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A TYPE OF MONOTONICITY ON THE L_p CENTROID BODY AND L_p PROJECTION BODY

WANG WEIDONG, LU FENGHONG AND LENG GANGSONG

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Abstract. Associated with L_p centroid body and L_p projection body, Lutwak, Yang and Zhang recently made a series of studies. In this paper, associated with the L_p mixed volume and dual mixed volumes, we establish several inequalities for the monotonicity of L_p centroid body and L_p projection body.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$, denote by S^{n-1} the unit sphere in $\mathbb{R}^n$. If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (0, \infty)$, is defined by

$$h(K,u) = \max\{u \cdot x : x \in K\}, u \in S^{n-1}$$

where $u \cdot x$ denotes the standard inner product of u and x and $h(K,u)$ is written as $h_K(u)$.

For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, denote by $V(K)$ its n-dimensional volume, for real number $p \geqslant 1$, the L_p centroid body of K, $\Gamma_p K$, is the convex body, whose support function is defined by [9,14]

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |\, u \cdot x \,|^p \, dx. \tag{1.1}$$

where the integration is with respect to Lebesgue measure and

$$c_{n,p} = \omega_{n+p}/\omega_2 \omega_n \omega_{p-1}, \tag{1.2}$$

with

$$\omega_n = \pi^{\frac{n}{2}}/\Gamma(1+\frac{n}{2}).$$

here ω_n is the n-dimensional volume of the unit ball B in $\mathbb{R}^n$.

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For L_p centroid body of K, Γ_pK, Lutwak, Yang and Zhang conjectured (see [14]) and proved (see [9]) the following result, respectively.

If K is a star body (about the origin) in $\mathbb{R}^n$, then for $p \geqslant 1$

$$V(\Gamma_pK) \geqslant V(K), \tag{1.3}$$

with equality if and only if K is an ellipsoid centered at the origin.

The proof of inequality (1.3) involves the L_p-analog of the Petty projection inequality. The authors first presented the notion of the $L_p(p > 1)$ projection body in [9].

For each convex body $K \subset \mathbb{R}^n$ and for each real number $p > 1$, define the L_p projection body, Π_pK, of K to be the origin-symmetric convex body whose support function is given by [9]

$$h^p_{\Pi_pK}(u) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} | u \cdot v |^p \, dS_p(K, v), \tag{1.4}$$

where $u, v \in S^{n-1}$, and $S_p(K, v)$ is a positive Borel measure on S^{n-1}.

We denote the polar body of Γ_pK, Γ^*_pK, rather than $(\Gamma_pK)^*$, and the polar body of Π_pK, Π^*_pK, rather than $(\Pi_pK)^*$, respectively. Regarding the polar body of Π_pK, Π^*_pK, Lutwak, Yang and Zhang proved the following the L_p-Petty projection inequality [9]:

If $K \in \mathscr{K}^n$ in $\mathbb{R}^n$, then for $p \geqslant 1$,

$$V(K)^{(n-p)/p} V(\Pi^*_pK) \leqslant \omega_n^{n/p}, \tag{1.5}$$

with equality if and only if K is an ellipsoid centered at the origin.

It is shown in [9] that the L_p-Petty projection inequality (1.5) implies inequality (1.3).

If $p = 1$, we denote that Γ_1K and Π_1K are classical centroid body ΓK and projection body ΠK, respectively.

It is well-known that the study of the classical centroid bodies and projection bodies is one of the most prominent events in the Brunn-Minkowski theory (see [1,3,4,5,6,15,16]). Recently, associated with the $L_p(p \geqslant 1)$ centroid body and L_p ($p > 1$) projection body, Lutwak, Yang and Zhang made a series of studies, the $L_p(p \geqslant 1)$ centroid body was studied to already get many results (see [2,8,9,10,11,13,16]), the study of the L_p ($p > 1$) projection body see [9,10,12].

