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GEOMETRIC INEQUALITIES FOR A SIMPLEX

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Abstract. In this paper, we study the problem of geometric inequalities for an n-simplex. Some new geometric inequalities for simplex are established. As the special case, some known inequalities are gotten.

1. Main results and applications

Let σ_n be an n-dimensional simplex in the n-dimensional Euclidean space E^n, $\tau = \{A_0, A_1, \ldots, A_n\}$ denote the vertex set of σ_n, V the volume of σ_n, R and r the circumradius and inradius of σ_n, respectively. For $i = 0, 1, \ldots, n$, let r_i be the radius of i-th escribed sphere of σ_n, F_i the area of the i-th face $f_i = A_0 \ldots A_{i-1}A_{i+1} \ldots A_n$ of σ_n. Let P be an arbitrary interior point of the simplex σ_n, d_i the distance from the point P to the i-th face f_i of σ_n, h_i the altitute of σ_n from vertex A_i for $i = 0$, 1, $\ldots$, n.

Let a_0, a_1 and a_2 denote the edge-lengths of triangle $A_0A_1A_2$ (2-dimensional simplex). An important inequality for a triangle was established by Janic [1] as follows

$$\frac{a_0^2}{r_1r_2} + \frac{a_1^2}{r_2r_0} + \frac{a_2^2}{r_0r_1} \geqslant 4. \tag{1}$$

Let P be an arbitrary interior point of the triangle ABC. Gerasimov [2] obtained an inequality for the triangle ABC as follows

$$\frac{d_1d_2}{a_1a_2} + \frac{d_2d_0}{a_2a_0} + \frac{d_0d_1}{a_0a_1} \leqslant \frac{1}{4}. \tag{2}$$

We extended inequalities (1) and (2) to an n-dimensional simplex. Our main results are the following theorems.

THEOREM 1. *For the n-dimensional simplex σ_n we have*

$$\sum_{i=0}^{n} \frac{F_i^{\frac{n}{n-1}}}{r_0 \ldots r_{i-1}r_{i+1} \ldots r_n} \geqslant \frac{(n-1)^n n^{\frac{3n^2}{2(n-1)}}}{n^n(n+1)^{\frac{n-2}{2}}(n!)^{\frac{n}{n-1}}}, \tag{3}$$

with equality if the simplex σ_n is regular.

If $n = 2$ in inequality (3), then inequality (1) is gotten from inequality (3).

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THEOREM 2. *Let P be an arbitrary interior point of the simplex σ_n and real number $\theta \in \langle 0,1]$, then we have*

$$\sum_{i=0}^{n} \frac{d_0 \dots d_{i-1} d_{i+1} \dots d_n}{(F_0 \dots F_{i-1} F_{i+1} \dots F_n)^{2\theta-1}} \leqslant \frac{(n!)^{2\theta}}{(n+1)^{(n-1)(1-\theta)} n^{n(3\theta-1)}} V^{n-2(n-1)\theta}, \qquad (4)$$

with equality if the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If take $\theta = \frac{n}{2(n-1)}$ in inequality (4), we get a corollary as follows.

COROLLARY 1. *Let P be an arbitrary interior point of the simplex σ_n, then we have*

$$\sum_{i=0}^{n} \frac{d_0 \dots d_{i-1} d_{i+1} \dots d_n}{(F_0 \dots F_{i-1} F_{i+1} \dots F_n)^{\frac{1}{n-1}}} \leqslant \frac{(n!)^{\frac{n}{n-1}}}{(n+1)^{\frac{n-2}{2}} n^{\frac{n(n+2)}{2(n-1)}}}, \qquad (5)$$

with equality if the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If $n = 2$ in inequality (5), then inequality (2) is gotten from inequality (5).
If taking $\theta = \frac{1}{2}$ in inequality (4), then we obtain a generalization of Gerber's inequality as follows.

COROLLARY 2. *Let P be an arbitrary interior point of the simplex σ_n, then*

$$\sum_{i=0}^{n} d_0 \dots d_{i-1} d_{i+1} \dots d_n \leqslant \frac{n!}{(n+1)^{\frac{n-1}{2}} n^{\frac{n}{2}}} V, \qquad (6)$$

with equality if the simplex σ_n is regular.

