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BOUNDS FOR EXTREME SINGULAR VALUES OF A COMPLEX MATRIX AND ITS APPLICATIONS

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Abstract. In this study, we have obtained bounds for extreme singular values of a complex matrix A of order $n \times n$.

In addition, we have found a bounds for the extreme singular values of Hilbert matrix, its Hadamard square root, Cauchy-Toeplitz matrix, Cauchy-Hankel matrix in the forms

$$H = (1(i+j-1))_{i,j=1}^{n}, \qquad H^{\circ 1/2} = (1\,(i+j-1)^{1/2})_{i,j=1}^{n},$$

$$T_n = [1(g+(i-j)h)]_{i,j=1}^{n} \quad \text{and} \quad H_n = [1(g+(i+j)h)]_{i,j=1}^{n},$$

respectively.

1. Introduction and preliminaries

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with all positive entries. Then the Hadamard inverse of A given by $A^{\circ(-1)} = (1/a_{ij})_{i,j=1}^{n}$ is positive semidefinite and the Hadamard square root by $A^{\circ 1/2} = (a_{ij}^{1/2})_{i,j=1}^{n}$ [6].

The matrix

$$H = (1/(i+j-1))_{i,j=1}^{n} \tag{1.1}$$

is well known as Hilbert matrix. Hence the Hadamard square root of Hilbert matrix, denoted by

$$H^{\circ 1/2} = (1/(i+j-1)^{1/2})_{i,j=1}^{n}. \tag{1.2}$$

Let $C = [1/(x_i - y_j)]_{i,j=1}^{n} (x_i \neq y_j)$ be a Cauchy matrix and $T_n = [t_{j-i}]_{i,j=0}^{n-1}$ be a Toeplitz matrix. In generally Cauchy-Toeplitz matrix is being defined as

$$T_n = \left[\frac{1}{g+(i-j)h}\right]_{i,j=1}^{n} \tag{1.3}$$

where $h \neq 0$, g and h are any numbers and g/h is not an integer. Toeplitz matrices are precisely those matrices that have constant values along all diagonals parallel to the main diagonal and thus a Toeplitz matrix is determined by its first row and column.

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On the other hand, let $H_n = [h_{i+j}]_{i,j=0}^{n-1}$ be a Hankel matrix. Every $n \times n$ Cauchy-Hankel matrix is of the form

$$H_n = \left[\frac{1}{g+(i+j)h}\right]_{i,j=1}^{n} \tag{1.4}$$

where $h \neq 0$, g and h are any numbers and g/h is not an integer. Hankel matrices are symmetrical.

Recently, there have been several papers on the norms of Cauchy-Toeplitz matrix and Cauchy-Hankel matrix $[1,2,4,5]$. Refs. $[4,5]$ are related to the spectral norm of Cauchy-Toeplitz matrix. In [5], a lower bound for the spectral norm of Cauchy-Toeplitz matrix was obtained by Tyrtyshnikov taking $g = 1/2$ and $h = 1$ in the (1.3). Parter proved that singular values could be related to eigenvalues of certain Hermitian Toeplitz matrices corresponding to Laurent-Fourier series [4]. Güngör [9] obtained lower bounds for the spectral norm and Euclidean norm of Cauchy-Toeplitz and Cauchy-Hankel matrices in the forms (1.3) and (1.4) by taking $g = 1/2$ and $h = 1$ using B matrix which is defined in [8].

In this paper, firstly, we have established a lower and upper bound extreme singular values of a complex matrix A of order $n \times n$ using B matrix is defined in [8]. In Section 3, we have obtained an upper and lower bound for extreme singular values of the Hilbert matrix and its Hadamard square root. In Section 4, we have obtained bounds for extreme singular values of Cauchy-Toeplitz and Cauchy-Hankel matrices. Consequently, we have given an example related to this bounds which are found.

