

SYMMETRIC NORMS AND REVERSE INEQUALITIES TO DAVIS AND HANSEN–PEDERSEN CHARACTERIZATIONS OF OPERATOR CONVEXITY

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*Dedicated to Françoise Piquard,
 respectfully and affectionately*

(communicated by F. Hansen)

Abstract. Let A, B, Z be n -by- n matrices. Suppose $AB \geq 0$ (positive semi-definite) and $Z > 0$ with extremal eigenvalues a and b . Then, the sharp inequality

$$\|ZAB\| \leq \frac{a+b}{2\sqrt{ab}} \|BZA\|$$

holds for every unitarily invariant norm. Among the consequences, we get the operator inequality $XZX \leq [(a+b)^2/4ab]Z$ for every $0 \leq X \leq I$, and some Kantorovich type inequalities (Mond-Pečarić inequalities). Also in connection, reverse inequalities of Davis and Hansen-Pedersen characterizations of operator convexity are established. For instance, given any operator convex function $f : [0, \infty) \rightarrow [0, \infty)$ and any subspace \mathcal{E} ,

$$f(Z_{\mathcal{E}}) \geq \frac{4ab}{(a+b)^2} (f(Z))_{\mathcal{E}}.$$

In passing, we point out a simplified proof of Hansen-Pedersen's inequality.

0. Introduction

Capital letters A, B, \dots, Z mean n -by- n complex matrices, or operators on a finite dimensional Hilbert space \mathcal{H} ; I stands for the identity. When A is positive semidefinite, resp. positive definite, we write $A \geq 0$, resp. $A > 0$. Let $\|\cdot\|$ be a general symmetric (or unitarily invariant) norm, i.e. $\|UAV\| = \|A\|$ for all A and all unitaries U, V . If A and B are such that the product AB is normal, then a classical inequality claims [1, p. 253]

$$\|AB\| \leq \|BA\| \tag{1}$$

Section 1 presents a generalization of (1) when $AB \geq 0$. Then, for $Z > 0$,

$$\|ZAB\| \leq \frac{a+b}{2\sqrt{ab}} \|BZA\| \tag{2}$$

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where a, b are the extremal eigenvalues of Z . Several sharp inequalities are derived. For instance, if $0 \leq X \leq I$, then

$$XZX \leq \frac{(a+b)^2}{4ab} Z.$$

Another example concerns compressions $Z_{\mathcal{E}}$ of Z onto subspaces $\mathcal{E} \subset \mathcal{H}$,

$$(Z_{\mathcal{E}})^{-1} \geq \frac{4ab}{(a+b)^2} (Z^{-1})_{\mathcal{E}}. \quad (3)$$

This Kantorovich type inequality is due to Mond-Pečarić. In Section 2 we extend (3) to all operator convex functions $f : [0, \infty) \rightarrow [0, \infty)$. Such inequalities are reverse inequalities to Davis' characterization of operator convexity via compressions. Equivalently, we show that, given any isometric column of operators $\{A_i\}_{i=1}^m$, i.e. $\sum A_i^* A_i = I$, we have

$$f\left(\sum A_i^* Z_i A_i\right) \geq \frac{4ab}{(a+b)^2} \sum A_i^* f(Z_i) A_i.$$

This is a reverse inequality to the Hansen-Pedersen inequality.

1. Norms inequalities

LEMMA 1.1. *Let $Z > 0$ with extremal eigenvalues a and b . Then, for every norm one vector h ,*

$$\|Zh\| \leq \frac{a+b}{2\sqrt{ab}} \langle h, Zh \rangle.$$

Proof. Let \mathcal{E} be any subspace of \mathcal{H} and let a' and b' be the extremal eigenvalues of $Z_{\mathcal{E}}$. Then $a \geq a' \geq b' \geq b$ and setting $t = \sqrt{a/b}$, $t' = \sqrt{a'/b'}$, we have $t \geq t' \geq 1$. Since $t \rightarrow t + 1/t$ increases on $[1, \infty)$ and