Using inequality (6) and the arithmetic-geometric mean inequality we get Gerber's inequality [3] as follows

$$\prod_{i=0}^{n} d_i \leqslant \frac{(n!)^{\frac{n+1}{n}}}{n^{\frac{n+1}{2}} (n+1)^{\frac{1}{2n}}} V^{\frac{n+1}{n}}. \qquad (7)$$

THEOREM 3. *Let P be an arbitrary interior point of the simplex σ_n, then we have*

$$\sum_{i=0}^{n} \frac{1}{d_0 \dots d_{i-1} d_{i+1} \dots d_n} \geqslant (n+1) n^{n+1} \frac{r}{R^{n+1}}, \qquad (8)$$

with equality if the simplex σ_n is regular and the point P is the circumcenter of σ_n.

If the point P is the incenter I of the simplex σ_n, then $d_i = r \ (i = 0, 1, \dots, n)$, and following n-dimensional Euler inequality in [4] is gotten from (8)

$$R \geqslant nr. \qquad (9)$$

2. Lemmas and proof of theorems

To prove theorems stated above, we need some lemmas as follows.

Let m_i $(i = 0, 1, \dots, n)$ be positive numbers, $V_{i_0 i_1 \dots i_k}$ denote the k-dimensional volume of the k-dimensional simplex $A_{i_0}A_{i_1}\dots A_{i_k}$ for $A_{i_0}, A_{i_1}, \dots, A_{i_k} \in \tau$. Put

$$M_k = \sum_{0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n} m_{i_0} m_{i_1} \dots m_{i_k} V^2_{i_0 i_1 \dots i_k} \qquad (1 \leqslant k \leqslant n),$$

$$M_0 = \sum_{i=0}^{n} m_i.$$

LEMMA 1. *For positive numbers m_i $(i = 0, 1, \dots, n)$ and the n-dimensional simplex σ_n, we have*

$$M_k^l \geqslant \frac{[(n-l)!(l!)^3]^k}{[(n-k)!(k!)^3]^l} (n! M_0)^{l-k} M_l^k \qquad (1 \leqslant k < l \leqslant n), \tag{10}$$

with equality if the simplex σ_n is regular and $m_0 = m_1 = \cdots = m_n$.

LEMMA 2.

$$\left(\prod_{i=0}^{n} F_i\right)^{\frac{n}{n^2-1}} \geqslant \frac{1}{(n+1)^{\frac{1}{2}}} \left(\frac{n^{3n}}{n!^2}\right)^{\frac{1}{2(n-1)}} V_n, \tag{11}$$

with equality if the simplex σ_n is regular.

For the proof of Lemmas 1 and 2, the reader is referred to [5] or [1].

LEMMA 3.

$$\sum_{i=0}^{n} \frac{h_0 \dots h_{i-1} h_{i+1} \dots h_n}{r_0 \dots r_{i-1} r_{i+1} \dots r_n} \geqslant (n+1)(n-1)^n, \tag{12}$$

with equality if the simplex σ_n is regular.

For the proof of Lemma 3, the reader is referred to [6].

LEMMA 4.

$$V \geqslant \frac{n^{\frac{n}{2}} (n+1)^{\frac{n+1}{2}}}{n!} r^n, \tag{13}$$

with equality if the simplex σ_n is regular.

For the proof of Lemma 4, the reader is referred to [5] or [1].

Proof of Theorem 1. Without lose generality, let $F_0 \leqslant F_1 \leqslant \cdots \leqslant F_n$.