In this paper, we shall, associated with the L_p mixed volume V_p and L_p dual mixed volume V_{-p}, continuously study the $L_p(p \geqslant 1)$ centroid body and L_p ($p > 1$) projection body, to obtain the following main results about the monotonicity of the $L_p(p \geqslant 1)$ centroid body and $L_p(p > 1)$ projection body.

THEOREM A. *If K, L are star body (about the origin) in $\mathbb{R}^n$, for $p \geqslant 1$ and any star body (about the origin) Q in $\mathbb{R}^n$, $V_{-p}(K, Q) \leqslant V_{-p}(L, Q)$, then*

$$\frac{V(\Gamma^*_pK)^{\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma^*_pL)^{\frac{p}{n}}}{V(L)},$$

with equality if and only if $K = L$.

THEOREM B. *If* $K,L \in \mathscr{K}^n$ *in* $\mathbb{R}^n$, $V_p(K,Q) \leqslant V_p(L,Q)$ *for* $p \geqslant 1$ *and each* $Q \in \mathscr{K}^n$ *in* $\mathbb{R}^n$*, then*

$$V(\Pi_p K) \leqslant V(\Pi_p L),$$

with equality if and only if $\Pi_p K = \Pi_p L$.

This paper, except for the introduction, is divided into three sections. Section 2 contains some notions and background materials. We shall prove Theorem A and its polar form in section 3. Last section, we shall study the monotonicity of the L_p projection body.

2. Background materials

Let K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0,+\infty)$, is defined by

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\}, u \in S^{n-1}$$

and $\rho(K,u)$ is written as $\rho_K(u)$. If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathscr{K}^n$, we define the polar body of K, K^*, by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}.$$

obviously, we have $(K^*)^* = K$.

If K is a convex body, then the support and radial functions of K^*, the polar body of K, are defined respectively by

$$h_{K^*} = \frac{1}{\rho_K} \quad \text{and} \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.1}$$

For L_p-mixed and dual mixed volumes, those formulae are directly given as follows.

If $K,L \in \mathscr{K}^n$ in $\mathbb{R}^n$, then for $p \geqslant 1$, the L_p-mixed volume, $V_p(K,L)$, of the K and L was defined by (see [8,9])

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(v)^p dS_p(K,v). \tag{2.2}$$

If K, L are star bodies in $\mathbb{R}^n$, for $p \geqslant 1$, the dual mixed volume, $V_{-p}(K,L)$, of the K and L was defined by (see [9])

$$V_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_L^{-p}(v)dS(v), \tag{2.3}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.2) of the mixed volume V_p it follows immediately that for each convex body K,

$$V_p(K,K) = V(K). \tag{2.4}$$

From the formula (2.3) of the dual mixed volume V_{-p} it follows immediately the for each star body K,

$$V_{-p}(K,K) = V(K). \tag{2.5}$$

We shall require two basic inequalities for the L_p -mixed volume V_p and the dual mixed volumes V_{-p}. The L_p -Minkowski inequality states that for convex bodies K, L (see [8])

$$V_p(K,L) \geqslant V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}, \tag{2.6}$$

with equality if and only if K and L are dilates. The L_p -Minkowski inequality for the dual mixed volumes V_{-p} is that for star bodies K,L (see [9])

$$V_{-p}(K,L) \geqslant V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}}, \tag{2.7}$$

with equality if and only if K and L are dilates.

3. The monotonicity for the L_p centroid body

The monotonicity of the L_p centroid body, except the case of $\Gamma_p^* K$ in theorem A, include the case of $\Gamma_p K$ ($\Gamma_p K$ is the polar body of $\Gamma_p^* K$). We, corresponding to Theorem A, shall establish a inequality which associated with the case of $\Gamma_p K$. Those results are stated by The Theorem 1 and Theorem $1'$ as follows.

