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NOTE ON SOME INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS

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Abstract. We give several Jensen's type inequalities for functions convex with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$. Results of Bessenyei and Páles from [1] are generalized.

1. Introduction

Let real function f be defined on some nonempty interval I of the real line $\mathbb{R}$. We say that function f is *convex* on I if inequality

$$f\left[\lambda x+(1-\lambda)y\right]\leqslant \lambda f\left(x\right)+(1-\lambda)f\left(y\right)$$

holds for all $x, y \in I$ and $\lambda \in (0,1)$. Geometrically, this means that if P, Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. In paper [2] Beckenbach generalized this geometric idea by replacing straight lines, i.e. elements of the family

$$\mathscr{F}_1=\{F:\mathbb{R}\to\mathbb{R}\mid F\left(x\right)=\alpha x+\beta,\ \alpha,\beta\in\mathbb{R}\},$$

by elements of a two parameter family $\mathscr{F}$ of continuous functions defined on I such that for any pairs $(x_1, y_1), (x_2, y_2) \in I\times\mathbb{R}$ with $x_1 \neq x_2$ there exists a unique element $F(\cdot\,; x_1, x_2) \in \mathscr{F}$ such that $F(x_i\,; x_1, x_2) = y_i,\ i = 1, 2.$

We say that a function $f : I \to \mathbb{R}$ is *convex with respect to* $\mathscr{F}$ if

$$f\left(x\right)\leqslant F\left(x\,;x_1,x_2\right)\ \text{for all}\ (x_1,x_2)$$

whenever $x_1 < x_2$, and $x_1, x_2 \in I$.

Special attention has been given to the case in which $\mathscr{F}$ is a *linear family*. By definition, this is a family $\mathscr{F}$ such that any $F \in \mathscr{F}$ may be expressed in the form

$$F=\alpha\omega_1+\beta\omega_2,$$

where α, β are real numbers and ω_1, ω_2 are two fixed continuous functions on I. The condition on the family $\mathscr{F}$ that for any pairs $(x_1, y_1), (x_2, y_2) \in I\times\mathbb{R}$ with $x_1 \neq x_2$

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there exists a unique element $F(\cdot\,; x_1, x_2) \in \mathscr{F}$ such that $F(x_i\,; x_1, x_2) = y_i,\ i = 1,2,$ turns out to be equivalent to the requirement

$$\begin{vmatrix} \omega_1(x_1) & \omega_1(x_2) \\ \omega_2(x_1) & \omega_2(x_2) \end{vmatrix} \neq 0 \tag{1.1}$$

whenever $x_1 \neq x_2$. Since this determinant is continuous function it must have the same sign for all $(x_1, x_2) \in I^2$, $x_1 \neq x_2$, and we shall assume the basis functions ω_1 and ω_2 chosen so that the determinant (1.1) is positive whenever $x_1 < x_2$. Such a set of functions $\{\omega_1, \omega_2\}$ is called a *Tchebycheff system*. Also, if $\{\omega_1, \omega_2\}$ is a Tchebycheff system on a interval I such that ω_1 is a positive function, then a simple calculation shows us that the function $\frac{\omega_2}{\omega_1}$ is continuous and strictly increasing on I.

We say that a function $f : I \to \mathbb{R}$ is *convex with respect to a Tchebycheff system* $\{\omega_1, \omega_2\}$ if

$$\begin{vmatrix} f(x_1) & f(x_2) & f(x) \\ \omega_1(x_1) & \omega_1(x_2) & \omega_1(x) \\ \omega_2(x_1) & \omega_2(x_2) & \omega_2(x) \end{vmatrix} \geqslant 0$$

whenever $x_1 < x_2 < x$, and $x_1, x_2, x \in I$. It can be easily seen that if for all $x \in I$ $\omega_1(x) = 1$ and $\omega_2(x) = x$, the notion of the convexity with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$ reduces to the notion of standard convexity.

These notions can be further generalized using n points instead of just two to determine the functions of the family $\mathscr{F}$, in which case we obtain an n-parameter family $\mathscr{F}_n$. More abut generalized convex functions can be found for example in [4].