$$\frac{a+b}{2\sqrt{ab}} = \frac{1}{2} \left(t + \frac{1}{t} \right), \quad \frac{a'+b'}{2\sqrt{a'b'}} = \frac{1}{2} \left(t' + \frac{1}{t'} \right),$$

we infer

$$\frac{a+b}{2\sqrt{ab}} \geq \frac{a'+b'}{2\sqrt{a'b'}}.$$

Therefore, it suffices to prove the lemma for $Z_{\mathcal{E}}$ with $\mathcal{E} = \text{span}\{h, Zh\}$. Hence, we may assume $\dim \mathcal{H} = 2$, $Z = ae_1 \otimes e_1 + be_2 \otimes e_2$ and $h = xe_1 + (\sqrt{1-x^2})e_2$. Setting $x^2 = y$ we have

$$\frac{\|Zh\|}{\langle h, Zh \rangle} = \frac{\sqrt{a^2 y + b^2 (1-y)}}{ay + b(1-y)}.$$

The right hand side attains its maximum on $[0, 1]$ at $y = b/(a+b)$, and then

$$\frac{\|Zh\|}{\langle h, Zh \rangle} = \frac{a+b}{2\sqrt{ab}}$$

proving the lemma. \square

THEOREM 1.2. *Let A, B such that $AB \geq 0$. Let $Z > 0$ with extremal eigenvalues a and b . Then, for every symmetric norm, the following sharp inequality holds*

$$\|ZAB\| \leq \frac{a+b}{2\sqrt{ab}} \|BZA\|.$$

Proof. For the sharpness see Remark 1.9 below.

It suffices to consider the Fan k -norms $\|\cdot\|_{(k)}$ [1, p. 93]. Fix k and let $\|\cdot\|_1$ denote the trace-norm. There exist two rank k projections E and F such that

$$\begin{aligned} \|ZAB\|_{(k)} &= \|ZABE\|_1 \\ &= \|Z(AB)^{1/2}F(AB)^{1/2}E\|_1 \\ &\leq \|Z(AB)^{1/2}F(AB)^{1/2}\|_1. \end{aligned}$$

Consider the canonical decomposition

$$(AB)^{1/2}F(AB)^{1/2} = \sum_{j=1}^k c_j h_j \otimes h_j$$

in which $\{h_j\}_{j=1}^k$ is an orthonormal system and $\{h_j \otimes h_j\}_{j=1}^k$ are the associated rank one projections. We have, using the triangle inequality and then the above lemma,

$$\begin{aligned} \|Z(AB)^{1/2}F(AB)^{1/2}\|_1 &\leq \sum_{j=1}^k c_j \|Zh_j \otimes h_j\|_1 \\ &= \sum_{j=1}^k c_j \|Zh_j\| \\ &\leq \frac{a+b}{2\sqrt{ab}} \sum_{j=1}^k c_j \langle h_j, Zh_j \rangle \\ &= \frac{a+b}{2\sqrt{ab}} \text{Tr}(AB)^{1/2}F(AB)^{1/2}Z. \end{aligned}$$

Next, there exists a rank k projection G such that

$$\begin{aligned} \frac{a+b}{2\sqrt{ab}} \text{Tr}(AB)^{1/2}F(AB)^{1/2}Z &= \frac{a+b}{2\sqrt{ab}} \text{Tr}(AB)^{1/2}F(AB)^{1/2}ZG \\ &\leq \frac{a+b}{2\sqrt{ab}} \text{Tr}GZ^{1/2}ABZ^{1/2}G \\ &\leq \frac{a+b}{2\sqrt{ab}} \|Z^{1/2}ABZ^{1/2}\|_{(k)} \\ &\leq \frac{a+b}{2\sqrt{ab}} \|BZA\|_{(k)} \end{aligned}$$

where at the last step we used the basic inequality (1). \square

One may ask whether our theorem can be improved to singular values inequalities. This is not possible as it is shown by the next example:

Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

Then the largest and smallest eigenvalues of Z are $a = 8$ and $b = 2$, so

$$\frac{a+b}{2\sqrt{ab}} = 1.25.$$

Besides, $\mu_2(ZAB) = 8$ and $\mu_2(AZB) = 4.604$, and since $4.604 \times 1.25 = 5.755 < 8$, Theorem 1.2 can not be extended to singular values inequalities.