Now, we give some preliminaries related to our study. Let A be an $n \times n$ complex matrix. Let $\sigma_i(A)$-s $(i = 1, \ldots, n)$ such that $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A)$ be the singular values of A .

A function Ψ is called a psi (or digamma) function if

$$\Psi(x) = \frac{d}{d(x)}\{\log[\Gamma(x)]\} \quad \text{where} \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

The n-th derivate of a psi function is called a polygamma function, i.e.

$$\Psi(n,x) = \frac{d}{dx^n}[\Psi(x)] = \frac{d}{dx^n}\left\{\frac{d}{dx}(\ln[\Gamma(x)])\right\} \text{ [5].}$$

If $n = 0$ then $\Psi(n,x) = \Psi(x) = \frac{d}{dx}\{\ln[\Gamma(x)]\}$. On the other hand, if $a > 0$ and b are any numbers and n is a positive integer, then

$$\lim_{n\to\infty} \Psi(a, n+b) = 0 \text{ [3].}$$

To minimize the numerical round-off errors in solving the system $Ax = b$, it is normally convenient that the rows of A be properly scaled before the solution procedure begins. One way is to premultiply by the diagonal matrix

$$D = \operatorname{diag}\left\{\frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \ldots, \frac{\alpha_n}{r_n(A)}\right\}, \tag{1.5}$$

where $r_i(A)$ is the Euclidean norm of the i-th row of A and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers such that

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n. \tag{1.6}$$

Clearly, the Euclidean norm of the coefficient matrix $B = DA$ of the scaled system is equal to $\sqrt{n}$ and if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ then each row of B is a unit vector in the Euclidean norm. Also, we can define $B = AD$,

$$D = \operatorname{diag}\left\{\frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \ldots, \frac{\alpha_n}{c_n(A)}\right\}, \tag{1.7}$$

where $c_i(A)$ is the Euclidean norm of the i-th columm of A. Again, $\| B\|_E = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ then each column of B is a unit vector in the Euclidean norm.

Since the matrices PA, AP and A have the same singular values for any permutation matrix P, we assume, without loss of generality, that the rows and columns of A are such that

$$r_1(A) \leqslant r_2(A) \leqslant \cdots \leqslant r_n(A), \tag{1.8}$$

$$c_1(A) \leqslant c_2(A) \leqslant \cdots \leqslant c_n(A), \tag{1.9}$$

and α_i-s in (1.6) are ordered in such a way that

$$0 < \alpha_n \leqslant \cdots \leqslant \alpha_2 \leqslant \alpha_1. \tag{1.10}$$

THEOREM 1. [10] *Assume that A and B are two arbitrary $n \times n$ matrices with the singular values*

$$\sigma_n(A) \leqslant \cdots \leqslant \sigma_1(A), \ \sigma_n(B) \leqslant \cdots \leqslant \sigma_1(B).$$

Then the singular values

$$\sigma_n(AB) \leqslant \cdots \leqslant \sigma_1(AB)$$

of the matrix AB satisfy

(a) $\sigma_i(AB)=\theta_i\sigma_i(A)=\eta_i\sigma_i(B), \sigma_n(B) \leqslant \theta_i \leqslant \sigma_1(B), \sigma_n(A) \leqslant \eta_i \leqslant \sigma_1(A), 1 \leqslant i \leqslant n,$

(b) $\sigma_i(AB) = w_i\sqrt{\sigma_i(A)\sigma_i(B)}, \sqrt{\sigma_n(A)\sigma_n(B)} \leqslant w_i \leqslant \sqrt{\sigma_1(A)\sigma_1(B)} 1 \leqslant i \leqslant n$

