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ON GENERALIZED STRONGLY NONLINEAR VARIATIONAL–LIKE INEQUALITIES

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Abstract. The purpose of this paper is to introduce and study a new class of generalized strongly nonlinear variational-like inequalities. The existence and uniqueness of solution and a new iterative algorithm for the generalized strongly nonlinear variational-like inequality are proved and suggested, respectively. Moreover, the convergence criteria of the sequence generated by the iterative algorithm are also given.

1. Introduction

Recently, variational inequality theory has been extended and applied in various directions, see [1]-[8], [10]-[33] and the references therein. It is worth mentioning that one of the most important problems in variational inequality theory is the development of efficient and implementable iterative algorithms for solving various variational inequalities. It is well known that there are a lot of iterative type algorithms for finding the approximate solutions of various variational inequalities in Hilbert spaces [3] and [7], [8], [10]-[33]. By using the auxiliary principle technique, Ding [4], [5] and Ding-Tan [6] studied some classes of general nonlinear mixed variational inequalities and variational-like inequalities in reflexive Banach spaces, and suggested some iterative algorithms to compute approximate solutions for these general nonlinear mixed variational inequalities and variational-like inequalities. Verma [27]-[31] proved the existence of solutions for several classes of nonlinear variational inequalities involving various nonlinear monotone mappings in Hilbert spaces.

In this paper, we introduce and study a new class of generalized strongly nonlinear variational-like inequalities. By applying a result due to Chang [1], we prove the existence and uniqueness of solution of the generalized strongly nonlinear variational-like inequality and suggest a new iterative algorithm for solving the class of generalized strongly nonlinear variational-like inequalities. The convergence criteria of the sequence generated by the iterative algorithm are given. The results presented in this paper extend and improve the corresponding results in [4], [5] and [33].

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2. Preliminaries

Let H be a real Hilbert space endowed with an inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H, $A,C,E,F : K \to H$, $N,M : H \times H \to H$ and $\eta : K \times K \to H$ be mappings. Suppose that $a : H \times H \to (-\infty,\infty)$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

$(C1)$ $a(v,v) \geqslant c\|v\|^2$, $\forall v \in H$;

$(C2)$ $a(u,v) \leqslant d\|u\|\|v\|$, $\forall u,v \in H$.

Clearly, $c \leqslant d$.

Let $b : H \times H \to (-\infty,+\infty)$ be nondifferential and satisfy the following conditions:

$(C3)$ b is linear in the first argument;

$(C4)$ b is convex in the second argument;

$(C5)$ b is bounded, that is, there exists a constant $q > 0$ satisfying

$$|b(u,v)| \leqslant q\|u\|\|v\|, \quad \forall u,v \in H;$$

$(C6)$ $b(u,v) - b(u,w) \leqslant b(u,v-w)$, $\forall u,v,w \in H$.

Now we consider the following generalized strongly nonlinear variational-like inequality problem: For a given $g \in H$, find $u \in K$ such that

$$\langle N(Au,Cu) - M(Eu,Fu) + g, \eta(v,u)\rangle + a(u,v-u) \geqslant b(u,u) - b(u,v), \quad \forall v \in K. \tag{2.1}$$

Special Cases

(A) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$, $M(Eu,Fu) = 0$ and $b(u,v) = f(v)$ for all $u,v \in K$, and $g = 0$, then the generalized strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, \eta(v,u)\rangle \geqslant f(u) - f(v), \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ding [4].

(B) If $a(u,v) = 0$ and $M(Eu,Fu) = 0$ for all $u,v \in K$, then the generalized strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle N(Au,Cu) + g, \eta(v,u)\rangle \geqslant b(u,u) - b(u,v), \quad \forall v \in K, \tag{2.3}$$

which was studied by Ding [5].

(C) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$, $M(Eu,Fu) = 0$, $\eta(u,v) = gu - gv$ and $b(u,v) = f(v)$ for all $u,v \in K$, and $g = 0$, then the generalized strongly nonlinear variational-like inequality (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, gv - gu)\rangle \geqslant f(u) - f(v), \quad \forall v \in K, \tag{2.4}$$

which was studied by Yao [33].