We denote by $\text{Sing}(X)$ the sequence of the singular values of X , arranged in decreasing order and counted with their multiplicities. Similarly, when X has only real eigenvalues, $\text{Eig}(X)$ stands for the sequence of X 's eigenvalues. Given two sequences of real numbers $\{a_j\}_{j=1}^n$ and $\{b_j\}_{j=1}^n$, we use the notation $\{a_j\}_{j=1}^n \prec_w \{b_j\}_{j=1}^n$ for weak-majorisation, that is $\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j$, $k = 1, \dots$.

A straightforward application of Theorem 1.2 is:

COROLLARY 1.3. *Let $A \geq 0$ and let $Z > 0$ with extremal eigenvalues a and b . Then,*

$$\text{Sing}(AZ) \prec_w \frac{a+b}{2\sqrt{ab}} \text{Eig}(AZ).$$

Proof. For each Fan norms, replace A and B by $A^{1/2}$ in Theorem 1.2. \square

Special cases of the above corollary are:

COROLLARY 1.4. *Let $A \geq 0$ and let $Z > 0$ with extremal eigenvalues a and b . Then,*

$$\|AZ\|_\infty \leq \frac{a+b}{2\sqrt{ab}} \rho(AZ)$$

and

$$\|AZ\|_1 \leq \frac{a+b}{2\sqrt{ab}} \text{Tr}AZ.$$

Here, $\|\cdot\|_\infty$ stands for the standard operator norm and $\rho(\cdot)$ for the spectral radius.

From the preceding result, one may derive an interesting operator inequality:

COROLLARY 1.5. *Let $0 \leq A \leq I$ and let $Z > 0$ with extremal eigenvalues a and b . Then,*

$$AZA \leq \frac{(a+b)^2}{4ab} Z.$$

Proof. The claim is equivalent to the operator norm inequalities

$$\|Z^{-1/2}AZA Z^{-1/2}\|_\infty \leq \frac{(a+b)^2}{4ab}$$

or

$$\|Z^{-1/2}AZ^{1/2}\|_\infty \leq \frac{a+b}{2\sqrt{ab}}.$$

But the previous corollary entails

$$\begin{aligned} \|Z^{-1/2}AZ^{1/2}\|_\infty &= \|Z^{-1/2}AZ^{-1/2}Z\|_\infty \\ &\leq \frac{a+b}{2\sqrt{ab}}\rho(Z^{-1/2}AZ^{-1/2}Z) \\ &= \frac{a+b}{2\sqrt{ab}}\|A\|_\infty \\ &\leq \frac{a+b}{2\sqrt{ab}}, \end{aligned}$$

hence, the result holds. \square

A special case of Corollary 1.5 gives a comparison between Z and the compression EZE , for an arbitrary projection E .

COROLLARY 1.6. *Let $Z > 0$ with extremal eigenvalues a and b and let E be any projection. Then,*

$$EZE \leq \frac{(a+b)^2}{4ab}Z.$$

We may then derive a classical inequality:

COROLLARY 1.7. *(Kantorovich) Let $Z > 0$ with extremal eigenvalues a and b and let h be norm one vector. Then,*

$$\langle h, Zh \rangle \langle h, Z^{-1}h \rangle \leq \frac{(a+b)^2}{4ab}.$$

Proof. Rephrase Corollary 1.6 as

$$\|Z^{-1/2}EZEZ^{-1/2}\|_\infty \leq \frac{(a+b)^2}{4ab}$$

and take $E = h \otimes h$. \square

A classical inequality in Matrix theory, for positive definite matrices, claims that "The inverse of a principal submatrix is less than or equal to the corresponding submatrix of the inverse" [6, p. 474]. In terms of compressions, this means

$$(Z_{\mathcal{E}})^{-1} \leq (Z^{-1})_{\mathcal{E}} \tag{4}$$

for every subspace \mathcal{E} and every $Z > 0$. Corollary 1.6 entails a reverse inequality, first proved by B. Mond and J.E. Pečarić [7]:

COROLLARY 1.8. *(Mond-Pečarić) Let $Z > 0$ with extremal eigenvalues a and b . Then, for every subspace \mathcal{E} ,*

$$(Z_{\mathcal{E}})^{-1} \geq \frac{4ab}{(a+b)^2}(Z^{-1})_{\mathcal{E}}.$$

Note that Corollary 1.8 implies Corollary 1.7.