In paper [1] Bessenyei and Pales considered the notion of (ω_1, ω_2)-convexity, which is a particular case of generalized convexity in the sense of Beckenbach [2], and hence a generalization of standard convexity. It can be easily seen that their definition of (ω_1, ω_2)-convexity, where (ω_1, ω_2) is a positive regular pair, is equivalent to the definition of the convexity with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$, so generally wherever is stated "... (ω_1, ω_2) *is a positive regular pair...*" can stay "... $\{\omega_1, \omega_2\}$ *is a Tchebycheff system...*". In that paper Bessenyei and Pales gave characterization theorems and Hadamard-type inequalities for (ω_1, ω_2)-convex functions. The main results from [1] are stated here as follows.

THEOREM A: *Let (ω_1, ω_2) be a positive regular pair on a nonempty interval I such that ω_1 is positive. The following statements are equivalent:*

(i) $f : I \to \mathbb{R}$ is (ω_1, ω_2)-convex;

(ii) for all $x, y, z \in I$ such that $x < y < z$ we have that

$$\frac{\begin{vmatrix} f(y) & f(z) \\ \omega_1(y) & \omega_1(z) \end{vmatrix}}{\begin{vmatrix} \omega_1(y) & \omega_1(z) \\ \omega_2(y) & \omega_2(z) \end{vmatrix}} \leqslant \frac{\begin{vmatrix} f(x) & f(y) \\ \omega_1(x) & \omega_1(y) \end{vmatrix}}{\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix}};$$

(iii) for all $x_0 \in Int(I)$ there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\omega_1(x_0) + \beta\omega_2(x_0) &= f(x_0), \\ \alpha\omega_1(x) + \beta\omega_2(x) &\leqslant f(x), \quad \forall x \in I; \end{aligned}$$

(iv) *for all* $n \in \mathbb{N}, x_0, x_1, ..., x_n \in I$ *and* $\lambda_1, ..., \lambda_n \geqslant 0$ *satisfying the conditions*

$$\sum_{k=1}^{n} \lambda_k \omega_1(x_k) = \omega_1(x_0)$$
$$\sum_{k=1}^{n} \lambda_k \omega_2(x_k) = \omega_2(x_0)$$

we have that

$$f(x_0) \leqslant \sum_{k=1}^{n} \lambda_k f(x_k); \tag{1.2}$$

(v) *for all* $x_0, x_1, x_2 \in I$ *and* $\lambda_1, \lambda_2 \geqslant 0$ *satisfying the conditions*

$$\lambda_1 \omega_j(x_1) + \lambda_2 \omega_j(x_2) = \omega_j(x_0), \qquad j = 1, 2$$

we have that

$$f(x_0) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

THEOREM B: *Let* (ω_1, ω_2) *be a positive regular pair on a nonempty open interval* I *such that* ω_1 *is positive. The function* $f : I \to \mathbb{R}$ *is* (ω_1, ω_2)*-convex if and only if the function* $g : \frac{\omega_2}{\omega_1}(I) \to \mathbb{R}$ *defined by*

$$g := \frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1} \tag{1.3}$$

is convex in the standard sense.

THEOREM C: *Let* (ω_1, ω_2) *be a positive regular pair on the interval* $[a, b]$ *such that* ω_1 *is positive on* $[a, b]$*. If* $f : [a, b] \to \mathbb{R}$ *is an* (ω_1, ω_2)*-convex function, then the inequalities*

$$cf(\xi) \leqslant \int_a^b f(x)\, dx \leqslant c_1 f(a) + c_2 f(b) \tag{1.4}$$

hold, where

$$\xi = \left(\frac{\omega_2}{\omega_1}\right)^{-1} \left(\frac{\int_a^b \omega_2(x)\, dx}{\int_a^b \omega_1(x)\, dx}\right), \quad c = \frac{\int_a^b \omega_1(x)\, dx}{\omega_1(\xi)}$$

and

$$c_1 = \frac{\begin{vmatrix} \int_a^b \omega_1(x)\, dx & \omega_1(b) \\ \int_a^b \omega_2(x)\, dx & \omega_2(b) \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} \omega_1(a) & \int_a^b \omega_1(x)\, dx \\ \omega_2(a) & \int_a^b \omega_2(x)\, dx \end{vmatrix}}{\begin{vmatrix} \omega_1(a) & \omega_1(b) \\ \omega_2(a) & \omega_2(b) \end{vmatrix}}.$$