DEFINITION 2.1. Let $A : K \to H$, $N : H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) A is said to be *Lipschitz continuous* with constant α if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leqslant \alpha \|u - v\|, \quad \forall u, v \in K;$$

(2) N is said to be *Lipschitz continuous* with constant β in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u,w) - N(v,w)\| \leqslant \beta \|u - v\|, \quad \forall u, v, w \in H;$$

(3) N is said to be *strongly monotone* with constant γ with respect to A in the second argument if

$$\langle N(w,Au) - N(w,Av), u - v\rangle \geqslant \gamma \|u - v\|^2, \quad \forall u, v \in K,\ w \in H;$$

(4) N is said to be *relaxed Lipschitz* with constant γ with respect to A in the second argument if

$$\langle N(w,Au) - N(w,Av), u - v\rangle \leqslant -\gamma \|u - v\|^2, \quad \forall u, v \in K,\ w \in H;$$

(5) N is said to be η*-monotone* with respect to A in the first argument if

$$\langle N(Au,w) - N(Av,w), \eta(u,v)\rangle \geqslant 0, \quad \forall u, v \in K,\ w \in H;$$

(6) N is said to be η*-strongly monotone* with constant ξ with respect to A in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Au,w) - N(Av,w), \eta(u,v)\rangle \geqslant \xi \|u - v\|^2, \quad \forall u, v \in K,\ w \in H;$$

(7) N is said to be η*-relaxed Lipschitz* with constant ζ with respect to A in the second argument if there exists a constant $\zeta > 0$ such that

$$\langle N(w,Au) - N(w,Av), \eta(u,v)\rangle \leqslant -\zeta \|u - v\|^2, \quad \forall u, v \in K,\ w \in H;$$

(8) η is said to be *Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leqslant \delta \|u - v\|, \quad \forall u, v \in K;$$

(9) η is said to be *strongly monotone* with constant ω if there exists a constant $\omega > 0$ such that

$$\langle u - v, \eta(u,v)\rangle \geqslant \omega \|u - v\|^2, \quad \forall u, v \in K.$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

LEMMA 2.1. [1] *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and $\phi, \psi : X \times X \to R$ be mappings satisfying the following conditions:*

(a) $\psi(x,y) \leqslant \phi(x,y)$, $\forall x, y \in X$, *and* $\psi(x,x) \geqslant 0$, $\forall x \in X$;

(b) *for each* $x \in X$, $\phi(x,\cdot)$ *is upper semicontinuous on* X;

(c) *for each* $y \in X$, *the set* $\{x \in X : \psi(x,y) < 0\}$ *is a convex set;*

(d) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0,y) < 0$, $\forall y \in X \setminus K$.

Then there exists $\hat{y} \in K$ *such that* $\phi(x,\hat{y}) \geqslant 0$, $\forall x \in X$.

LEMMA 2.2. [9] *Let* $\{\alpha_n\}_{n\geqslant 0}$, $\{\beta_n\}_{n\geqslant 0}$ *and* $\{\gamma_n\}_{n\geqslant 0}$ *be nonnegative sequences satisfying*

$$\alpha_{n+1} \leqslant (1-\lambda_n)\alpha_n + \beta_n\lambda_n + \gamma_n, \quad \forall n \geqslant 0,$$

where $\{\lambda_n\}_{n\geqslant 0} \subset [0,1]$, $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n < \infty$. *Then*

$$\lim_{n\to\infty}\alpha_n = 0.$$

3. Existence Theorems

In this section, we give two existence theorems of solutions for the generalized strongly nonlinear variational-like inequality (2.1).

THEOREM 3.1. *Let* $a : H\times H \to (-\infty,\infty)$ *be a coercive continuous bilinear form with* $(C1)$ *and* $(C2)$ *and* $b : H\times H \to (-\infty,\infty)$ *satisfy* $(C3)-(C6)$. *Suppose that* $A,C,E : K \to H$ *and* $N,M : H\times H \to H$ *are continuous mappings,* $\eta : K\times K \to H$ *is Lipschitz continuous with constant* δ *and strongly monotone with constant* ω, *for each* $v \in K$, $\eta(\cdot,v)$ *is continuous and* $\eta(v,u) = -\eta(u,v)$ *for all* $u,v \in K$. *Assume that* N *is* η*-strongly monotone with constant* ξ *with respect to* A *in the first argument and* η*-monotone with respect to* C *in the second argument. Let* $F : K \to H$ *be Lipschitz continuous with constant* l, M *be* η*-relaxed Lipschitz with constant* ϱ *with respect to* E *in the first argument, relaxed Lipschitz with constant* τ *with respect to* F *in the second argument and Lipschitz continuous with constant* ϑ *in the second argument. Suppose that for given* $x,y \in H$ *and* $v \in K$, *the mappings* $u \mapsto \langle N(x,y)+g, \eta(v,u)\rangle$ *and* $u \mapsto \langle M(x,y), \eta(u,v)\rangle$ *be concave and upper semicontinuous. Let* $k = \vartheta l$, $j = \frac{\xi+\varrho}{\delta}$ *and* $p = \frac{\omega-d-q}{\delta}$. *If there exists a constant* $\mu > 0$ *satisfying*