2. Bounds for extreme singular values of a complex matrix

THEOREM 2. *Let A be an $n \times n$ complex matrix. Let α_i-s and $r_i(A)$-s $(c_i(A)$-s$)$ be as in (1.10) and (1.8) $((1.9))$, respectively. Then*

$$\sigma_n(A) \leqslant \left[n \left(\sum_{i=1}^{n} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} \right)^{-1} \right]^{1/2} \tag{2.1}$$

and

$$\left[n \left(\sum_{i=1}^{n} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} \right)^{-1} \right]^{1/2} \leqslant \sigma_1(A). \tag{2.2}$$

Proof. Firstly, we can write Theorem 1 (a) in the form

$$\sigma_i(AB) = \theta_i \sigma_i(A) = \eta_i \sigma_i(B)$$

where $\sigma_n(B) \leqslant \theta_i \leqslant \sigma_1(B), \sigma_n(A) \leqslant \eta_i \leqslant \sigma_1(A), 1 \leqslant i \leqslant n.$ Hence, we obtain that

$$\sigma_n(B)\sigma_i(A) \leqslant \sigma_i(AB) \leqslant \sigma_1(B)\sigma_i(A) \tag{2.3}$$

and

$$\sigma_n(A)\sigma_i(B) \leqslant \sigma_i(AB) \leqslant \sigma_1(A)\sigma_i(B). \tag{2.4}$$

By applying $B = DA$ and $B = AD$ matrices to (2.3) and (2.4), respectively, we have

$$\sigma_n(A)\sigma_i(D) \leqslant \sigma_i(B) \leqslant \sigma_1(A)\sigma_i(D).$$

Hence, (2.1) and (2.2) are obvious. □

3. Bounds for extreme singular values of Hilbert matrix and its Hadamard square root

THEOREM 3. *Let the matrix H and α_i-s $(i = 0, \ldots, n-1$ and $\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{n-1}^2 = n)$ be as in (1.1) and (1.10), respectively. Then*

$$\sigma_n(H) \leqslant \left[n\left(\sum_{i=0}^{n-1} \frac{\alpha_i^2}{-\Psi(1,n+i+1) + \Psi(1,i+1)}\right)^{-1} \right]^{1/2} \leqslant \sigma_1(H). \tag{3.1}$$

Proof. For the matrix H in (1.1) we have

$$\sum_{i=0}^{n-1} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} = \sum_{i=0}^{n-1} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} = \sum_{i=0}^{n-1} \frac{\alpha_i^2}{-\Psi(1,n+i+1) + \Psi(1,i+1)}.$$

By applying this equality to Theorem 2, we obtain (3.1). □

THEOREM 4. *Let the matrix $H^{\circ 1/2}$ and α_i-s $(i = 0, \ldots, n-1$ and $\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_{n-1}^2 = n)$ be as in (1.2) and (1.10), respectively. Then*

$$\sigma_n\left(H^{\circ 1/2}\right) \leqslant \left[n\left(\sum_{i=0}^{n-1} \frac{\alpha_i^2}{\Psi(n+i+1) - \Psi(i+1)}\right)^{-1} \right]^{1/2} \leqslant \sigma_1\left(H^{\circ 1/2}\right). \tag{3.2}$$

Proof. For the matrix $H^{\circ 1/2}$ in (1.2) we have

$$\sum_{i=0}^{n-1} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} = \sum_{i=0}^{n-1} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} = \sum_{i=0}^{n-1} \frac{\alpha_i^2}{\Psi(n+i+1) - \Psi(i+1)}.$$

By applying this equality to Theorem 2, we obtain (3.2). □

4. Bounds for extreme singular values of Cauchy-Toeplitz and Cauchy-Hankel matrices

THEOREM 5. *Let the matrix T_n and α_i-s be as in (1.3) and (1.10), respectively. Then*

$$\sigma_n(T_n) \leqslant \begin{cases} \sqrt{\dfrac{n}{a_1+a_2}}, & \text{if } n \text{ is odd} \\ \sqrt{\dfrac{n}{a_3}}, & \text{if } n \text{ is even} \end{cases} \tag{4.1}$$