It can be easily seen that if function $f : [a, b] \to \mathbb{R}$ is an $(1, x)$-convex function (i.e. convex in the standard sense), then (1.4) becomes well-known Hadamard's inequality for convex functions (see [3, p. 137] or [5, p. 10])

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}.$$

We may note here that the main results of paper [1] are Theorem A and Theorem C, but the use of Theorem B will enable us to generalize some standard inequalities for convex functions. In Section 2 we give several inequalities of Jensen's type for functions convex with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$ (i.e. for (ω_1, ω_2)-convex functions), and two of them are generalizations of (1.2) and (1.4). In Section 3 we give one inequality of Giaccardi's type for functions convex with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$.

2. Two inequalities of Jensen's type

Let $(\Omega, \mathscr{A}, \nu)$ be a measure space with $0 < \nu(\Omega) < \infty$, and let $h : \Omega \to I$, $I \subset \mathbb{R}$, be a function from $L^1(\nu)$. Then for any convex function $\varphi : I \to \mathbb{R}$ inequality

$$\varphi\left(\frac{1}{\nu(\Omega)}\int_\Omega h\,d\nu\right) \leqslant \frac{1}{\nu(\Omega)}\int_\Omega (\varphi \circ h)\,d\nu \tag{2.1}$$

holds. This inequality is well known as the integral Jensen's inequality (see [3, p. 45] or [5, p. 10]). If $I = [c,d]$, function h is measurable and function φ is convex and continuous on I, then the converse Jensen's inequality (see [3, p. 98]) states

$$\frac{1}{\nu(\Omega)}\int_\Omega (\varphi \circ h)\,d\nu \leqslant \frac{d-\overline{h}}{d-c}\varphi(c) + \frac{\overline{h}-c}{d-c}\varphi(d), \tag{2.2}$$

where $\overline{h} = \frac{1}{\nu(\Omega)}\int_\Omega h\,d\nu$. We will use (2.1), (2.2) and Theorem B to obtain two Jensen's type inequalities for functions convex with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$.

Throughout the rest of the paper, we assume that:

(i) $(\Omega, \mathscr{A}, \mu)$ is a measure space with $0 < \mu(\Omega) < \infty$;

(ii) $f : [m,M] \to \mathbb{R}$, $m < M$, is a continuous and convex function with respect to a Tchebycheff system $\{\omega_1, \omega_2\}$ on $[m,M]$, where ω_1 is positive on $[m,M]$;

(iii) $u : \Omega \to [m,M]$ is a measurable function.

THEOREM 1. *Let functions f and u be as the above. Then the inequalities*

$$kf(\xi) \leqslant \int_\Omega (f \circ u)\,d\mu \leqslant c_1 f(m) + c_2 f(M) \tag{2.3}$$

hold, where

$$P_{\omega_1} = \int_\Omega (\omega_1 \circ u)\,d\mu,$$

$$P_{\omega_2} = \int_\Omega (\omega_2 \circ u)\,d\mu,$$

$$\xi = \left(\frac{\omega_2}{\omega_1}\right)^{-1}\left(\frac{P_{\omega_2}}{P_{\omega_1}}\right), \quad k = \frac{P_{\omega_1}}{\omega_1(\xi)}$$

and

$$c_1 = \frac{\begin{vmatrix} P_{\omega_1} & \omega_1(M) \\ P_{\omega_2} & \omega_2(M) \end{vmatrix}}{\begin{vmatrix} \omega_1(m) & \omega_1(M) \\ \omega_2(m) & \omega_2(M) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} \omega_1(m) & P_{\omega_1} \\ \omega_2(m) & P_{\omega_2} \end{vmatrix}}{\begin{vmatrix} \omega_1(m) & \omega_1(M) \\ \omega_2(m) & \omega_2(M) \end{vmatrix}}.$$

