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GRONWALL INEQUALITIES ON TIME SCALES

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Dedicated to Professor V. Lakshmikantham
on his 80th Birthday

(*communicated by V. Lakshmikantham*)

Abstract. The authors use several different methods to extend Gronwall's inequality to more general cases on a time scale. Two applications are also given.

1. Introduction

To unify the theory of continuous and discrete dynamic systems, in 1990, Hilger [7] proposed the study of dynamic systems on a time scale and developed necessary calculus for functions on a time scale (that is , any closed subset of reals).

Recently, the well-known Gronwall integral inequality has been established on a time scale , see for example, Agarwal, Bohner and Peterson [1], Kaymakcalan, Ozgun and Zafer [9] and the books of Lakshmikantham, Sivasun and Kaymakcalan [12] and Bohner and Peterson [4].

In this paper, we use several different methods to extend the above-mentioned results to more general cases on a time scale. Applying these results, we can obtain that an initial value problem has at most one solution. For other related results, we refer to [2, 3, 5, 6, 10, 11, 13, 14, 15, 16, 17].

2. Preliminaries and lemmas

We first briefly introduce the time scales calculus.

By a times scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$ with order and topological structure in a canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

DEFINITION. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale, then two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} | s > t\}$$

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are called the jump operators.

If $\sigma(t) > t,\ t \in \mathbb{T}$, we say t is *right-scattered*. If $\rho(t) < t,\ t \in \mathbb{T}$, we say t is *left-scattered*. If $\sigma(t) = t,\ t \in \mathbb{T}$ we say t is *right-dense*. If $\rho(t) = t,\ t \in \mathbb{T}$, we say t is *left-dense*.

DEFINITION. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if

(a) f is continuous at each right-dense point or maximal point of $\mathbb{T}$.

(b) $\lim_{s \to t^-} g(s) = g(t^-)$ exists for each left-dense point $t \in \mathbb{T}$.

The set of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}[\mathbb{T}, \mathbb{R}]$. Let

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m. \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

DEFINITION. The function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for each} \quad t \in \mathbb{T},$$

where $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

DEFINITION. Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, then we define $f^{\Delta}(t)$ to be the number (if it exists) with property that for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leqslant \epsilon |\sigma(t) - s|$$

for all $s \in U$. In this case $f^{\Delta}(t)$ is called the *delta*-derivative of $f(t)$ at t. If f is differentiable at each $t \in \mathbb{T}$, then f is called *delta*-differentiable on $\mathbb{T}$.

DEFINITION. A function $g : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, and in this case, we define the integral of f by

$$\int_s^t f(u)\Delta u = g(t) - g(s)$$

for all $s, t \in T$, and we say that f is integrable on $\mathbb{T}$.

Throughout this paper, we suppose that

(a) $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$;

(b) $\mathbb{T}$ is a time scale with a as minimal element;

(c) $\Re^+ = \{p : \mathbb{T} \to \mathbb{R} \mid 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$, that is, the set $\Re^+$ is all positively regressive functions;

(d) an interval means the intersection of a real interval with the given time scale.

For further information concerning time scales, see [4] and [12]. In order to establish our main results, we need the following two lemmas which can be found in [1], [4] and [12].

LEMMA A. *Let* $y,\, f\, \in C_{rd}[\mathbb{T},\mathbb{R}]$ *and* $p \in \Re^{+}$. *If*

$$y^{\Delta}(t) \leqslant p(t)y(t) + f(t), \quad t \in \mathbb{T},$$

then

$$y(t) \leqslant y(a)e_p(t,a) + \int_a^t e_p(t,\sigma(s))f(s)\Delta s, \quad t \in \mathbb{T},$$

where $e_p(t,a)$ *is the solution of the initial value problem*

$$y^{\Delta}(t) = p(t)y(t), \quad y(a) = 1.$$

LEMMA B. (Gronwall's inequality [1, 4, 12]) *Let* $y,\, f\, \in C_{rd}[\mathbb{T},\mathbb{R}]$ *and* $p \in \Re^{+}$ *with* $p \geqslant 0$. *If*

$$y(t) \leqslant f(t) + \int_a^t y(s)p(s)\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then

$$y(t) \leqslant f(t) + \int_a^t e_p(t,\sigma(s))f(s)p(s)\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k.$$

3. Main results

We now can state and prove our main results as follows.