and

$$\left.\begin{matrix} \sqrt{\dfrac{n}{a_4+a_5}}, & \text{if } n \text{ is odd} \\ \sqrt{\dfrac{n}{a_6}}, & \text{if } n \text{ is even} \end{matrix}\right\} \leqslant \sigma_1(T_n). \tag{4.2}$$

$$a_1 = \sum_{j=1}^{(n-1)/2} \frac{\alpha_j^2}{-\Psi\left(1, n+\frac{3}{2}-j\right)+\Psi\left(1, \frac{3}{2}-j\right)} + \frac{\alpha_{\frac{(n+1)}{2}}^2}{-\Psi\left(1, \frac{n}{2}\right)+\Psi\left(1, -\frac{n}{2}\right)},$$

$$a_2 = \sum_{i=(n+3)/2}^{n} \frac{\alpha_i^2}{-\Psi\left(1, n+\frac{1}{2}-i\right)+\Psi\left(1, \frac{1}{2}-i\right)},$$

$$a_3 = \sum_{j=1}^{n/2} \frac{\alpha_j^2}{-\Psi\left(1, n+\frac{3}{2}-j\right)+\Psi\left(1, \frac{3}{2}-j\right)} + \sum_{i=\frac{n}{2}+1}^{n} \frac{\alpha_i^2}{-\Psi\left(1, n+\frac{1}{2}-i\right)+\Psi\left(1, \frac{1}{2}-i\right)},$$

$$a_4 = \sum_{i=1}^{(n-1)/2} \frac{\alpha_i^2}{-\Psi\left(1, n+\frac{1}{2}-i\right)+\Psi\left(1, \frac{1}{2}-i\right)} + \frac{\alpha_{\frac{(n+1)}{2}}^2}{-\Psi\left(1, \frac{n}{2}\right)+\Psi\left(1, -\frac{n}{2}\right)},$$

$$a_5 = \sum_{j=(n+3)/2}^{n} \frac{\alpha_j^2}{-\Psi\left(1, n+\frac{3}{2}-j\right)+\Psi\left(1, \frac{3}{2}-j\right)}$$

and

$$a_6 = \sum_{i=1}^{n/2} \frac{\alpha_i^2}{-\Psi\left(1, n+\frac{1}{2}-i\right)+\Psi\left(1, \frac{1}{2}-i\right)} + \sum_{j=\frac{n}{2}+1}^{n} \frac{\alpha_j^2}{-\Psi\left(1, n+\frac{3}{2}-j\right)+\Psi\left(1, \frac{3}{2}-j\right)}.$$

Proof. For the matrix T_n in (1.3) we have

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} = \begin{cases} a_1+a_2, & \text{if } n \text{ is odd} \\ a_3, & \text{if } n \text{ is even} \end{cases}$$

and

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} = \begin{cases} a_4+a_5, & \text{if } n \text{ is odd} \\ a_6, & \text{if } n \text{ is even} \end{cases}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are as in Theorem 5.

By applying this equality to Theorem 2, we obtain (4.1) and (4.2). □

THEOREM 6. *Let the matrix H_n and α_i-s be as in (1.4) and (1.10), respectively. Then*

$$\sigma_n(H_n) \leqslant \left[n \left(\sum_{i=1}^{n} \frac{\alpha_i^2}{-\Psi(1, n+\frac{3}{2}+i) + \Psi(1, \frac{3}{2}+i)} \right)^{-1} \right]^{1/2} \leqslant \sigma_1(H_n). \qquad (4.3)$$

Proof. For the matrix H_n in (1.4) we have

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} = \sum_{i=1}^{n} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} = \sum_{i=1}^{n} \frac{\alpha_i^2}{-\Psi(1, n+\frac{3}{2}+i) + \Psi(1, \frac{3}{2}+i)}.$$