THEOREM 1. *If K, L are star bodies (about the origin) in $\mathbb{R}^n$, for $p \geqslant 1$ and any star body (about the origin) Q in $\mathbb{R}^n$, $V_{-p}(K,Q) \leqslant V_{-p}(L,Q)$, then*

$$\frac{V(\Gamma_p^* K)^{\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p^* L)^{\frac{p}{n}}}{V(L)}, \tag{3.1}$$

with equality if and only if $K = L$.

THEOREM $1'$. *If K, L are star bodies (about the origin) in $\mathbb{R}^n$, for $p \geqslant 1$ and any star body (about the origin) Q in $\mathbb{R}^n$, $V_{-p}(K,Q) \leqslant V_{-p}(L,Q)$, then*

$$\frac{V(\Gamma_p K)^{-\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p L)^{-\frac{p}{n}}}{V(L)}, \tag{3.2}$$

with equality if and only if $K = L$.

In order to prove these Theorems, we will require lemmas as follows.

LEMMA 1. [9]. *If K is a star body (about the origin) and L is a convex body in $\mathbb{R}^n$, then for $p \geqslant 1$*

$$V_p(L, \Gamma_p K) = \frac{\omega_n}{V(K)} V_{-p}(K, \Pi_p^* L). \tag{3.3}$$

LEMMA 2. *If K, L are star bodies (about the origin) in $\mathbb{R}^n$, then for $p \geqslant 1$,*

$$\frac{V_{-p}(K, \Gamma_p^* L)}{V(K)} = \frac{V_{-p}(L, \Gamma_p^* K)}{V(L)}. \tag{3.4}$$

Proof. Using (1.1) and (2.1), we have

$$\rho_{\Gamma_p^* K}^{-p}(u) = h_{\Gamma_p K}^{p}(u) = \frac{1}{c_{n,p}V(K)} \int_K | u \cdot x |^p \, dx$$
$$= \frac{1}{(n+p)c_{n,p}V(K)} \int_{S^{n-1}} | u \cdot v |^p \, \rho_K^{n+p}(v) dS(v).$$

According to (2.3) we obtain

$$V_{-p}(L, \Gamma_p^* K) = \frac{1}{n} \int_{S^{n-1}} \rho_L^{n+p}(u) \rho_{\Gamma_p^* K}^{-p}(u) dS(u)$$
$$= \frac{1}{n(n+p)c_{n,p}V(K)} \int_{S^{n-1}} \int_{S^{n-1}} \rho_L^{n+p}(u) \rho_K^{n+p}(v) \, | u \cdot v |^p \, dS(v) dS(u)$$
$$= \frac{V(L)}{nV(K)} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_{\Gamma_p^* L}^{-p}(v) dS(v)$$
$$= \frac{V(L)}{V(K)} V_{-p}(K, \Gamma_p^* L). \quad \square$$

Proof of Theorem 1. Since $p \geqslant 1$, $V_{-p}(K,Q) \leqslant V_{-p}(L,Q)$ for any star body Q in $\mathbb{R}^n$, taking $Q = \Gamma_p^* M$ for any star body $M \in \mathbb{R}^n$. we have

$$V_{-p}(K, \Gamma_p^* M) \leqslant V_{-p}(L, \Gamma_p^* M), \tag{3.5}$$

with equality if and only if $K = L$.

Associated with inequality (3.5) and equality (3.4) in Lemma 2, we get that

$$\frac{V(K)V_{-p}(M, \Gamma_p^* K)}{V(M)} \leqslant \frac{V(L)V_{-p}(M, \Gamma_p^* L)}{V(M)}. \tag{3.6}$$

Taking $M = \Gamma_p^* L$ and using (2.5) and (2.7), we obtain that

$$\begin{aligned} V(L)V(\Gamma_p^* L) &\geqslant V(K)V_{-p}(\Gamma_p^* L, \Gamma_p^* K) \\ &\geqslant V(\Gamma_p^* L)^{\frac{n+p}{n}} V(\Gamma_p^* K)^{-\frac{p}{n}} V(K), \end{aligned} \tag{3.7}$$

with equality if and only if $\Gamma_p^* K$ and $\Gamma_p^* L$ are dilates in the second inequality of (3.7).