$$\mu j + p > 0, \tag{3.1}$$

and one of the following conditions:

$$\left|\mu - \frac{\tau - jp}{k^2-j^2}\right| < \frac{\sqrt{(\tau-jp)^2-(k^2-j^2)(1-p^2)}}{k^2-j^2}, \qquad k > j, \quad |\tau - jp| > \sqrt{(k^2-j^2)(1-p^2)}; \tag{3.2}$$

$$\left|\mu - \frac{\tau - jp}{k^2-j^2}\right| > \frac{\sqrt{(\tau-jp)^2+(j^2-k^2)(1-p^2)}}{j^2-k^2}, \quad k < j, \tag{3.3}$$

then for any given $g \in H$, *the generalized strongly nonlinear variational-like inequality* (2.1) *has a unique solution in* K.

Proof. First of all we show that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that

$$\begin{aligned}\langle \hat{w}, \eta(v,\hat{w})\rangle \geqslant{}& \langle \hat{u}, \eta(v,\hat{w})\rangle - \mu\langle N(A\hat{w},C\hat{w}) - M(E\hat{w},F\hat{u}) + g, \eta(v,\hat{w})\rangle \\ &- \mu a(\hat{u}, v-\hat{w}) - \mu b(\hat{u},v) + \mu b(\hat{u},\hat{w}), \quad \forall v \in K,\end{aligned} \tag{3.4}$$

where $\mu > 0$ is a constant. Let $\hat{u}$ be in K. Define the functionals ϕ and $\psi : K \times K \to R$ by

$$\begin{aligned}\phi(v,w) = {} & \langle v, \eta(v,w)\rangle - \langle \hat{u}, \eta(v,w)\rangle \\ & + \mu\langle N(Av,Cv) - M(Ev,F\hat{u}) + g, \eta(v,w)\rangle \\ & + \mu a(\hat{u}, v-w) + \mu b(\hat{u}, v) - \mu b(\hat{u}, w)\end{aligned}$$

and

$$\begin{aligned}\psi(v,w) = {} & \langle w, \eta(v,w)\rangle - \langle \hat{u}, \eta(v,w)\rangle \\ & + \mu\langle N(Aw,Cw) - M(Ew,F\hat{u}) + g, \eta(v,w)\rangle \\ & + \mu a(\hat{u}, v-w) + \mu b(\hat{u}, v) - \mu b(\hat{u}, w)\end{aligned}$$

for all $v, w \in K$.

We check that the functionals ϕ and ψ satisfy all the conditions of Lemma 2.1 in the weak topology. It is easy to see for all $v, w \in K$,

$$\begin{aligned}\phi(v,w) - \psi(v,w) = {} & \langle v-w, \eta(v,w)\rangle + \mu\langle N(Av,Cv) - N(Aw,Cv), \eta(v,w)\rangle \\ & + \mu\langle N(Aw,Cv) - N(Aw,Cw), \eta(v,w)\rangle \\ & - \mu\langle M(Ev,F\hat{u}) - M(Ew,F\hat{u}), \eta(v,w)\rangle \\ \geqslant {} & [\omega + \mu(\xi + \varrho)]\|v-w\|^2 \geqslant 0,\end{aligned}$$

which yields that ϕ and ψ satisfy the condition (a) of Lemma 2.1. Note that b is convex and lower semicontinuous with respect to the second argument and for given $x, y \in H$, $v \in K$, the mappings $u \mapsto \langle N(x,y)+g, \eta(v,u)\rangle$ and $u \mapsto \langle M(x,y), \eta(u,v)\rangle$ are concave and upper semicontinuous. It follows that $\phi(v,w)$ is weakly upper semicontinuous with respect to w and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. Therefore the conditions (b) and (c) of Lemma 2.1 hold. Put