By the formula [1]

$$r_i = \frac{nV}{\sum_{j=0}^{n} F_j - 2F_i} \qquad (i = 0, 1, \dots, n), \tag{14}$$

we know that $r_0 \leqslant r_1 \leqslant \cdots \leqslant r_n$ and

$$\frac{1}{\sum\limits_{j=1}^{n} F_j r_j} \leqslant \frac{1}{\sum\limits_{\substack{j=0\\ j\neq 1}}^{n} F_j r_j} \leqslant \ldots \leqslant \frac{1}{\sum\limits_{\substack{j=0\\ j\neq n}}^{n} F_j r_j}.$$

Using Chebyshev inequality we have

$$\begin{aligned} \sum_{i=0}^{n} \frac{F_i^{\frac{n}{n-1}}}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} r_j} &= \left(\prod_{i=0}^{n} F_i\right) \sum_{i=0}^{n} \frac{F_i^{\frac{1}{n-1}}}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} F_j r_j} \\ &\geqslant \frac{1}{n+1}\left(\prod_{i=0}^{n} F_i\right)\left(\sum_{i=0}^{n} F_i^{\frac{1}{n-1}}\right)\left(\sum_{i=0}^{n} \frac{1}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} F_j r_j}\right). \end{aligned} \tag{15}$$

Substituting $F_j = \frac{nV}{h_j}$ $(j = 0, 1, \ldots, n)$ into the right of inequality (15) and using the arithmetic-geometric mean inequality we get

$$\begin{aligned} \sum_{i=0}^{n} \frac{F_i^{\frac{n}{n-1}}}{\prod\limits_{\substack{j=0\\ j\neq i}}^{n} r_j} &\geqslant \frac{1}{n+1}\left(\prod_{i=0}^{n} F_i\right)\left(\sum_{i=0}^{n} F_i^{\frac{1}{n-1}}\right) \cdot \frac{1}{(nV)^n} \sum_{i=0}^{n} \frac{h_0 \ldots h_{i-1} h_{i+1} \ldots h_n}{r_0 \ldots r_{i-1} r_{i+1} \ldots r_n} \\ &\geqslant \frac{\left(\prod\limits_{i=0}^{n} F_i\right)^{\frac{n^2}{n^2-1}}}{(nV)^n} \sum_{i=0}^{n} \frac{h_0 \ldots h_{i-1} h_{i+1} \ldots h_n}{r_0 \ldots r_{i-1} r_{i+1} \ldots r_n}. \end{aligned} \tag{16}$$

Using inequalities (16), (11) and (12) we obtain inequality (3). It is easy to see that equality in (3) holds if the simplex σ_n is regular. □

Proof of Theorem 2. Taking $k = n-1$, $l = n$ in inequality (10), we have

$$\left(\sum_{i=0}^{n} m_0 \ldots m_{i-1} m_{i+1} \ldots m_n F_i^2\right)^n \geqslant \frac{n^{3n}}{n!^2}\left(\sum_{i=0}^{n} m_i\right)\left(\prod_{i=0}^{n} m_i\right)^{n-1} V^{2(n-1)}. \tag{17}$$

Put $m_0 \ldots m_{i-1} m_{i+1} \ldots m_n = \lambda_i F_i^{-2}$ $(i = 0, 1, \ldots, n)$ in equality (17), we get

$$\left(\frac{1}{n}\sum_{i=0}^{n} \lambda_i\right)^n \left(\prod_{i=0}^{n} F_i^2\right) \geqslant \frac{(nV)^{2(n-1)}}{(n-1)!^2}\left(\prod_{i=0}^{n} \lambda_i\right)\left(\sum_{i=0}^{n} \frac{F_i^2}{\lambda_i}\right). \tag{18}$$

Now we prove the following inequality (19) is valid for any number $\theta \in \langle 0, 1]$.

$$\left(\frac{1}{n}\sum_{i=0}^{n} \lambda_i\right)^n \prod_{i=0}^{n} F_i^{2\theta} \geqslant \left(\prod_{i=0}^{n} \lambda_i\right)\left(\sum_{i=0}^{n} \frac{F_i^{2\theta}}{\lambda_i}\right) \frac{(n+1)^{2(n-1)\theta}}{n^{n(1-\theta)}} \cdot \frac{(nV)^{2(n-1)\theta}}{(n-1)!^{2\theta}}. \tag{19}$$

When $\theta = 1$, inequalities (19) and (18) are same, so inequality (19) is valid for $\theta = 1$. For $\theta \in \langle 0, 1\rangle$, using inequality (18) we have