Using (3.7), we have that

$$\frac{V(\Gamma_p^* K)^{\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p^* L)^{\frac{p}{n}}}{V(L)},$$

We know that inequality (3.5) and (3.6) are equivalent by Lemma 1, but with equality if and only if $K = L$ in (3.5), and $K = L$ implies the equality holds in the second inequality of (3.7), we get that equality holds in (3.1) if and only if $K = L$. $\square$

Proof of Theorem 1$'$. For $p \geqslant 1$ and any star body Q in $\mathbb{R}^n$, since $V_{-p}(K,Q) \leqslant V_{-p}(L,Q)$, here taking for $Q = \Pi_p^* M$ where M is any convex body, we have

$$V_{-p}(K, \Pi_p^* M) \leqslant V_{-p}(L, \Pi_p^* M), \tag{3.8}$$

with equality if and only if $K = L$.

According to Lemma 1, we have

$$V(K)V_p(M,\Gamma_p K) \leqslant V_p(M,\Gamma_p L)V(L).$$

Taking $M = \Gamma_p L$ and using (2.4) and (2.6), we obtain

$$V(L)V(\Gamma_p L) \geqslant V(\Gamma_p L)^{\frac{n-p}{n}} V(\Gamma_p K)^{\frac{p}{n}} V(K), \tag{3.9}$$

with equality if and only if $\Gamma_p K$ and $\Gamma_p L$ dilates.

Using (3.9), we obtain that

$$\frac{V(\Gamma_p K)^{-\frac{p}{n}}}{V(K)} \geqslant \frac{V(\Gamma_p L)^{-\frac{p}{n}}}{V(L)}.$$

According to the case of equality holds in (3.8) and (3.9), we know that with equality if and only if $K = L$ in (3.2) of Theorem 1′. □

4. The monotonicity for the L_p projection body

This section, we shall study the monotonicity for L_p projection body and its polar body. Except Theorem B, we shall give a result which associated with the polar body $\Pi_p^* K$ of $\Pi_p K$. It is shown that the monotonicity of the L_p projection body and its polar body by the following two theorems.

THEOREM 2. *If $K,\ L \in \mathscr{K}^n$, $V_p(K,Q) \leqslant V_p(L,Q)$ for $p \geqslant 1$ and any $Q \in \mathscr{K}^n$, then*

$$V(\Pi_p K) \leqslant V(\Pi_p L), \tag{4.1}$$

with equality if and only if $\Pi_p K = \Pi_p L$.

THEOREM **2′** *If $K,\ L \in \mathscr{K}^n$, $V_p(K,Q) \leqslant V_p(L,Q)$ for $p \geqslant 1$ and any $Q \in \mathscr{K}^n$, then*

$$V(\Pi_p^* K) \geqslant V(\Pi_p^* L), \tag{4.2}$$

with equality if and only if $\Pi_p^* K = \Pi_p^* L$.

The proof of these theorems will use the following Lemma.

LEMMA 3. *Suppose $K,\ L \in \mathscr{K}^n$, then for $p \geqslant 1$,*

$$V_p(K,\Pi_p L) = V_p(L,\Pi_p K). \tag{4.3}$$

Proof. For $p = 1$ the identity of (4.3) was presented in [5].