$$D = [\omega + \mu(\xi + \varrho)]^{-1}[\delta\mu(\|N(Av^*,Cv^*)\| + \|M(Ev^*,F\hat{u}) - g\|) - \mu(1+d+q)\|\hat{u}\|]$$

and

$$T = \{w \in K : \|w - v^*\| \leqslant D\}.$$

Clearly, T is a weakly compact subset of K and for any $w \in K \setminus T$

$$\begin{aligned}\psi(v^*,w) = {} & -\langle w, \eta(w,v^*)\rangle + \langle \hat{u}, \eta(w,v^*)\rangle \\ & - \mu\langle N(Aw,Cw) - N(Av^*,Cw), \eta(w,v^*)\rangle \\ & - \mu\langle N(Av^*,Cw) - N(Av^*,Cv^*), \eta(w,v^*)\rangle \\ & - \mu\langle N(Av^*,Cv^*), \eta(w,v^*)\rangle + \mu\langle M(Ew,F\hat{u}) - M(Ev^*,F\hat{u}), \eta(w,v^*)\rangle \\ & + \mu\langle M(Ev^*,F\hat{u}) - g, \eta(w,v^*)\rangle + \mu a(\hat{u}, -w) + \mu b(\hat{u}, v^*) - \mu b(\hat{u}, w) \\ \leqslant {} & -\|w - v^*\|\{[\omega + \mu(\xi + \varrho)]\|w - v^*\| - \delta\mu(\|N(Av^*,Cv^*)\| \\ & + \|M(Ev^*,F\hat{u}) - g\|) - \mu(1+d+q)\|\hat{u}\|\} \\ < {} & 0,\end{aligned}$$

which means that the condition (d) of Lemma 2.1 holds. Thus Lemma 2.1 ensures that there exists a $\hat{w} \in K$ such that $\phi(v,\hat{w}) \geqslant 0$ for all $v \in K$, that is,

$$\begin{aligned}\langle v, \eta(v,\hat{w})\rangle \geqslant {} & \langle \hat{u}, \eta(v,\hat{w})\rangle - \mu\langle N(Av,Cv) - M(Ev,F\hat{u}) + g, \eta(v,\hat{w})\rangle \\ & - \mu a(\hat{u}, v-\hat{w}) - \mu b(\hat{u}, v) + \mu b(\hat{u}, \hat{w}).\end{aligned} \tag{3.5}$$

Set t be in $(0,1]$ and v be in K. Replacing v by $v_t = tv + (1-t)\hat{w}$ in (3.5), we know that

$$\begin{aligned}\langle v_t, \eta(v_t, \hat{w})\rangle \geqslant \langle \hat{u}, \eta(v_t, \hat{w})\rangle &- \mu\langle N(Av_t, Cv_t) - M(Ev_t, F\hat{u}) + g, \eta(v_t, \hat{w})\rangle \\ &- \mu a(\hat{u}, t(v-\hat{w})) - \mu b(\hat{u}, v_t) + \mu b(\hat{u}, \hat{w}).\end{aligned} \tag{3.6}$$

Notice that a is bilinear and b is convex with respect to the second argument. From (3.6) we deduce that

$$\begin{aligned}t[\langle v_t, \eta(v, \hat{w})\rangle] \geqslant t[\langle \hat{u}, \eta(v, \hat{w})\rangle &- \mu\langle N(Av_t, Cv_t) - M(Ev_t, F\hat{u}) + g, \eta(v, \hat{w})\rangle \\ &- \mu a(\hat{u}, v-\hat{w}) - \mu b(\hat{u}, v) + \mu b(\hat{u}, \hat{w})],\end{aligned}$$

which implies that

$$\begin{aligned}\langle v_t, \eta(v, \hat{w})\rangle \geqslant \langle \hat{u}, \eta(v, \hat{w})\rangle &- \mu\langle N(Av_t, Cv_t) - M(Ev_t, F\hat{u}) + g, \eta(v, \hat{w})\rangle \\ &- \mu a(\hat{u}, v-\hat{w}) - \mu b(\hat{u}, v) + \mu b(\hat{u}, \hat{w}).\end{aligned}$$

Letting $t \to 0^+$ in the above inequality, we conclude that

$$\begin{aligned}\langle \hat{w}, \eta(v, \hat{w})\rangle \geqslant \langle \hat{u}, \eta(v, \hat{w})\rangle &- \mu\langle N(A\hat{w}, C\hat{w}) - M(E\hat{w}, F\hat{u}) + g, \eta(v, \hat{w})\rangle \\ &- \mu a(\hat{u}, v-\hat{w}) - \mu b(\hat{u}, v) + \mu b(\hat{u}, \hat{w}), \quad \forall v \in K.\end{aligned}$$