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n \prod_{i=0}^{n} F_i^{2\theta} = \left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n \prod_{i=0}^{n} F_i^{2}\right]^\theta \left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\right]^{1-\theta}$$
$$\geqslant \left[\frac{(nV)^{2(n-1)}}{(n-1)!^2}\left(\prod_{i=0}^{n}\lambda_i\right)\left(\sum_{i=0}^{n}\frac{F_i^2}{\lambda_i}\right)\right]^\theta \left[\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n\right]^{1-\theta}. \tag{20}$$

By Marclaurin inequality [1] we have

$$\left(\frac{1}{n+1}\sum_{i=0}^{n}\lambda_0\ldots\lambda_{i-1}\lambda_{i+1}\ldots\lambda_n\right)^{\frac{1}{n}} \leqslant \frac{1}{n+1}\sum_{i=0}^{n}\lambda_i,$$

i. e.

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n \geqslant \frac{(n+1)^{n-1}}{n^n}\left(\prod_{i=0}^{n}\lambda_i\right)\left(\sum_{i=0}^{n}\frac{1}{\lambda_i}\right). \tag{21}$$

From (20) and (21) we get

$$\left(\frac{1}{n}\sum_{i=0}^{n}\lambda_i\right)^n \prod_{i=0}^{n} F_i^{2\theta} \geqslant \left(\sum_{i=0}^{n}\lambda_i\right)\left[\sum_{i=0}^{n}\left(\frac{F_i^{2\theta}}{\lambda_i^{\theta}}\right)^{\frac{1}{\theta}}\right]^\theta \left[\sum_{i=0}^{n}\left(\frac{1}{\lambda_i^{1-\theta}}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \times$$
$$\times \left[\frac{(n+1)^{n-1}}{n^n}\right]^{1-\theta}\left[\frac{(nV)^{2(n-1)}}{(n-1)!^2}\right]^\theta. \tag{22}$$

By Hölder inequality [1] we have

$$\left[\sum_{i=0}^{n}\left(\frac{F_i^{2\theta}}{\lambda_i^{\theta}}\right)^{\frac{1}{\theta}}\right]^\theta \left[\sum_{i=0}^{n}\left(\frac{1}{\lambda_i^{1-\theta}}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \geqslant \sum_{i=0}^{n}\frac{F_i^{2\theta}}{\lambda_i}. \tag{23}$$

Using (22) and (23) we get inequality (18).

Taking $\lambda_i = d_iF_i$ $(i = 0, 1, \ldots, n)$ in equality (18) and noting the fact $\sum_{i=0}^{n} d_iF_i = nV$, we get inequality (4). It is easy to prove that equality in (4) holds if the simplex σ_n is regular and the point P is the circumcenter of σ_n. □

Proof of Theorem 3. Inequality (18) can be written as

$$\frac{n^{3n}}{n!^2}V^{2(n-1)}\sum_{i=0}^{n}\lambda_0\ldots\lambda_{i-1}\lambda_{i+1}\ldots\lambda_n F_i^2 \leqslant \left(\sum_{i=0}^{n}\lambda_i\right)^n \prod_{i=0}^{n} F_i. \tag{24}$$

Let V' denote the volume of the n-dimensional simplex $\sigma_n' = A_0'A_1' \dots A_n'$, F_i' the area of the i-th face f_i' of σ_n'. By Cauchy inequality and inequality (24), we have

$$\begin{aligned} &\frac{n^{3n}}{n!^2} V^{n-1} V'^{n-1} \sum_{i=0}^{n} \lambda_0 \dots \lambda_{i-1}\lambda_{i+1} \dots \lambda_n F_i F_i' \\ &\quad \leqslant \left[\frac{n^{3n}}{n!^2} V^{2(n-1)} \sum_{i=0}^{n} \lambda_0 \dots \lambda_{i-1}\lambda_{i+1} \dots \lambda_n F_i^2\right]^{\frac{1}{2}} \times \\ &\quad \times \left[\frac{n^{3n}}{n!^2} V'^{2(n-1)} \sum_{i=0}^{n} \lambda_0 \dots \lambda_{i-1}\lambda_{i+1} \dots \lambda_n F'^2_i\right]^{\frac{1}{2}} \\ &\quad \leqslant \left(\sum_{i=0}^{n} \lambda_i\right)^n \left(\prod_{i=0}^{n} F_i\right) \left(\prod_{i=0}^{n} F_i'\right). \end{aligned} \tag{25}$$