THEOREM 1. *Let* $y \in C_{rd}(\mathbb{T},\mathbb{R}^{+})$. *Suppose that* $h,\ g \in \Re^{+}$ *with* $h \geqslant 0$, $g \geqslant 0$ *and* $y_0 \in \mathbb{R}$. *If*

$$y(t) \leqslant y_0 + \int_a^t h(s)\Big[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\Big]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then

$$y(t) \leqslant y_0\Big[1 + \int_a^t h(s)e_{h+g}(s,a))\Delta s\Big] \quad \textit{for} \quad t \in \mathbb{T}^k.$$

In particular, if $y(a) = y_0 = 0$, then $y(t) \equiv 0$ on $\mathbb{T}^k$.

Proof. Let

$$x(t) = y_0 + \int_a^t h(s)\Big[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\Big]\Delta s \quad \text{for} \quad t \in \mathbb{T}^k.$$

Then

$$y(t) \leqslant x(t) \quad \text{for} \quad t \in \mathbb{T}^k$$

and

$$x^{\Delta}(t) = h(t)\Big[y(t) + \int_a^t g(s)y(s)\Delta s\Big] \leqslant h(t)m(t) \quad \text{for} \quad t \in \mathbb{T}^k, \tag{1}$$

where

$$m(t) = x(t) + \int_a^t g(s)x(s)\Delta s \quad \text{for} \quad t \in \mathbb{T}^k. \tag{2}$$

Then

$$x(t) \leqslant m(t), \quad m(a) = x(a) = y_0.$$

It follows from (1), (2) and $x(t) \leqslant m(t)$ that

$$\begin{aligned} m^{\Delta}(t) &= x^{\Delta}(t) + g(t)x(t) \\ &\leqslant h(t)m(t) + g(t)m(t) \\ &= [h(t) + g(t)]m(t). \end{aligned}$$

Applying Lemma A with $f \equiv 0$,

$$m(t) \leqslant m(a)e_{h+g}(t,a) = y_0 e_{h+g}(t,a).$$

It follows from (1) and the above inequality that

$$x^{\Delta}(t) \leqslant h(t)y_0 e_{h+g}(t,a).$$

Integrating the above inequality from a to t,

$$x(t) - x(a) \leqslant y_0 \int_a^t h(s)e_{h+g}(s,a)\Delta s,$$

which implies that

$$x(t) \leqslant y_0 \Big[1 + \int_a^t h(s)e_{h+g}(s,a)\Delta s\Big].$$

Hence we obtain the desired result.

THEOREM 2. *Let $y, f \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ with $f(t)$ be a nondecreasing function and $g, h \in \Re^+$ with $g \geqslant 0$, $h \geqslant 0$. If*

$$y(t) \leqslant f(t) + \int_a^t h(s)\Big[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\Big]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then the following two inequalities hold:

(a) $y(t) \leqslant f(t)\Big[1 + \int_a^t h(s)e_{h+g}(s,a)\Delta s\Big]$ *for* $t \in \mathbb{T}^k$.

(b) $y(t) \leqslant f(t)e_{h+g}(t,a)$ *for* $t \in \mathbb{T}^k$.

In particular, if $f(t) \equiv 0$, then $y(t) \equiv 0$ for $t \in \mathbb{T}^k$.