That is, $\hat{w}$ is a solution of (3.4). Now we prove the uniqueness. For any two solutions $w_1, w_2 \in K$ of (3.4), we know that

$$\begin{aligned}\langle w_1, \eta(v, w_1)\rangle \geqslant \langle \hat{u}, \eta(v, w_1)\rangle &- \mu\langle N(Aw_1, Cw_1) - M(Ew_1, F\hat{u}) + g, \eta(v, w_1)\rangle \\ &- \mu a(\hat{u}, v-w_1) - \mu b(\hat{u}, v) + \mu b(\hat{u}, w_1)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\langle w_2, \eta(v, w_2)\rangle \geqslant \langle \hat{u}, \eta(v, w_2)\rangle &- \mu\langle N(Aw_2, Cw_2) - M(Ew_2, F\hat{u}) + g, \eta(v, w_2)\rangle \\ &- \mu a(\hat{u}, v-w_2) - \mu b(\hat{u}, v) + \mu b(\hat{u}, w_2)\end{aligned} \tag{3.8}$$

for all $v \in K$. Taking $v = w_2$ in (3.7) and $v = w_1$ in (3.8), we get that

$$\begin{aligned}\langle w_1, \eta(w_2, w_1)\rangle \geqslant \langle \hat{u}, \eta(w_2, w_1)\rangle &- \mu\langle N(Aw_1, Cw_1) - M(Ew_1, F\hat{u}) + g, \eta(w_2, w_1)\rangle \\ &- \mu a(\hat{u}, w_2-w_1) - \mu b(\hat{u}, w_2) + \mu b(\hat{u}, w_1)\end{aligned}$$

and

$$\begin{aligned}\langle w_2, \eta(w_1, w_2)\rangle \geqslant \langle \hat{u}, \eta(w_1, w_2)\rangle &- \mu\langle N(Aw_2, Cw_2) - M(Ew_2, F\hat{u}) + g, \eta(w_1, w_2)\rangle \\ &- \mu a(\hat{u}, w_1-w_2) - \mu b(\hat{u}, w_1) + \mu b(\hat{u}, w_2).\end{aligned}$$

Adding these inequalities, we deduce that

$$\begin{aligned}\omega\|w_1 - w_2\|^2 &\leqslant -\mu\langle N(Aw_1, Cw_1) - N(Aw_2, Cw_1), \eta(w_1, w_2)\rangle \\ &\quad - \mu\langle N(Aw_2, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2)\rangle \\ &\quad + \mu\langle M(Ew_1, F\hat{u}) - M(Ew_2, F\hat{u}), \eta(w_1, w_2)\rangle \\ &\leqslant -\mu(\xi + \varrho)\|w_1 - w_2\|^2,\end{aligned}$$

which yields that $w_1 = w_2$. That is, $\hat{w}$ is the unique solution of (3.4). This means that there exists a mapping $G : K \to K$ satisfying $G(\hat{u}) = \hat{w}$, where $\hat{w}$ is the unique solution of (3.4) for each $\hat{u} \in K$.

Next we show that G is a contraction mapping. Let u_1 and u_2 be arbitrary elements in K. Using (3.4), we see that

$$\begin{aligned}\langle Gu_1, \eta(v, Gu_1)\rangle \geqslant &\langle u_1, \eta(v, Gu_1)\rangle \\ &- \mu\langle N(A(Gu_1), C(Gu_1)) - M(E(Gu_1), Fu_1) + g, \eta(v, Gu_1)\rangle \\ &- \mu a(u_1, v - Gu_1) - \mu b(u_1, v) + \mu b(u_1, Gu_1)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\langle Gu_2, \eta(v, Gu_2)\rangle \geqslant &\langle u_2, \eta(v, Gu_2)\rangle \\ &- \mu\langle N(A(Gu_2), C(Gu_2)) - M(E(Gu_2), Fu_2) + g, \eta(v, Gu_2)\rangle \\ &- \mu a(u_2, v - Gu_2) - \mu b(u_2, v) + \mu b(u_2, Gu_2)\end{aligned} \tag{3.10}$$

for all $v \in K$. Letting $v = Gu_2$ in (3.9) and $v = Gu_1$ in (3.10), and adding these inequalities, we arrive at