If take σ_n' is a regular simplex and $F_0' = F_1' = \dots = F_n' = 1$, then

$$V' = (n+1)^{\frac{1}{2}} \left(\frac{n!^2}{n^{3n}}\right)^{\frac{1}{2(n-1)}},$$

and inequality (25) is

$$\frac{(n+1)^{\frac{n-1}{2}} n^{\frac{3n}{2}}}{n!} V^{n-1} \sum_{i=0}^{n} \lambda_0 \dots \lambda_{i-1}\lambda_{i+1} \dots \lambda_n F_i \leqslant \left(\sum_{i=0}^{n} \lambda_i\right) \prod_{i=0}^{n} F_i. \tag{26}$$

Put $\lambda_0 = \lambda_1 = \dots = \lambda_n = 1$ in inequality (26), we get

$$\frac{1}{V} \geqslant \frac{n^{\frac{3n}{2(n-1)}}}{n!^{\frac{1}{n-1}} (n+1)^{\frac{n+1}{2(n-1)}}} \left(\sum_{i=0}^{n} \frac{1}{F_0 \dots F_{i-1}F_{i+1} \dots F_n}\right)^{\frac{1}{n-1}}. \tag{27}$$

By Cauchy inequality we have

$$\left(\sum_{i=0}^{n} d_0 \dots d_{i-1}d_{i+1} \dots d_n\right) \left(\sum_{i=0}^{n} \frac{1}{d_0 \dots d_{i-1}d_{i+1} \dots d_n}\right) \geqslant (n+1)^2,$$

i. e.

$$\sum_{i=0}^{n} \frac{1}{d_0 \dots d_{i-1}d_{i+1} \dots d_n} \geqslant \frac{(n+1)^2}{\sum_{i=0}^{n} d_0 \dots d_{i-1}d_{i+1} \dots d_n}. \tag{28}$$

Using (28), (6) and (27), we get

$$\begin{aligned} \sum_{i=0}^{n} \frac{1}{d_0 \dots d_{i-1}d_{i+1} \dots d_n} &\geqslant \frac{(n+1)^{\frac{n+3}{2}} n^{\frac{n}{2}}}{n!} \cdot \frac{1}{V} \\ &\geqslant \frac{(n+1)^{\frac{n^2+n-4}{2(n-1)}} n^{\frac{n^2}{2(n-1)}}}{(n-1)!^{\frac{n}{n-1}}} \left(\sum_{i=0}^{n} F_i\right)^{\frac{1}{n-1}} \left(\frac{1}{\sum_{i=0}^{n} F_i}\right)^{\frac{1}{n-1}}. \end{aligned} \tag{29}$$

By inequality (29), formula [1] $\sum_{i=0}^{n} F_i = \frac{nV}{r}$ and following inequality [1]:

$$\prod_{i=0}^{n} F_i \leqslant \frac{(n+1)^{\frac{n^2-1}{2}}}{n!^{n+1} n^{\frac{n^2-3n-4}{2}}} R^{n^2-1}, \tag{30}$$

we get

$$\sum_{i=0}^{n} \frac{1}{d_0 \dots d_{i-1} d_{i+1} \dots d_n} \geqslant (n+1)^{\frac{n-3}{2(n-1)}} n^{\frac{2n^2-n-2}{2(n-1)}} n!^{\frac{1}{n-1}} \left(\frac{V}{r}\right)^{\frac{1}{n-1}} \cdot \frac{1}{R^{n+1}}. \tag{31}$$

From (31) and (13) we get inequality (8). It is easy to prove that equality in (8) holds if the simplex σ_n is regular and the point P is the circumcenter of σ_n. □

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