Proof. From Theorem B we know that function g defined by (1.3) is continuous and convex in the standard sense. We define functions ν on $\mathscr{A}$ and h on Ω as

$$d\nu = (\omega_1 \circ u)\, d\mu, \tag{2.4}$$

$$h(x) = \left(\frac{\omega_2}{\omega_1} \circ u\right)(x). \tag{2.5}$$

Since functions ω_1 and $\frac{\omega_2}{\omega_1}$ are continuous on $[m,M]$ and function u is measurable, we know that function ν defined with (2.4) is a measure on $\mathscr{A}$ and that $0 < \nu(\Omega) < \infty$, and we also know that function h defined with (2.5) is measurable. Now we can apply integral Jensen's inequality (2.1) on function $\varphi = g$ to obtain

$$\begin{aligned} g\left(\frac{P_{\omega_2}}{P_{\omega_1}}\right) &= g\left(\frac{1}{P_{\omega_1}}\int_\Omega (\omega_1 \circ u)\left(\frac{\omega_2}{\omega_1} \circ u\right) d\mu\right) \\ &= g\left(\frac{1}{\nu(\Omega)}\int_\Omega \left(\frac{\omega_2}{\omega_1} \circ u\right) d\nu\right) \leqslant \frac{1}{\nu(\Omega)}\int_\Omega \left(g \circ \frac{\omega_2}{\omega_1} \circ u\right) d\nu \\ &= \frac{1}{\nu(\Omega)}\int_\Omega (f \circ u)\, d\mu = \frac{1}{P_{\omega_1}}\int_\Omega (f \circ u)\, d\mu. \end{aligned}$$

On the other hand, from the definition of function g we know that

$$g\left(\frac{P_{\omega_2}}{P_{\omega_1}}\right) = \left[\frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1}\right]\left(\frac{P_{\omega_2}}{P_{\omega_1}}\right),$$

so we obtain

$$\frac{f(\xi)}{\omega_1(\xi)} \leqslant \frac{1}{P_{\omega_1}}\int_\Omega (f \circ u)\, d\mu,$$

from which we can easily get the left side of (2.3).

To obtain the right side of (2.3) we use (2.2) for the same functions φ, h and ν as before. We obtain

$$\begin{aligned} \frac{1}{P_{\omega_1}}\int_\Omega (f \circ u)\, d\mu &= \frac{1}{P_{\omega_1}}\int_\Omega \left(g \circ \frac{\omega_2}{\omega_1} \circ u\right) d\nu \\ &\leqslant \frac{d - \frac{P_{\omega_2}}{P_{\omega_1}}}{d-c} g(c) + \frac{\frac{P_{\omega_2}}{P_{\omega_1}} - c}{d-c} g(d), \end{aligned} \tag{2.6}$$

since in this case $\overline{h} = \frac{P_{\omega_2}}{P_{\omega_1}}$. Multiplying (2.6) by P_{ω_1} and using the monotonicity property of the function $\frac{\omega_2}{\omega_1}$ we obtain

$$\begin{aligned}
\int_{\Omega}(f\circ u)\,d\mu &= \int_{\Omega}\left(g\circ\frac{\omega_2}{\omega_1}\circ u\right)d\nu \\
&\leqslant \frac{dP_{\omega_1}-P_{\omega_2}}{d-c}g(c)+\frac{P_{\omega_2}-cP_{\omega_1}}{d-c}g(d) \\
&= \frac{\frac{\omega_2(M)}{\omega_1(M)}P_{\omega_1}-P_{\omega_2}}{\frac{\omega_2(M)}{\omega_1(M)}-\frac{\omega_2(m)}{\omega_1(m)}}\cdot\frac{f(m)}{\omega_1(m)}+\frac{P_{\omega_2}-\frac{\omega_2(m)}{\omega_1(m)}P_{\omega_1}}{\frac{\omega_2(M)}{\omega_1(M)}-\frac{\omega_2(m)}{\omega_1(m)}}\frac{f(M)}{\omega_1(M)} \\
&= c_1 f(m)+c_2 f(M).
\end{aligned}$$

This completes the proof.

REMARK 1. If in Theorem 1 μ is Lebesgue measure on set $\Omega=[a,b]\subset\mathbb{R},$ and if function u is defined as

$$u(x)=x,\quad x\in[a,b],$$

then inequalities (2.3) become inequalities (1.4) from Theorem C.