$$\begin{aligned}\omega\|Gu_1 - Gu_2\|^2 \leqslant &-\mu\langle N(A(Gu_1), C(Gu_1)) - N(A(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2)\rangle \\ &- \mu\langle N(A(Gu_2), C(Gu_1)) - N(A(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2)\rangle \\ &+ \mu\langle M(E(Gu_1), Fu_1) - M(E(Gu_2), Fu_1), \eta(Gu_1, Gu_2)\rangle \\ &+ \langle u_1 - u_2 + \mu(M(E(Gu_2), Fu_1) - M(E(Gu_2), Fu_2)), \eta(Gu_1, Gu_2)\rangle \\ &+ \mu a(u_1 - u_2, Gu_1 - Gu_2) + \mu b(u_1 - u_2, Gu_2 - Gu_1) \\ \leqslant &-\mu(\xi + \varrho)\|Gu_1 - Gu_2\|^2 \\ &+ [\delta\sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2} + \mu(d + q)]\|u_1 - u_2\|\|Gu_1 - Gu_2\|,\end{aligned}$$

that is,

$$\|Gu_1 - Gu_2\| \leqslant \theta\|u_1 - u_2\|,$$

where

$$\theta = \frac{\delta\sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2} + \mu(d + q)}{\omega + \mu(\xi + \varrho)} < 1 \tag{3.11}$$

by (3.1) and one of (3.2) and (3.3), which yields that $G : K \to K$ is a contraction mapping and hence it has a unique fixed point $u \in K$, which is a unique solution of the generalized strongly nonlinear variational-like inequality (2.1). This completes the proof.

THEOREM 3.2. *Let $a, b, A, E, F, N, M, \eta, k$ and p be as in Theorem 3.1. Suppose that N is Lipschitz continuous with constant ζ in the second argument, $C : K \to H$ is Lipschitz continuous with constant ε and $j = \frac{\xi+\varrho-\delta\zeta\varepsilon}{\delta} > 0$. If there exists a constant $\mu > 0$ satisfying (3.1) and one of (3.2) and (3.3), then for any given $g \in H$, the generalized strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

Proof. Put

$$D=[\omega+\mu(\xi+\varrho-\delta\zeta\varepsilon)]^{-1}[\delta\mu(\|N(Av^*,Cv^*)\|+\|M(Ev^*,F\hat{u})-g\|)-\mu(1+d+q)\|\hat{u}\|]$$

and

$$T=\{w\in K:\|w-v^*\|\leqslant D\}.$$

As in the proof of Theorem 3.1, we conclude that

$$\begin{aligned}\psi(v^*,w)=&-\langle w,\eta(w,v^*)\rangle+\langle \hat{u},\eta(w,v^*)\rangle\\&-\mu\langle N(Aw,Cw)-N(Av^*,Cw),\eta(w,v^*)\rangle\\&-\mu\langle N(Av^*,Cw)-N(Av^*,Cv^*),\eta(w,v^*)\rangle\\&-\mu\langle N(Av^*,Cv^*),\eta(w,v^*)\rangle+\mu\langle M(Ew,F\hat{u})-M(Ev^*,F\hat{u}),\eta(w,v^*)\rangle\\&+\mu\langle M(Ev^*,F\hat{u})-g,\eta(w,v^*)\rangle+\mu a(\hat{u},-w)+\mu b(\hat{u},v^*)-\mu b(\hat{u},w)\\\leqslant&-\|w-v^*\|\{[\omega+\mu(\xi+\varrho-\delta\zeta\varepsilon)]\|w-v^*\|-\delta\mu(\|N(Av^*,Cv^*)\|\\&+\|M(Ev^*,F\hat{u})-g\|)-\mu(1+d+q)\|\hat{u}\|\}\\<&\ 0\end{aligned}$$

for any $w\in K\setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted.

REMARK 3.2 Theorems 3.1 and 3.2 extend Theorem 3.1 in [33].

4. Algorithm and Convergence Theorems

Using Theorem 3.1, we suggest the following iterative algorithm.