For $p > 1$, by using (1.2) we obtain easily

$$nc_{n-2,p} = (n+p)c_{n,p}.$$

Associated with (1.4) we have that

$$h^p_{\Pi_p K}(u) = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |\ u \cdot v\ |^p \, dS_p(K,v).$$

According to (2.2) we get that

$$\begin{aligned}V_p(L,\Pi_pK) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_pK}(u)dS_p(L,u)\\ &= \frac{1}{n(n+p)c_{n,p}\omega_n}\int_{S^{n-1}}\int_{S^{n-1}} \mid u\cdot v\mid^p dS_p(K,v)dS_p(L,u)\\ &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Pi_pL}(v)dS_p(K,v)\\ &= V_p(K,\Pi_pL). \quad \square\end{aligned}$$

Taking $L=\Pi_pK$ in Lemma 3 and using (2.4), we have that

CORROLARY 1. *Suppose* $K\in\mathscr{K}^n$*, then for* $p\geqslant 1$*,*

$$V(\Pi_pK)=V_p(K,\Pi_p\Pi_pK)$$

Proof of the Theorem 2. Since K, $L\in\mathscr{K}^n$ and $V_p(K,Q)\leqslant V_p(L,Q)$ for $p\geqslant 1$ and any $Q\in\mathscr{K}^n$, taking $Q=\Pi_pM$ for any $M\in\mathscr{K}^n$, we get

$$V_p(K,\Pi_pM)\leqslant V_p(L,\Pi_pM), \tag{4.4}$$

using Lemma 3 then

$$V_p(M,\Pi_pK)\leqslant V_p(M,\Pi_pL), \tag{4.5}$$

taking $M=\Pi_pL$ and associated with (2.4) and (2.6), the result is that

$$V(\Pi_pL)\geqslant V_p(\Pi_pL,\Pi_pK)\geqslant V(\Pi_pL)^{\frac{n-p}{n}}V(\Pi_pK)^{\frac{p}{n}}, \tag{4.6}$$

thus

$$V(\Pi_pK)\leqslant V(\Pi_pL).$$

Because (4.4) and (4.5) are equivalent, but $V_p(M,\Pi_pK)=V_p(M,\Pi_pL)$ for any $M\in\mathscr{K}^n$ if and only if $\Pi_pK=\Pi_pL$ (Π_pK and Π_pL are special dilates), and equality holds in (4.6) if and only if Π_pK and Π_pL are dilates by (2.6), thus with equality if and only if $\Pi_pK=\Pi_pL$. $\square$

Proof of the Theorem $2'$. Because of $V_p(K,Q)\leqslant V_p(L,Q)$ for $p\geqslant 1$ and any $Q\in\mathscr{K}^n$, let $Q=\Gamma_pM$ for any $M\in\mathscr{K}^n$, we obtain

$$V_p(K,\Gamma_pM)\leqslant V_p(L,\Gamma_pM). \tag{4.7}$$

by Lemma 1, we have

$$V_{-p}(M,\Pi^*_pK)\leqslant V_{-p}(M,\Pi^*_pL), \tag{4.8}$$

taking $M=\Pi^*_pL$ and using (2.5) and (2.7) we get that

$$V(\Pi^*_pL)\geqslant V_{-p}(\Pi^*_pL,\Pi^*_pK)\geqslant V(\Pi^*_pL)^{\frac{n+p}{n}}V(\Pi^*_pK)^{-\frac{p}{n}}, \tag{4.9}$$

thus

$$V(\Pi^*_pK)\geqslant V(\Pi^*_pL).$$

According to Lemma 1 we know that (4.7) and (4.8) are equivalent, but with equality if and only if $\Pi_p^* K = \Pi_p^* L$, in (4.8), and equality holds of the second inequality in (4.9) if and only if $\Pi_p^* K$ and $\Pi_p^* L$ are dilates by (2.7), thus with equality if and only if $\Pi_p^* K = \Pi_p^* L$. □

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Wang Weidong
Department of Mathematics
Hubei Institute for nationalities
Hubei Enshi, 445000
P. R. China
Department of Mathematics
Shanghai University
Shanghai, 200436
P. R. China
e-mail: gleng@mail.shu.edu.cn
e-mail: wangwd722@eyou.com

Lu Fenghong
Department of Mathematics
Shanghai University
Shanghai, 200436
P. R. China

Leng Gangsong
Department of Mathematics
Shanghai University
Shanghai, 200436
P. R. China