Proof. Let E be the projection onto \mathcal{E} . By Corollary 1.6, for every $r > 0$, there exists $x > 0$ such that

$$EZE + xE^\perp \leq \frac{(a+b)^2}{4ab}(Z+rI).$$

Since $t \rightarrow -1/t$ is operator monotone we deduce

$$(EZE + xE^\perp)^{-1} \geq \frac{4ab}{(a+b)^2}(Z+rI)^{-1}$$

so that

$$(Z_{\mathcal{E}})^{-1} \geq \frac{4ab}{(a+b)^2}\{(Z+rI)^{-1}\}_{\mathcal{E}}$$

and the result follows by letting $r \rightarrow 0$. \square

REMARK 1.9. All the previous inequalities are sharp. Indeed, let h be a norm one vector for which equality occurs in Lemma 1.1. Then, replacing A, B, E by $h \otimes h$ and \mathcal{E} by $\text{span}\{h\}$ in the above statements, yields equality cases.

REMARK 1.10. As for a standard proof of (1) [1, p. 253], it is tempting to first prove Theorem 1.2 for the operator norm and then to use an antisymmetric tensor product argument to derive the general case. Such an approach seems impossible. Indeed if a_k and b_k are the extremal eigenvalues of $\wedge^k(Z)$, then the relation

$$\frac{(a_k + b_k)^2}{4a_k b_k} \leq \left(\frac{(a+b)^2}{4ab} \right)^k$$

is not true in general.

The next result states a companion inequality to Corollary 1.8.

PROPOSITION 1.11. *Let $Z > 0$ with extremal eigenvalues a and b and let $1 \leq p \leq 2$. Then, for every subspace \mathcal{E} ,*

$$(Z_{\mathcal{E}})^p \geq \frac{4ab}{(a+b)^2}(Z^p)_{\mathcal{E}}.$$

Proof. Let E be the projection onto \mathcal{E} . For any norm one vector $h \in \mathcal{E}$, Lemma 1.1 implies

$$\begin{aligned} \langle h, (Z^p)_{\mathcal{E}} h \rangle &= \langle h, EZ^p E h \rangle \\ &= \|Z^{p/2} h\|^2 \\ &\leq \frac{(a+b)^2}{4ab} \langle h, Z^{p/2} h \rangle^2. \end{aligned}$$

Then, using the concavity of $t \rightarrow t^{p/2}$ and next the convexity of $t \rightarrow t^p$, we deduce

$$\begin{aligned} \langle h, (Z^p)_{\mathcal{E}} h \rangle &\leq \frac{(a+b)^2}{4ab} \langle h, Zh \rangle^p \\ &= \frac{(a+b)^2}{4ab} \langle h, EZEh \rangle^p \\ &\leq \frac{(a+b)^2}{4ab} \langle h, (Z_{\mathcal{E}})^p h \rangle. \end{aligned}$$

and the proof is complete. \square

2. Operator convexity

Davis' characterization of operator convexity [2] claims: f is an operator convex on $[a, b]$ if and only if for every subspace \mathcal{E} and every Hermitian Z with spectrum in $[a, b]$,

$$f(Z_{\mathcal{E}}) \leq (f(Z))_{\mathcal{E}} \tag{D}$$

Since $t \rightarrow t^p$, $1 \leq p \leq 2$ and $t \rightarrow 1/t$ are operators convex on $(0, \infty)$, both Proposition 1.11 and Corollary 1.8 are reverse inequalities to Davis' characterization of operator convexity.

Proposition 1.11 is a special case of the next theorem.