Algorithm. Suppose that $a:H\times H\to(-\infty,\infty)$ is a coercive continuous bilinear form with $(C1)$ and $(C2)$ and $b:H\times H\to(-\infty,\infty)$ satisfies $(C3)-(C6)$. Let $A,C,E,F:K\to H$, $N,M:H\times H\to H$ and $\eta:K\times K\to H$ be mappings, and $g\in H$ be given. For any given $u_0\in K$, compute sequences $\{u_n\}_{n\geqslant0}$ and $\{w_n\}_{n\geqslant0}$ by the following iterative schemes

$$\begin{aligned}\langle w_n,\eta(v,w_n)\rangle\geqslant&(1-\alpha_n)\langle u_n,\eta(v,w_n)\rangle\\&+\alpha_n\langle u_n-\mu N(Aw_n,Cw_n)+\mu M(Ew_n,Fu_n)-\mu g,\eta(v,w_n)\rangle\\&-\alpha_n\mu a(u_n,v-w_n)-\alpha_n\mu b(u_n,v)+\alpha_n\mu b(u_n,w_n)+\langle r_n,\eta(v,w_n)\rangle\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\langle u_{n+1},\eta(v,u_{n+1})\rangle\geqslant&(1-\beta_n)\langle w_n,\eta(v,u_{n+1})\rangle\\&+\beta_n\langle w_n-\mu N(Au_{n+1},Cu_{n+1})+\mu M(Eu_{n+1},Fw_n)-\mu g,\eta(v,u_{n+1})\rangle\\&-\beta_n\mu a(w_n,v-u_{n+1})-\beta_n\mu b(w_n,v)+\beta_n\mu b(w_n,u_{n+1})\\&+\langle s_n,\eta(v,u_{n+1})\rangle\end{aligned}\tag{4.2}$$

for all $v\in K$ and $n\geqslant0$, where $\{\alpha_n\}_{n\geqslant0}$, $\{\beta_n\}_{n\geqslant0}\subset[0,1]$ and $\{r_n\}_{n\geqslant0}$, $\{s_n\}_{n\geqslant0}\in H$.

THEOREM 4.1. *Let* $a, b, A, C, E, F, N, M, \eta, k, j$ *and* p *be as in Theorem* 3.1. *Assume that*

$$\lim_{n\to\infty} \|r_n\| = \lim_{n\to\infty} \|s_n\| = 0 \tag{4.3}$$

and

$$\inf\{\alpha_n, \beta_n : n \geqslant 0\} > 0. \tag{4.4}$$

If there exists a constant $\mu > 0$ *satisfying* (3.1) *and*

$$\frac{\delta - \omega}{(\xi + \varrho)\inf\{\alpha_n, \beta_n : n \geqslant 0\}} \leqslant \mu < \min\left\{\frac{\delta}{d+q}, \frac{2\delta(\delta\tau - d - q)}{(\delta\vartheta l)^2 - (d+q)^2}\right\}, \tag{4.5}$$

then for any given $g \in H$, *the generalized strongly nonlinear variational-like inequality* (2.1) *possesses a unique solution* $u \in K$ *and the iterative sequence* $\{u_n\}_{n\geqslant 0}$ *generated by Algorithm converges strongly to* u.

Proof. Put

$$\theta_1 = \frac{\delta}{\omega + \mu(\xi + \varrho)\inf\{\alpha_n, \beta_n : n \geqslant 0\}}$$

and

$$\theta_2 = \frac{\mu(d+q)}{\delta} + \sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2}.$$

In view of (3.1), (3.11) and (4.5), we conclude easily that

$$\begin{aligned}
\theta &= \frac{\delta\sqrt{1 - 2\mu\tau + (\mu\vartheta l)^2} + \mu(d+q)}{\omega + \mu(\xi + \varrho)} \\
&= \frac{\delta\theta_2}{\omega + \mu(\xi + \rho)} \\
&\leqslant \frac{\delta\theta_2}{\omega + \mu(\xi + \rho)\inf\{\alpha_n, \beta_n : n \geqslant 0\}} \\
&\leqslant \theta_2 < 1,
\end{aligned}$$

which yields that one of (3.2) and (3.3) holds. It follows from Theorem 3.1 that for any given $g \in H$, the generalized strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$ such that

$$\begin{aligned}
\langle u, \eta(v,u)\rangle \geqslant{} & (1-\alpha_n)\langle u, \eta(v,u)\rangle \\
& + \alpha_n\langle u - \mu N(Au, Cu) + \mu M(Eu, Fu) - \mu g, \eta(v,u)\rangle \\
& - \alpha_n\mu a(u, v-u) - \alpha_n\mu b(u,v) + \alpha_n\mu b(u,u)
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\langle u, \eta(v,u)\rangle \geqslant{} & (1-\beta_n)\langle u, \eta(v,u)\rangle \\
& + \beta_n\langle u - \mu N(Au, Cu) + \mu M(Eu, Fu) - \mu g, \eta(v,u)\rangle \\
& - \beta_n\mu a(u, v-u) - \beta_n\mu b(u,v) + \beta_n\mu b(u,u)
\end{aligned} \tag{4.7}$$

