of Theorem 1 from [11], function a, as a solution of equation (1), has to be additive .

Letting $n \to +\infty$ in (16) we get

$$\|f(x)-a(x)\| \leqslant K \quad \text{for } x \in S. \tag{17}$$

Moreover, one can easily verify that the constant K in (16) may be replaced by ε if x satisfies (i). Therefore

$$\|f(x)-a(x)\| \leqslant \varepsilon \quad \text{for } x \in S \text{ satisfying (i).} \tag{18}$$

Step 3. We will show that if $a(x) \neq 0$ then for a sufficiently large $n \in \mathbb{N}$ and all $p \in \{1,\ldots,2^{n-1}\}$, we have

$$\|f((2^n-(p-1))x)+f((2^n-p)x)+f(x)\| > \delta. \tag{19}$$

Suppose, on the contrary, that there exists an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ and $p_{n_k} \in \{1,2,...,2^{n_k-1}\}$ such that for all $k \in \mathbb{N}$

$$\left\|f((2^{n_k}-(p_{n_k}-1))x)+f((2^{n_k}-p_{n_k})x)+f(x)\right\| \leqslant \delta. \tag{20}$$

By (17), with x replaced by $(2^{n_k}-(p_{n_k}-1))x$ we have

$$\left\|f((2^{n_k}-(p_{n_k}-1))x)-a((2^{n_k}-(p_{n_k}-1))x)\right\| \leqslant K.$$

Similarly

$$\left\|f((2^{n_k}-p_{n_k}x))-a((2^{n_k}-p_{n_k})x)\right\| \leqslant K.$$

Adding the above inequalities and (20), side by side, we get

$$\|a((2^{n_k+1}-2p_{n_k}+1)x)+f(x)\| \leqslant 2K+\delta.$$

which immediately yields $a(x)=0$ and contradicts the assumption.

Step 4. We will show that if $a(x) \neq 0$ then

$$\|f(x)-a(x)\| \leqslant \varepsilon. \tag{21}$$

Consider $x \in S$ with $a(x) \neq 0$. For a sufficiently large $n \in \mathbb{N}$ it is $\|f(2^n x)\| > \delta$, hence applying (14) with x and y replaced by $2^{n-1}x$ we get

$$\|f(2^n x)-2f(2^{n-1}x)\| \leqslant \varepsilon. \tag{22}$$

Replacing x sequentially by $(2^n-1)x,(2^n-2)x,\ldots,2^{n-1}x$ and y by x in (14) and taking into account (19) we obtain

$$\begin{gathered}
\|f(2^n x)-f((2^n-1)x)-f(x)\| \leqslant \varepsilon,\\
\|f((2^n-1)x)-f((2^n-2)x)-f(x)\| \leqslant \varepsilon,\\
\vdots\\
\|f((2^{n-1}+1)x)-f(2^{n-1}x)-f(x)\| \leqslant \varepsilon,
\end{gathered}$$

respectively. Adding the above inequalities up, side by side, we have

$$\|f(2^n x)-f(2^{n-1}x)-2^{n-1}f(x)\| \leqslant 2^{n-1}\varepsilon.$$

Using the inequality above and (22), we have

$$\left\|\frac{f(2^{n-1}x)}{2^{n-1}}-f(x)\right\| \leqslant \left(1+\frac{1}{2^{n-1}}\right)\varepsilon.$$

Letting $n \to \infty$ we obtain (21).

Step 5. We will prove (15).

It suffices to note that (15) results immediately from Lemma (3), assertions of which are fulfilled according to (17), (18) and (21).

The uniqueness of a function a satisfying assertions of our theorem follows immediately from its additivity and from (15). □

REMARK 1. Let us recall the following theorem from [3] concerning the stability of equation (3) on a restricted domain:

THEOREM BT (cf. [3], Theorem 1) *Let* $(S,+)$ *be a commutative semigroup and let* $V \subset S$ *be a weakly bounded set, i.e. such that for an arbitrary* $x \in S \setminus \{0\}$ *there exists* $n \in \mathbb{N}$ *with*

$$kx \notin V \quad \text{for } k \geqslant n.$$

If a function $f : S \to \mathbb{R}$ *satisfies the inequality*

$$\big|\,|f(x+y)| - |f(x)+f(y)|\,\big| \leqslant \varepsilon,$$

for some $\varepsilon > 0$ *and every* $(x,y) \in S \times S \setminus V \times V$, *then there exists a unique additive function* $a : S \to \mathbb{R}$ *such that*

$$|f(x) - a(x)| \leqslant 3\varepsilon \quad \text{for } x \in S.$$

Moreover, if $V = \emptyset$, *then*

$$|f(x) - a(x)| \leqslant \varepsilon \quad \text{for } x \in S.$$

Apart from the restriction of the domain, the above theorem becomes an immediate corollary of Theorem 1.

Now, as a direct consequence of Theorem 1 we obtain the stability of equation (2) in the Hyers-Ulam sense.

THEOREM 2. *Let* $(S,+)$ *be an abelian semigroup. If for a given* $\varepsilon \geqslant 0$ *a function* $f : S \to \mathbb{C}$ *satisfies*

$$|(f(x+y))^2 - (f(x)+f(y))^2| \leqslant \varepsilon \quad \text{for } x,y \in S, \tag{23}$$

then there exists a unique additive function $a : S \to \mathbb{C}$ *such that*

$$|f(x) - a(x)| \leqslant \sqrt{\varepsilon} \quad \text{for } x \in S.$$

It suffices to apply Theorem 1, with δ and ε replaced by $\sqrt{\varepsilon}$. □

REMARK 2. Let us note that there is a stability result concerning equation (2) belonging to P. W. Cholewa (cf. [5]), who proved that this equation is superstable, i.e. any function f satisfying (23) has to be bounded or additive. Superstability of equation (2) can also be proved, in a quite simple way, starting from Theorem 2 (cf. [13]).

REMARK 3. The assumption that $(S,+)$ is abelian, in both Theorem 1 and Theorem 2, may be replaced by the weak substitute for commutativity of S (cf. [12]), i.e.

$$(x+y)^k = x^k + y^k \quad \text{for } x,y \in S,$$

where $k \geqslant 2$ is a fixed integer.

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