COROLLARY 1. *Let functions f and u be as in Theorem 1. If $x_0\in[m,M]$ satisfies the conditions*

$$\frac{P_{\omega_1}}{\mu(\Omega)} = \omega_1(x_0) \tag{2.7}$$

$$\frac{P_{\omega_2}}{\mu(\Omega)} = \omega_2(x_0), \tag{2.8}$$

then the inequalities

$$\begin{aligned}
f(x_0) &\leqslant \frac{1}{\mu(\Omega)}\int_{\Omega}(f\circ u)\,d\mu \\
&\leqslant \frac{\omega_1(x_0)\,\omega_2(M)-\omega_1(M)\,\omega_2(x_0)}{\omega_1(m)\,\omega_2(M)-\omega_1(M)\,\omega_2(m)}f(m) \\
&\quad+\frac{\omega_1(m)\,\omega_2(x_0)-\omega_1(x_0)\,\omega_2(m)}{\omega_1(m)\,\omega_2(M)-\omega_1(M)\,\omega_2(m)}f(M)
\end{aligned}\tag{2.9}$$

hold.

Proof. If conditions (2.7) and (2.8) are satisfied, then inequality (2.3) reduces to (2.9) because of

$$\xi=x_0,\qquad k=\mu(\Omega),$$

and

$$c_1=\frac{\begin{vmatrix}\mu(\Omega)\,\omega_1(x_0) & \omega_1(M)\\ \mu(\Omega)\,\omega_2(x_0) & \omega_2(M)\end{vmatrix}}{\begin{vmatrix}\omega_1(m) & \omega_1(M)\\ \omega_2(m) & \omega_2(M)\end{vmatrix}},\quad c_2=\frac{\begin{vmatrix}\omega_1(m) & \mu(\Omega)\,\omega_1(x_0)\\ \omega_2(m) & \mu(\Omega)\,\omega_2(x_0)\end{vmatrix}}{\begin{vmatrix}\omega_1(m) & \omega_1(M)\\ \omega_2(m) & \omega_2(M)\end{vmatrix}}.$$

REMARK 2. If $\omega_1(x) = 1$ and $\omega_2(x) = x$ for all $x \in \Omega$, conditions (2.7) and (2.8) become

$$\begin{aligned}
\omega_1(x_0) &= \frac{1}{\mu(\Omega)} \int_\Omega (\omega_1 \circ u)\, d\mu = \frac{1}{\mu(\Omega)} \int_\Omega d\mu = 1, \\
\omega_2(x_0) &= \frac{1}{\mu(\Omega)} \int_\Omega (\omega_2 \circ u)\, d\mu = \frac{1}{\mu(\Omega)} \int_\Omega u d\mu = \overline{u} = x_0 \in [m, M],
\end{aligned}$$

so if function $f : [m, M] \to \mathbb{R}$ is convex on $[m, M]$ in the standard sense, then inequalities (2.9) give us

$$\begin{aligned}
f\left(\frac{1}{\mu(\Omega)} \int_\Omega u d\mu\right) &\leqslant \frac{1}{\mu(\Omega)} \int_\Omega (f \circ u)\, d\mu \\
&\leqslant \frac{M - \overline{u}}{M - m} f(m) + \frac{\overline{u} - m}{M - m} f(M),
\end{aligned}$$

i.e. inequalities (2.9) reduce to the integral Jensen's inequality (2.1) and its converse (2.2).

THEOREM 2. *Let* $\{\omega_1, \omega_2\}$ *be a Tchebycheff system on an interval* $[m, M]$ *such that* ω_1 *is positive on* $[m, M]$. *If* $f : [m, M] \to \mathbb{R}$ *is an* (ω_1, ω_2)*-convex function,* $x_k \in [m, M]$ $(k = 1, 2, ..., n)$ *and* $\mathbf{p} = (p_1, \cdots, p_n)$ *a nonnegative* n*-tuple such that* $P_n = \sum_{k=1}^n p_k \neq 0$, *then the inequalities*

$$kf(\xi) \leqslant \sum_{k=1}^n p_k f(x_k) \leqslant c_1 f(m) + c_2 f(M) \tag{2.10}$$

hold, where

$$P_{\omega_1} = \sum_{k=1}^n p_k \omega_1(x_k),$$

$$P_{\omega_2} = \sum_{k=1}^n p_k \omega_2(x_k),$$

$$\xi = \left(\frac{\omega_2}{\omega_1}\right)^{-1} \left(\frac{P_{\omega_2}}{P_{\omega_1}}\right), \quad k = \frac{P_{\omega_1}}{\omega_1(\xi)}$$