By applying this equality to Theorem 2, we obtain (4.3). □

THEOREM 7. *Let T_n and H_n be Cauchy-Toeplitz and Cauchy-Hankel matrices as in (1.3) and (1.4), respectively, where $g = 1/2$ and $h = 1$. Let α_i-s $(i = 1, \ldots, n$ and $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n)$ be as in (1.10). Let the operation "$\circ$" be a Hadamard product i.e., if $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices, then $A \circ B = (a_{ij} b_{ij})$. Then*

$$\sigma_n(T_n \circ H_n) \leqslant \left[n \left(\frac{\alpha_1^2}{a} + \sum_{i=2}^{n} \frac{\alpha_i^2}{b+c} \right)^{-1} \right]^{1/2} \qquad (4.4)$$

and

$$\left[n \left(\frac{\alpha_1^2}{d+e} + \sum_{j=2}^{n} \frac{\alpha_j^2}{f+g} \right)^{-1} \right]^{1/2} \leqslant \sigma_1(T_n \circ H_n) \qquad (4.5)$$

where

$$a = \frac{2\left[-n + 4(n+1)^2 + 4(n+1)^3\right]}{(3+2n)^2 (2n+1)^2} - \frac{1}{2}\Psi\left(1, n+\frac{1}{2}\right) - \frac{16}{9} + \frac{\pi^2}{4},$$

$$b = \frac{2\left[\Psi\left(i+\frac{3}{2}+n\right) - \Psi\left(-i+\frac{1}{2}+n\right) - \Psi\left(i+\frac{3}{2}\right) + \Psi\left(-i+\frac{1}{2}\right)\right]}{8i^3 + 12i^2 + 6i + 1},$$

$$c = \frac{-\Psi\left(1, i+\frac{3}{2}+n\right) - \Psi\left(1, -i+\frac{1}{2}+n\right) + \Psi\left(1, i+\frac{3}{2}\right) + \Psi\left(1, -i+\frac{1}{2}\right)}{4i^2 + 4i + 1},$$

$$d = \frac{8}{27} \frac{-26 - 53n + 144(n+1)^2 - 88(n+1)^3 - 48(n+1)^4 + 48(n+1)^5}{(3+2n)^2 (2n+1)^2 (-1+2n)^2},$$

$$e = -\frac{2}{9}\Psi\left(1, n-\frac{1}{2}\right) - \frac{8}{81} + \frac{\pi^2}{9},$$

$$f = \frac{1}{4j^3}\left[-\Psi\left(-j+\frac{3}{2}+n\right) + \Psi\left(j+\frac{3}{2}+n\right) + \Psi\left(-j+\frac{3}{2}\right) - \Psi\left(j+\frac{3}{2}\right)\right]$$

and

$$g = \tfrac{1}{4j^2}\left[-\Psi\left(1, -j+\tfrac{3}{2}+n\right) - \Psi\left(1, j+\tfrac{3}{2}+n\right) + \Psi\left(1, -j+\tfrac{3}{2}\right) + \Psi\left(1, j+\tfrac{3}{2}\right)\right].$$

Proof. For the matrix $T_n \circ H_n$, we have

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{\min\{r_i^2, c_i^2\}} = \frac{\alpha_1^2}{a} + \sum_{i=2}^{n} \frac{\alpha_i^2}{b+c}$$

and

$$\sum_{i=1}^{n} \frac{\alpha_i^2}{\max\{r_i^2, c_i^2\}} = \frac{\alpha_1^2}{d+e} + \sum_{j=2}^{n} \frac{\alpha_j^2}{f+g}$$

where a, b, c, d, e, f and g are as in Theorem 7.

By applying this equalities to (2.1) and (2.2), we obtain (4.4) and (4.5). □

5. Numerical results

We will take $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ in the following examples.