Proof. Since $f(t)$ is nondecreasing, we see that, for any $\epsilon > 0$,

$$\begin{aligned} p(t) :&= \frac{y(t)}{f(t) + \epsilon} \\ &\leqslant 1 + \int_a^t h(s)\frac{y(s)}{f(t)+\epsilon}\Delta s + \int_a^t h(s)\Big(\int_a^s g(\tau)\frac{y(\tau)}{f(t)+\epsilon}\Delta\tau\Big)\Delta s \\ &\leqslant 1 + \int_a^t h(s)\frac{y(s)}{f(s)+\epsilon}\Delta s + \int_a^t h(s)\Big(\int_a^s g(\tau)\frac{y(\tau)}{f(\tau)+\epsilon}\Delta\tau\Big)\Delta s \\ &= 1 + \int_a^t h(s)p(s)\Delta s + \int_a^t h(s)\Big(\int_a^s g(\tau)p(\tau)\Delta\tau\Big)\Delta s := u(t). \end{aligned} \tag{3}$$

(a) It follows from Theorem 1 that

$$p(t) \leqslant 1 + \int_a^t h(s)e_{h+g}(s,a)\Delta s.$$

Hence

$$y(t) \leqslant (f(t)+\epsilon)\Big[1 + \int_a^t h(s)e_{h+g}(s,a)\Delta s\Big].$$

Letting $\epsilon \to 0$, we obtain the desired result (a).

(b) By (3), for $t \in \mathbb{T}^k$,

$$\begin{aligned} u^{\Delta}(t) &= h(t)p(t) + h(t)\int_a^t g(s)p(s)\Delta s \\ &\leqslant h(t)\Big[u(t) + \int_a^t g(s)u(s)\Delta s\Big] \\ &:= h(t)v(t). \end{aligned}$$

Thus, for $t \in \mathbb{T}^k$,

$$v(a) = u(a) = 1, \quad u(t) \leqslant v(t)$$

and

$$v^{\Delta}(t) = u^{\Delta}(t) + g(t)u(t) \leqslant [h(t)+g(t)]v(t).$$

However, it follows from Lemma A and (3) that

$$\frac{y(t)}{f(t)+\epsilon} = p(t) \leqslant u(t) \leqslant v(t) \leqslant e_{h+g}(t,a),$$

which implies

$$y(t) \leqslant (f(t)+\epsilon)e_{h+g}(t,a).$$

Since $\epsilon > 0$ was arbitrary, thus, we have the desired result (b).

In the following theorem we delete the nondecreasing property of $f(t)$ in Theorem 2.

THEOREM 3. *Let* $y,\ f \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ *and* $h,\ g \in \Re^+$ *with* $h \geqslant 0,\ g \geqslant 0$. *If*

$$y(t) \leqslant f(t) + \int_a^t h(s)\Big[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\Big]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k, \tag{4}$$

then

$$y(t) \leqslant f(t) + \int_a^t e_{h+g}(t,\sigma(s))f(s)[h(s)+g(s)]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k.$$

In particular, if $f(t) \equiv 0$, then $y(t) \equiv 0$ on $\mathbb{T}^k$.

Proof. Let

$$x(t) = y(t) + \int_a^t g(s)y(s)\Delta s \quad \text{for} \quad t \in \mathbb{T}^k. \tag{5}$$

Then

$$y(t) \leqslant x(t) \quad \text{for} \quad t \in \mathbb{T}^k.$$

By (4) and (5),

$$y(t) = x(t) - \int_a^t g(s)y(s)\Delta s \leqslant f(t) + \int_a^t h(s)x(s)\Delta s.$$

Thus,

$$\begin{aligned} x(t) &\leqslant f(t) + \int_a^t h(s)x(s)\Delta s + \int_a^t g(s)y(s)\Delta s \\ &\leqslant f(t) + \int_a^t h(s)x(s)\Delta s + \int_a^t g(s)x(s)\Delta s \\ &= f(t) + \int_a^t [h(s) + g(s)]x(s)\Delta s. \end{aligned}$$

Therefore, it follows from Lemma B that

$$y(t) \leqslant x(t) \leqslant f(t) + \int_a^t e_{h+g}(t,\sigma(s))f(s)[h(s)+g(s)]\Delta s \quad \text{for} \quad t \in \mathbb{T}^k.$$

Thus, the proof is complete.