and

$$c_1 = \frac{\begin{vmatrix} P_{\omega_1} & \omega_1(M) \\ P_{\omega_2} & \omega_2(M) \end{vmatrix}}{\begin{vmatrix} \omega_1(m) & \omega_1(M) \\ \omega_2(m) & \omega_2(M) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} \omega_1(m) & P_{\omega_1} \\ \omega_2(m) & P_{\omega_2} \end{vmatrix}}{\begin{vmatrix} \omega_1(m) & \omega_1(M) \\ \omega_2(m) & \omega_2(M) \end{vmatrix}}. \tag{2.11}$$

Proof. Directly from Theorem 1. We simply choose

$$\begin{aligned}
\Omega &= \{1, 2, ..., n\}, \\
\mu(\{i\}) &= p_i, \ \ i = 1, 2, ..., n, \\
u(i) &= x_i, \ \ i = 1, 2, ..., n.
\end{aligned}$$

COROLLARY 2. *Let* $\{\omega_1, \omega_2\}$ *be a Tchebycheff system on an interval* $[m, M]$ *such that* ω_1 *is positive on* $[m, M]$. *If* $f : [m, M] \to \mathbb{R}$ *is an* (ω_1, ω_2)*-convex function,* $\mathbf{p}$=$(p_1, \cdots, p_n)$ *a nonnegative n-tuple such that* $P_n = \sum_{k=1}^{n} p_k \neq 0$ *and* $x_k \in [m, M]$ $(k = 0, 1, 2, ..., n)$ *satisfying the conditions*

$$\frac{P_{\omega_1}}{P_n} = \omega_1 (x_0) \tag{2.12}$$

$$\frac{P_{\omega_2}}{P_n} = \omega_2 (x_0), \tag{2.13}$$

then the inequalities

$$\begin{aligned} f(x_0) &\leqslant \frac{1}{P_n} \sum_{k=1}^{n} p_k f(x_k) \\ &\leqslant \frac{\omega_1 (x_0) \omega_2 (M) - \omega_1 (M) \omega_2 (x_0)}{\omega_1 (m) \omega_2 (M) - \omega_1 (M) \omega_2 (m)} f(m) \\ &\quad + \frac{\omega_1 (m) \omega_2 (x_0) - \omega_1 (x_0) \omega_2 (m)}{\omega_1 (m) \omega_2 (M) - \omega_1 (M) \omega_2 (m)} f(M) \end{aligned} \tag{2.14}$$

hold.

Proof. Directly from Corollary 1 for Ω, μ and u as in Theorem 2.

REMARK 3. We may note here that the left hand side inequality in (2.14) is inequality (1.2) from Theorem A.

REMARK 4. If function $f : [m, M] \to \mathbb{R}$ is convex in the standard sense then, similarly as in Remark 2, from inequality (2.14) we obtain

$$\begin{aligned} f\left(\frac{1}{P_n} \sum_{k=1}^{n} p_k x_k\right) &\leqslant \frac{1}{P_n} \sum_{k=1}^{n} p_k f(x_k) \\ &\leqslant \frac{M - \overline{x}}{M - m} f(m) + \frac{\overline{x} - m}{M - m} f(M), \end{aligned}$$

where

$$\overline{x} = \frac{1}{P_n} \sum_{k=1}^{n} p_k x_k,$$

i.e. we obtain discrete Jensen's inequality and its converse (see for example [5, p. 69]).

3. One inequality of Giaccardi's type

THEOREM 3. *Assume that* $\{\omega_1, \omega_2\}$ *is a Tchebycheff system on interval* $[m, M]$ *such that* ω_1 *is positive on* $[m, M]$, *and that* $f : [m, M] \to \mathbb{R}$ *is an* (ω_1, ω_2)*-convex function. Let* $\mathbf{p}$=$(p_1, \cdots, p_n)$ *be a nonnegative* n*-tuple such that* $P_n = \sum_{k=1}^{n} p_k \neq 0$ *and* x_k $(k = 0, 1, ..., n)$ *be real numbers such that* $x_0, \widetilde{x} = \sum_{k=1}^{n} p_k x_k \in [m, M]$. *If*