for all $v \in K$ and $n \geqslant 0$. Taking $v = u$ in (4.1), $v = w_n$ in (4.6) and adding these inequalities, we get that

$$\begin{aligned}
\omega\|w_n - u\|^2 \leqslant\ & (1-\alpha_n)\langle u_n - u, \eta(w_n,u)\rangle \\
& - \alpha_n\mu\langle N(Aw_n, Cw_n) - N(Au, Cw_n), \eta(w_n,u)\rangle \\
& - \alpha_n\mu\langle N(Au, Cw_n) - N(Au, Cu), \eta(w_n,u)\rangle \\
& + \alpha_n\mu\langle M(Ew_n, Fu_n) - M(Eu, Fu_n), \eta(w_n,u)\rangle \\
& + \alpha_n\langle u_n - u + \mu(M(Eu, Fu_n) - M(Eu, Fu)), \eta(w_n,u)\rangle \\
& - \alpha_n\mu a(u_n - u, w_n - u) + \alpha_n\mu b(u_n - u, u - w_n) + \langle r_n, \eta(w_n,u)\rangle \\
\leqslant\ & (1-\alpha_n)\delta\|u_n - u\|\|w_n - u\| - \alpha_n\xi\mu\|w_n - u\|^2 - \alpha_n\rho\mu\|w_n - u\|^2 \\
& + \alpha_n\delta\sqrt{1 - 2\mu\tau + \mu^2\vartheta^2 l^2}\|u_n - u\|\|w_n - u\| \\
& + \alpha_n d\mu\|u_n - u\|\|w_n - u\| + \alpha_n q\mu\|u_n - u\|\|w_n - u\| \\
& + \delta\|r_n\|\|w_n - u\|, \quad \forall n \geqslant 0,
\end{aligned}$$

which means that

$$\begin{aligned}
\|w_n - u\| &\leqslant \frac{\delta}{\omega + \mu(\xi+\rho)\alpha_n}[1 - \alpha_n(1-\theta_2)]\|u_n - u\| + \frac{\delta}{\omega + \mu(\xi+\rho)\alpha_n}\|r_n\| \\
&\leqslant \theta_1[1 - \alpha_n(1-\theta_2)]\|u_n - u\| + \theta_1\|r_n\| \\
&\leqslant [1 - \alpha_n(1-\theta_2)]\|u_n - u\| + \|r_n\| \\
&\leqslant \|u_n - u\| + \|r_n\|, \quad \forall n \geqslant 0.
\end{aligned} \tag{4.8}$$

From (4.2), (4.7) and (4.8), we deduce similarly that

$$\begin{aligned}
\|u_{n+1} - u\| &\leqslant [1 - \beta_n(1-\theta_2)]\|w_n - u\| + \|s_n\| \\
&\leqslant [1 - \beta_n(1-\theta_2)]\|u_n - u\| + \|s_n\| + \|r_n\|, \quad \forall n \geqslant 0.
\end{aligned} \tag{4.9}$$

It follows from (4.3), (4.4), (4.9) and Lemma 2.2 that $\lim_{n\to\infty}\|u_{n+1} - u\| = 0$. This completes the proof.

Similarly we have the following result.

THEOREM 4.2. *Let $a, b, A, C, E, F, N, M, \eta, k, j$ and p be as in Theorem 3.2 and (4.3) and (4.4) hold. If there exists a constant $\mu > 0$ satisfying (3.1) and*

$$\frac{\delta - \omega}{(\xi + \varrho - \delta\zeta\varepsilon)\inf\{\alpha_n, \beta_n : n \geqslant 0\}} \leqslant \mu < \min\left\{\frac{\delta}{d+q}, \frac{2\delta(\delta\tau - d - q)}{(\delta\vartheta l)^2 - (d+q)^2}\right\}, \tag{4.10}$$

then for any given $g \in H$, the generalized strongly nonlinear variational-like inequality (2.1) possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geqslant 0}$ generated by Algorithm converges strongly to u.

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