COROLLARY 4. *Let* $y, f \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ *with* f *be nondecreasing in* $\mathbb{T}$ *and* $h, g \in \Re^+$ *with* $h \geqslant 0$*,* $g \geqslant 0$*. If*

$$y(t) \leqslant f(t) + \int_a^t h(s)\Big[y(s) + \int_a^s g(\tau)y(\tau)\Delta\tau\Big]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then

$$y(t) \leqslant f(t)\Big[1 + \int_a^t e_{h+g}(t,\sigma(s))[h(s)+g(s)]\Delta s\Big] \quad \textit{for} \quad t \in \mathbb{T}^k.$$

DEFINITION. A function $H \in C(\mathbb{R}^+,\mathbb{R}^+)$ is said to belong to the class S if $H(u)$ is nondecreasing and $H(u) > 0$ for $u > 0$.

We are in a position to generalize LaSalle's integral inequality [10] (which was established by LaSalle in 1949) on time scales as follows:

THEOREM 5. (LASALLE'S INEQUALITY) *Let* $f \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ *be nondecreasing and* $g \in \Re^+$ *with* $g \geqslant 0$*. If* $H \in S$ *and* $y \in C_{rd}(\mathbb{T},\mathbb{R}^+)$ *satisfies*

$$y(t) \leqslant f(t) + \int_a^t g(s)H(y(s))\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k, \tag{6}$$

then

$$\int_{f(t)}^{y(t)} \frac{\Delta s}{H(s)} \leqslant \int_a^t g(s)\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k.$$

In particular, if $f(t) \equiv 0$ *for* $t \in \mathbb{T}^k$ *and* $\displaystyle\int_0^c \frac{\Delta s}{H(s)} = \infty$ *for all* $c > 0$*, then* $y(t) \equiv 0$ *for* $t \in \mathbb{T}^k$*.*

Proof. For any given $\epsilon > 0$ and any fixed $x \in \mathbb{T}^k$. Let

$$p(t) := f(x) + \epsilon + \int_a^t g(s)H(y(s))\Delta s \quad \text{on} \quad \mathbb{T}^k \cap [a,x].$$

Then, $p(t) > 0$ and is nondecreasing on $\mathbb{T}^k \cap [a,x]$. Thus, it follows from (6) that

$$\begin{aligned} y(t) &< f(t) + \epsilon + \int_a^t g(s)H(y(s))\Delta s, \\ &\leqslant f(x) + \epsilon + \int_a^t g(s)H(y(s))\Delta s = p(t), \quad \text{on} \quad \mathbb{T}^k \cap [a,x]. \end{aligned}$$

Hence

$$p^{\Delta}(t) = g(t)H(y(t)) \leqslant g(t)H(p(t)).$$

Therefore,

$$\frac{p^{\Delta}(t)}{H(p(t))} \leqslant g(t).$$

Integrating the above inequality from a to x and using Theorem 1.98 in [2], we obtain that

$$\int_{f(x)+\epsilon}^{y(x)} \frac{\Delta s}{H(s)} \leqslant \int_{p(a)}^{p(x)} \frac{\Delta s}{H(s)} = \int_a^x \frac{p^{\Delta}(s)}{H(p(s))}\Delta s \leqslant \int_a^x g(s)\Delta s.$$

Letting $\epsilon \to 0$ in the above inequality, we obtain that

$$\int_{f(x)}^{y(x)} \frac{\Delta s}{H(s)} \leqslant \int_a^x g(s)\Delta s.$$

This completes our proof.