THEOREM 2.1. *Let $f : [0, \infty) \rightarrow [0, \infty)$ be operator convex and let $Z > 0$ with extremal eigenvalues a and b . Then, for every subspace \mathcal{E} ,*

$$f(Z_{\mathcal{E}}) \geq \frac{4ab}{(a+b)^2} (f(Z))_{\mathcal{E}}.$$

Proof. We have the integral representation [1]

$$f(t) = \alpha + \beta t + \gamma t^2 + \int_0^{\infty} \frac{\lambda t^2}{\lambda + t} d\mu(\lambda),$$

where α, β, γ are nonnegative scalars and μ is a positive finite measure. Therefore, it suffices to prove the result for

$$\alpha + \beta t + \gamma t^2$$

and

$$f_{\lambda}(t) = \frac{\lambda t^2}{\lambda + t}.$$

The quadratic case is a straightforward application of Proposition 1.11. To prove the f_{λ} case, note that f_{λ} is convex meanwhile $f_{\lambda}^{1/2}$ is concave and then proceed as in the proof of Proposition 1.11. \square

Davis' characterization (D) of operator convexity is equivalent to the following result of Hansen-Pedersen [5].

Recall that a family $\{A_i\}_{i=1}^m$ form an isometric column when $\sum A_i^* A_i = I$.

THEOREM 2.2. (*Hansen – Pedersen*) Let $\{Z_i\}_{i=1}^m$ be Hermitians with spectrum lying in $[a, b]$ and let f be operator convex $[a, b]$. Then, for every isometric column $\{A_i\}_{i=1}^m$,

$$f\left(\sum A_i^* Z_i A_i\right) \leq \sum A_i^* f(Z_i) A_i. \quad (\text{Jo})$$

(Jo) is the operator version of Jensen's inequality: operator convex combinations and operator convex functions replace the ordinary ones. As a straightforward consequence, we have the following contractive version of (Jo):

COROLLARY 2.3. (*Hansen-Pedersen*) Let $\{Z_i\}_{i=1}^m$ be Hermitians with spectrum lying in $[a, b]$ and let f be operator convex $[a, b]$ with $0 \in [a, b]$ and $f(0) \leq 0$. Then, for every contraction A ,

$$f(A^* Z A) \leq A^* f(Z) A. \quad (\text{C})$$

Exactly as Theorem 2.1 is a reverse inequality to (D), the following result is a reverse inequality to (Jo).

THEOREM 2.4. Let $f : [0, \infty) \rightarrow [0, \infty)$ be operator convex and let $\{Z_i\}_{i=1}^m$ be positive with spectrum lying in $[a, b]$, $a > 0$. Then, for every isometric column $\{A_i\}_{i=1}^m$,

$$f\left(\sum A_i^* Z_i A_i\right) \geq \frac{4ab}{(a+b)^2} \sum A_i^* f(Z_i) A_i.$$

Let us consider a very special case: For every $A, B > 0$ with spectrum lying on $[r, 2r]$, $r > 0$, and for every operator convex $f : [0, \infty) \rightarrow [0, \infty)$, we have

$$\frac{8}{9} \cdot \frac{f(A) + f(B)}{2} \leq f\left(\frac{A+B}{2}\right) \leq \frac{f(A) + f(B)}{2}.$$

The left inequality gives a negative answer to an approximation problem: Let f be an operator convex function on $[a, b]$, $0 < a < b$, and let $\varepsilon > 0$. Then, in general, there is no operator convex function g on $[0, \infty)$ such that

$$\max_{x \in [a, b]} |f(x) - g(x)| < \varepsilon.$$

From Theorem 2.4 we obtain a reverse inequality to (C):

COROLLARY 2.5. Let $f : [0, \infty) \rightarrow [0, \infty)$ be operator convex and let $Z > 0$ with extremal eigenvalues a and b . Then, for every contraction A ,

$$f(A^* Z A) \geq \frac{4ab}{(a+b)^2} A^* f(Z) A.$$

We turn to the proof of Theorem 2.4 and Corollary 2.5.