EXAMPLE 8. Let

$$x_1 = \left[n \left(\sum_{i=0}^{n-1} \frac{1}{-\Psi(1, n+i+1) + \Psi(1, i+1)} \right)^{-1} \right]^{1/2}.$$

For extreme singular values of the matrix H in (1.1), we have the following values:

n	$\sigma_n(H)$	x_1	$\sigma_1(H)$
5	$0.6869025215.10^{-5}$	0.4861024028	1.567050691
10	$0.00001397578013I$	0.3456005779	1.751919670
20	$0.00002191694535I$	0.2447829834	1.907134720
50	$0.00003397388936I$	0.1548996056	2.076296683

EXAMPLE 9. Let

$$x_2 = \left[n \left(\sum_{i=0}^{n-1} \frac{1}{\Psi(n+i+1) - \Psi(i+1)} \right)^{-1} \right]^{1/2}.$$

For extreme singular values of the matrix $H^{\circ 1/2}$ in (1.2), we have the following values:

n	$\sigma_n\left(H^{\circ 1/2}\right)$	x_2	$\sigma_1\left(H^{\circ 1/2}\right)$
5	$0.8127124207.10^{-5}$	1.052417644	2.533602599
10	$0.00003360072413I$	1.056539471	3.638302962
20	$0.00005898325435I$	1.058039521	5.180584414
50	$0.0001170827199I$	1.058039521	8.219123246

EXAMPLE 10. Let

$$x_3 = \begin{cases} \sqrt{\dfrac{n}{a_1 + a_2}}, & \text{if } n \text{ is odd} \\ \sqrt{\dfrac{n}{a_3}}, & \text{if } n \text{ is even} \end{cases} \quad \text{and} \quad x_4 = \begin{cases} \sqrt{\dfrac{n}{a_4 + a_5}}, & \text{if } n \text{ is odd} \\ \sqrt{\dfrac{n}{a_6}}, & \text{if } n \text{ is even} \end{cases}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are as in Theorem 5 . For extreme singular values of the matrix T_n in (1.3), we have the following values:

n	$\sigma_n(T_n)$	x_3	x_4	$\sigma_1(T_n)$
5	1.461864493	2.559714228	2.983318099	3.141589238
10	1.300969070	2.794198068	3.045265297	3.141592654
20	1.170504602	2.944783251	3.083324183	3.141592654
50	1.032417049	3.053228776	3.112592124	3.141592655

EXAMPLE 11. Let

$$x_5 = \left[n \left(\sum_{i=1}^{n} \frac{1}{-\Psi(1, n + \frac{3}{2} + i) + \Psi(1, \frac{3}{2} + i)} \right)^{-1} \right]^{1/2}.$$

For extreme singular values of the matrix H_n in (1.4), we have the following values:

n	$\sigma_n(H_n)$	x_5	$\sigma_1(H_n)$
5	$0.8421550433.10^{-5}I$	0.3612400002	0.928688586
10	$0.8548979537.10^{-5}I$	0.2937871206	1.167692602
20	$0.00002039511516I$	0.2247586482	1.387415414
50	$0.00002991650413I$	0.1495377488	1.639373809

EXAMPLE 12. Let

$$x_6 = \left[n \left(\frac{1}{a} + \sum_{i=2}^{n} \frac{1}{b+c} \right)^{-1} \right]^{1/2} \quad \text{and} \quad x_7 = \left[n \left(\frac{1}{d+e} + \sum_{j=2}^{n} \frac{1}{f+g} \right)^{-1} \right]^{1/2}$$

where a, b, c, d, e, f and g are as in Theorem 7. For extreme singular values of matrix $T_n \circ H_n$, we have the following values:

n	$\sigma_n(T_n \circ H_n)$	x_6	x_7	$\sigma_1(T_n \circ H_n)$
5	0.2018166931	0.3548145602	0.4524704029	0.999474704
10	0.0996616243	0.2120012139	0.2477198597	1.000163696
20	0.0477063748	0.1181900193	0.1294206340	1.000261912
50	0.0176375260	0.0510986596	0.0532203114	1.000276124

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