In 1995, Ozgun, A. Zafer and B. Kaymakcalem [9] established Bihari's inequality [3] (which was established by Bihari in 1956) on time scales as follows:

THEOREM A. *Let $H \in S$ and $g \in R^+$ with $g \geqslant 0$. If $y(t) \in C_{rd}[\mathbb{T}, \mathbb{R}^+]$ satisfies*

$$y(t) \leqslant M + \int_a^t g(s)H(y(s))\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

where $M > 0$ is a constant, then

$$y(t) \leqslant G^{-1}\Big(G(M) + \int_a^t g(s)\Delta s\Big), \quad \in \mathbb{T}^k,$$

where G satisfies condition:

(C) G is a solution of $G^{\Delta}(u(t)) = \dfrac{u^{\Delta}(t)}{H(u(t))}$ and G is strictly increasing with $G(M) + \int_a^t g(s)\Delta s$ which is in the domain of G^{-1} for $t \in \mathbb{T}^k$.

In order to generalize Bihari's inequality [3], Dhongade and Deo [5] introduced the following class S^*.

DEFINITION. A function $H \in C(\mathbb{R}^+, \mathbb{R}^+)$ is said to belong to the class S^* if $H \in S$ and

$$\frac{H(u)}{t} \leqslant H(\frac{u}{t}) \quad \text{for} \quad u \geqslant 0 \text{ and } t \geqslant 0. \tag{$*$}$$

However, looking closely at condition $(*)$ and letting $x = \frac{u}{t}$ for $u > 0$ and $t > 0$, we see that

$$\frac{1}{t} = \frac{x}{u}.$$

Hence, condition $(*)$ reduces to

$$xH(u) \leqslant uH(x).$$

In this case, the function H is limited to linear form: $H(u) := cu$ on $[0, \infty)$ for some nonzero constant c.

In order to eliminate this disadvantage, we modify the above definition as follows:

DEFINITION. A function $H \in C_{rd}(\mathbb{R}^+, \mathbb{R}^+)$ is said to belong to the class S^*_τ if $H \in S$ and there exists a positive number τ such that

$$\frac{H(u)}{t} \leqslant H(\frac{u}{t}),$$

for all $u \geqslant 0$ and $t \in (0, \tau]$.

For example, if $H(u) = u^p,\ p \geqslant 1$, then

$$H(\frac{u}{t}) = \frac{u^p}{t^p} \geqslant \frac{u^p}{t} = \frac{H(u)}{t} \quad \text{for} \quad t \in (0, 1].$$

Then we can improve Bihari's inequality as follows:

THEOREM 6. (BIHARI'S INEQUALITY) *Let* $f \in C_{rd}(\mathbb{T}, [0, \tau))$ *be nondecreasing and* $g \in \Re^+$ *with* $g \geqslant 0$. *If* $H \in S^*_\tau$ *and* $y \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *satisfies*

$$y(t) \leqslant f(t) + \int_a^t g(s)H(y(s))\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then

$$y(t) \leqslant f(t)G^{-1}\Big(G(1) + \int_a^t g(s)\Delta s\Big) \quad \textit{for} \quad t \in \mathbb{T}^k,$$

where G *satisfies condition* (C) *and* $G(1) + \int_a^t g(s)\Delta s$ *is in the domain of* G^{-1}. *In particular, if* $f(t) \equiv 0$ *for* $t \in \mathbb{T}^k$, *then* $y(t) \equiv 0$ *for* $t \in \mathbb{T}^k$.

Proof. It follows from (6) and $H \in S^*_\tau$ that, for any given $\epsilon > 0$ with $f(t) + \epsilon \leqslant \tau$,

$$y(t) < (f(t) + \epsilon) + \int_a^t g(s)H(y(s))\Delta s \quad \text{for} \quad t \in \mathbb{T}^k,$$

which implies

$$\frac{y(t)}{f(t)+\epsilon} < 1 + \int_a^t g(s)H\Big(\frac{y(s)}{f(s)+\epsilon}\Big)\Delta s := u(t).$$