Proof. Consider the following operators acting on $\oplus^m \mathcal{H}$,

$$V = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_m & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} Z_1 & & & \\ & \ddots & & \\ & & & Z_m \end{pmatrix}$$

and note that V is a partial isometry. Denoting by \mathcal{H} the first summand of the direct sum $\oplus^m \mathcal{H}$ and by $X: \mathcal{H}$ the restriction of X to \mathcal{H} , we observe that

$$f\left(\sum A_i^* Z_i A_i\right) = f(V^* \tilde{Z} V): \mathcal{H} = V^* f(\tilde{Z}_{V(\mathcal{H})}) V: \mathcal{H}.$$

Applying Theorem 2.1 with $\mathcal{E} = V(\mathcal{H})$, we get

$$\begin{aligned} f\left(\sum A_i^* Z_i A_i\right) &\geq \frac{4ab}{(a+b)^2} V^* f(\tilde{Z})_{V(\mathcal{H})} V: \mathcal{H} \\ &= \frac{4ab}{(a+b)^2} \sum A_i^* f(Z_i) A_i. \end{aligned}$$

and the proof of Theorem 2.4 is complete. To obtain its corollary, take an operator B such that $A^*A + B^*B = I$. Then, note that, using $f(0) \geq 0$,

$$\begin{aligned} f(A^*ZA) = f(A^*ZA + B^*0B) &\geq \frac{4ab}{(a+b)^2} \{A^*f(Z)A + B^*f(0)B\} \\ &\geq \frac{4ab}{(a+b)^2} A^*f(Z)A \end{aligned}$$

by application of Theorem 2.4. \square

REMARK 2.6. Corollary 1.8 and Proposition 1.11 for $p = 2$ have been obtained by Mond-Pecaric in the more general form of Theorem 2.4. Note that Proposition 1.11 with $p = 2$ immediately implies Lemma 1.1; hence, we have no pretention of originality in establishing this basic lemma.

REMARK 2.7. Hansen-Pedersen first prove the contractive version (C) in [4] and then, some twenty years later [5], prove the more general form (Jo). When proving (C) they noted a technical difficulty to derive (Jo) when $0 \notin [a, b]$. In fact, this difficulty can be easily overcome: Note that if (Jo) is valid for every operator convex functions on an interval $[a, b]$, then (Jo) is also valid on every interval of the type $[a + r, b + r]$.

REMARK 2.8. (D), (Jo), (C) are equivalent statements. Similarly, Theorems 2.1, 2.4 and Corollary 2.5 are equivalent.

Clearly, the previous results can be suitably restated for operators acting on infinite dimensional spaces.

Inspired by the seminal paper [3], we note that Corollary 2.5 can be stated in a still more general framework. Let $B(\mathcal{H})$ denote the algebra of all (bounded) linear operators on a separable Hilbert space \mathcal{H} .

COROLLARY 2.9. *Let $\Phi: \mathcal{L} \rightarrow B(\mathcal{H})$ be a positive, linear contraction on a C^* -algebra \mathcal{L} . Let $Z \in \mathcal{L}$, $Z > 0$ with $\text{Sp}(Z) \subset [a, b]$, $a > 0$. Then, for every operator convex function $f: [0, \infty) \rightarrow [0, \infty)$,*

$$f \circ \Phi(Z) \geq \frac{4ab}{(a+b)^2} \Phi \circ f(Z).$$

Proof. Restricting Φ to the commutative C^* -subalgebra generated by Z , one may suppose Φ completely positive. By Stinepring's dilation Theorem [8], there exist a larger Hilbert space $\mathcal{F} \supset \mathcal{H}$, a linear contraction $A : \mathcal{H} \rightarrow \mathcal{F}$ and a $*$ -homomorphism $\pi : \mathcal{L} \rightarrow \mathbf{B}(\mathcal{F})$ such that $\Phi(\cdot) = A^*(\pi(\cdot))A$. Therefore

$$\begin{aligned} f \circ \Phi(Z) &= f(A^*\pi(Z)A) \\ &\geq \frac{4ab}{(a+b)^2} A^* f(\pi(Z)) A \\ &= \frac{4ab}{(a+b)^2} A^* \pi(f(Z)) A \\ &= \frac{4ab}{(a+b)^2} \Phi \circ f(Z) \end{aligned}$$

where at the second step we apply Corollary 2.5 which can be extended to this situation by inspection of the proof of Theorem 2.4 and Corollary 2.5. \square

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