$$(x_i - x_0)(\widetilde{x} - x_i) \geqslant 0 \quad (i = 1, 2, ..., n), \quad \widetilde{x} \neq x_0, \tag{3.1}$$

then the inequality

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant \widetilde{A} f(\widetilde{x}) + \tilde{B} f(x_0) \tag{3.2}$$

holds, where

$$\begin{aligned} \widetilde{A} &= \frac{\omega_1(x_0) P_{\omega_2} - \omega_2(x_0) P_{\omega_1}}{\omega_1(x_0)\,\omega_2(\widetilde{x}) - \omega_1(\widetilde{x})\,\omega_2(x_0)}, \\ \widetilde{B} &= \frac{\omega_2(\widetilde{x}) P_{\omega_1} - \omega_1(\widetilde{x}) P_{\omega_2}}{\omega_1(x_0)\,\omega_2(\widetilde{x}) - \omega_1(\widetilde{x})\,\omega_2(x_0)}. \end{aligned}$$

Proof. We can easily see that conditions (3.1) imply either

$$x_0 \leqslant x_i \leqslant \tilde{x}, \quad i = 1, 2, ..., n,$$

or

$$\tilde{x} \leqslant x_i \leqslant x_0, \quad i = 1, 2, ..., n.$$

Since $x_0, \tilde{x} \in [m, M]$, this means that we can apply Theorem 2 on (ω_1, ω_2)-convex function f either on the subinterval $[x_0, \tilde{x}] \subseteq [m, M]$ or on the subinterval $[\tilde{x}, x_0] \subseteq [m, M]$. Therefore, in case when $x_0 < \tilde{x}$, the right hand side inequality in (2.10) with m and M replaced by x_0 and $\tilde{x}$ respectively, gives

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant c_1 f(x_0) + c_2 f(\tilde{x}),$$

where by (2.11):

$$c_1 = \frac{\begin{vmatrix} P_{\omega_1} & \omega_1(\tilde{x}) \\ P_{\omega_2} & \omega_2(\tilde{x}) \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\tilde{x}) \\ \omega_2(x_0) & \omega_2(\tilde{x}) \end{vmatrix}} = \tilde{B}, \quad c_2 = \frac{\begin{vmatrix} \omega_1(x_0) & P_{\omega_1} \\ \omega_2(x_0) & P_{\omega_2} \end{vmatrix}}{\begin{vmatrix} \omega_1(x_0) & \omega_1(\tilde{x}) \\ \omega_2(x_0) & \omega_2(\tilde{x}) \end{vmatrix}} = \tilde{A},$$

which means that inequality (3.2) is valid in this case. Similarly, in case when $\tilde{x} < x_0$, we apply the right hand side inequality in (2.10) with m and M replaced by $\tilde{x}$ and x_0 respectively, to obtain

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant c_1 f(\tilde{x}) + c_2 f(x_0).$$

We can easily check, using (2.11) again, that in this case $c_1 = \tilde{A}$ and $c_2 = \tilde{B}$, which means that (3.2) remains valid.

REMARK 5. Assume that $f : [m, M] \to \mathbb{R}$ is a convex function. Let $\mathbf{p}$=$(p_1, \cdots, p_n)$ be a nonnegative n-tuple such that $P_n = \sum_{k=1}^{n} p_k \neq 0$ and let $x_k \quad (k = 0, 1, ..., n)$ be real numbers such that $x_0, \sum_{k=1}^{n} p_k x_k \in [m, M]$. If conditions (3.1) are satisfied, we can apply Theorem 3 on such f, $\boldsymbol{x}$ and $\boldsymbol{p}$, where $\omega_1(x) = 1$ and $\omega_2(x) = x$ for all $x \in [m, M]$. It can be easily checked that in this case we obtain

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant Af\left(\sum_{k=1}^{n} p_k x_k\right) + B(P_n - 1) f(x_0)$$

where

$$A = \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\sum_{k=1}^{n} p_k x_k - x_0}, \quad B = \frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k x_k - x_0},$$

i.e., we obtain well known Giaccardi's inequality (see [5, p. 11]).

From these results we see that if we combine Theorem B and some standard inequalities for convex functions we can obtain in rather easy way variants of those inequalities for (ω_1, ω_2)-convex functions.

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