Thus,

$$u(a) = 1,\ u^{\Delta}(t) = g(t)H\Big(\frac{y(t)}{f(t)+\epsilon}\Big) \leqslant g(t)H(u(t)),$$

and hence

$$G^{\Delta}(u(t)) = \frac{u^{\Delta}(t)}{H(u(t))} \leqslant g(t).$$

Integrating the above inequality from a to $t \in \mathbb{T}^k$, we obtain that

$$G(u(t)) - G(1) \leqslant \int_a^t g(s)\Delta s,$$

which implies

$$\frac{y(t)}{f(t)+\epsilon} \leqslant u(t) \leqslant G^{-1}\Big(G(1) + \int_a^t g(s)\Delta s\Big).$$

Therefore

$$y(t) \leqslant (f(t)+\epsilon)G^{-1}\Big(G(1) + \int_a^t g(s)\Delta s\Big).$$

Letting $\epsilon \to 0$, we obtain the desired result.

THEOREM 7. *Let* $f \in C_{rd}(\mathbb{T},[0,\tau))$ *be nondecreasing,* $g \in \Re^{+}$ *with* $g \geqslant 0$ *and* $H \in S_{\tau}^{*}$. *If* $y \in C_{rd}(\mathbb{T},\mathbb{R}^{+})$ *satisfies*

$$y(t) \leqslant f(t) + \int_a^t g(s)\Big[y(s) + \int_a^s g(r)H(y(r))\Delta r\Big]\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k,$$

then

$$y(t) \leqslant f(t)\Big[1 + \int_a^t g(s)G^{-1}(G(1) + \int_a^t g(r)\Delta r)\Delta s\Big] \quad \textit{for} \quad t \in \mathbb{T}^k,$$

where G *satisfies condition* (C^{*}) G *is a solution of* $G^{\Delta}(u(t)) = \dfrac{u^{\Delta}(t)}{u(t)+H(u(t))}$ *and* G *is strictly increasing with* $G(M) + \int_a^t g(s)\Delta s$ *which is in the domain of* G^{-1} *for* $t \in \mathbb{T}^k$.

Proof. Since $f(t)$ is nonnegative, nondecreasing and $H \in S_{\tau}^{*}$, it follows that for any given $\epsilon > 0$ with $f(t)+\epsilon \leqslant \tau$,

$$\begin{aligned} p(t) := \frac{y(t)}{f(t)+\epsilon} &\leqslant 1 + \int_a^t g(s)\Big[\frac{y(s)}{f(s)+\epsilon} + \int_a^s \frac{g(r)H(y(r))}{f(r)+\epsilon}\Delta r\Big]\Delta s \\ &\leqslant 1 + \int_a^t g(s)\Big[\frac{y(s)}{f(s)+\epsilon} + \int_a^s g(r)H\Big(\frac{y(r)}{f(r)+\epsilon}\Big)\Delta r\Big]\Delta s \qquad (7) \\ &= 1 + \int_a^t g(s)\Big[p(s) + \int_a^s g(r)H(p(r))\Delta r\Big]\Delta s := v(r). \end{aligned}$$

Then

$$v^{\Delta}(t) = g(t)\Big[p(t) + \int_a^t g(s)H\Big(p(s)\Big)\Delta s\Big], \quad v(a) = 1. \tag{8}$$

It follows from $H \in S_\tau^*$, (7) and (8) that

$$v^{\Delta}(t) \leqslant g(t)\Big[v(t) + \int_a^t g(s)H(v(s))\Delta s\Big] := g(t)m(t), \tag{9}$$

where

$$m(t) = v(t) + \int_a^t g(s)H(v(s))\Delta s. \tag{10}$$

Then $m(a) = v(a) = 1$. It follows from (9), (10) and $v(t) \leqslant m(t)$ that

$$m^{\Delta}(t) \leqslant g(t)\Big(m(t) + H(m(t))\Big). \tag{11}$$

Dividing both sides of (11) by $m(t) + H(m(t))$ and integrating it from a to t,

$$G(m(t)) - G(m(a)) \leqslant \int_a^t g(s)\Delta s. \tag{12}$$

Then from (9) and (12),

$$v^{\Delta}(t) \leqslant g(t)G^{-1}\Big(G(1) + \int_a^t g(s)\Delta s\Big). \tag{13}$$

Integrating both sides of (13) from a to t and using (7), we see that

$$\frac{y(t)}{f(t)+\epsilon} \leqslant v(t) \leqslant 1 + \int_a^t g(s)G^{-1}\Big(G(1) + \int_a^s g(r)\Big)\Delta s,$$

which implies

$$y(t) \leqslant (f(t)+\epsilon)\Big[1 + \int_a^t g(s)G^{-1}\Big(G(1) + \int_a^s g(r)\Delta r\Big)\Delta s\Big].$$

Letting $\epsilon \to 0$ in the above inequality, we obtain the desired result.

4. Some applications

THEOREM 8. *Suppose that* $r, g \in C_{rd}([a,b],[0,\infty))$ *with* $r > 0$ *on* $[a,b]$, $g \in \Re^+$ *with* $g \geqslant 0$. *If* $F : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|F(x) - F(y)| \leqslant H(|x-y|) \quad \textit{for} \quad x,\ y \in \mathbb{R}$$

for some $H \in S$ *with*

$$\int_0^c \frac{\Delta t}{H(t)} = \infty \quad \textit{for each} \quad c > 0,$$

then the initial value problem

$$(IVP)\begin{cases} r(t)x^{\Delta}(t) = r(a)x^{\Delta}(a) - \int_a^t g(s)F(x(s))\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k, \\ x(a) = \alpha, \quad x^{\Delta}(a) = \beta \end{cases}$$

has at most one solution.

Proof. Suppose that the (IVP) has two solutions $x(t)$ and $y(t)$, then

$$x^{\Delta}(t) = \frac{1}{r(t)}\Big[r(s)x^{\Delta}(a) - \int_a^t g(s)F(x(s))\Delta s\Big]$$

and

$$y^{\Delta}(t) = \frac{1}{r(t)}\Big[r(s)y^{\Delta}(a) - \int_a^t g(s)F(y(s))\Delta s\Big].$$

Hence

$$x(t) = \alpha + \int_a^t \frac{1}{r(s)}\Big[r(s)\beta - \int_a^s g(u)F(x(u))\Delta u\Big]\Delta s,$$
$$y(t) = \alpha + \int_a^t \frac{1}{r(s)}\Big[r(s)\beta - \int_a^s g(u)F(y(u))\Delta u\Big]\Delta s,$$

which imply

$$\begin{aligned} |x(t) - y(t)| &\leqslant \int_a^t \frac{1}{r(s)} \int_a^s g(u)H(|x(u) - y(u)|)\Delta u \Delta s \\ &\leqslant \Big(\int_a^b \frac{\Delta s}{r(s)}\Big)\Big(\int_a^t g(u)H(|x(u) - y(u)|)\Delta u\Big). \end{aligned}$$

By LaSalle's inequality, $|x(t) - y(t)| \equiv 0$ on $\mathbb{T}^k$. Therefore

$$x(t) = y(t) \quad \text{on} \quad \mathbb{T}^k.$$

Similarly, using Theorem 6, we can prove the following:

THEOREM 9. *Suppose that* $r, g \in C_{rd}([a,b],[0,\infty))$ *with* $r > 0$ *on* $[a,b]$, $g \in \Re^+$ *with* $g \geqslant 0$. *If* $F : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|F(x) - F(y)| \leqslant H(|x - y|) \quad \textit{for} \quad x,\ y \in \mathbb{R}$$

for some $H \in S_{\tau}^{*}$, *then the initial value problem*

$$(IVP)\begin{cases} r(t)x^{\Delta}(t) = r(a)x^{\Delta}(a) - \int_a^t g(s)F(x(s))\Delta s \quad \textit{for} \quad t \in \mathbb{T}^k, \\ x(a) = \alpha, \quad x^{\Delta}(a) = \beta \end{cases}$$

has at most one solution.

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