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SHARPENING OF JORDAN'S INEQUALITIES AND ITS APPLICATIONS

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Abstract. In this paper, we establish the following inequalities

$$\frac{\sin r}{r}+\frac{\sin r-r\cos r}{2r^3}(r^2-x^2)\leqslant\frac{\sin x}{x}\leqslant\frac{\sin r}{r}+\frac{r-\sin r}{r^3}(r^2-x^2)$$

for $x\in(0,r], r\leqslant\pi/2$. An application of inequalities above leads to the following refinement of Yang Le inequalities:

$$4C_n^2[\frac{\sin r}{r}\frac{\lambda}{2}\pi+\frac{\sin r-r\cos r}{2r^3}(r^2\frac{\lambda}{2}\pi-\frac{\lambda^3}{8}\pi^3)]^2\cos^2\frac{\lambda}{2}\pi$$
$$\leqslant(n-1)\sum_{k=1}^{n}\cos^2\lambda A_k-2\cos\lambda\pi\sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j$$
$$\leqslant 4C_n^2[\frac{\sin r}{r}\frac{\lambda}{2}\pi+\frac{r-\sin r}{r^3}(r^2\frac{\lambda}{2}\pi-\frac{\lambda^3}{8}\pi^3)]^2,$$

where, $A_i>0(i=1,2,\cdots,n), \sum_{i=1}^{n}A_i\leqslant\pi, 0\leqslant\lambda\leqslant 1$ and $n\geqslant 2$ is a natural number.

1. Introduction

The following Theorem is know as Jordan's inequalities [1]:

THEOREM 1. *If* $0<x\leqslant\pi/2$, *then*

$$\frac{2}{\pi}\leqslant\frac{\sin x}{x}<1. \tag{1}$$

The equality in (1) *holds if and only if* $x=\pi/2$.

Qi, Cui and Xu [2] showed a new lower and upper bounds for the function $\frac{\sin x}{x}$, and obtained the following results.

THEOREM 2. *If* $0<x\leqslant\pi/2$, *then*

$$\frac{3}{\pi}-\frac{4}{\pi^3}x^2\leqslant\frac{\sin x}{x}\leqslant 1-\frac{4(\pi-2)}{\pi^3}x^2. \tag{2}$$

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Theorem 2 is equivalent to the following

THEOREM 3. *If* $0 < x \leqslant \pi/2$, *then*

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \leqslant \frac{\sin x}{x} \leqslant \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2). \tag{3}$$

Debnath and Zhao [3] proved the left inequality in (3) using other method. In fact, let $g(x) = \frac{\frac{\sin x}{x} - \frac{2}{\pi}}{\pi^2 - 4x^2}$,then $\lim_{x \longrightarrow 0^+} g(x) = \frac{\pi-2}{\pi^3}$ and $\lim_{x \longrightarrow \frac{\pi}{2}^-} g(x) = \frac{1}{\pi^3}$, we find that $\frac{1}{\pi^3}$ and $\frac{\pi-2}{\pi^3}$ are best constants in (3).

Now, let $0 < x \leqslant r \leqslant \frac{\pi}{2}$, then $(\frac{\sin x}{x})' = \frac{\cos x(x - \tan x)}{x^2} < 0$, so $\frac{\sin x}{x}$ is decreasing on $(0, r]$, we obtain

THEOREM 4. *If* $0 < x \leqslant r \leqslant \pi/2$, *then*

$$\frac{\sin r}{r} \leqslant \frac{\sin x}{x} < 1. \tag{4}$$

The equality in (4) *holds if and only if* $x = r$.

Put $r = \pi/2$ in (4), then (1) holds by (4).

In this paper, we obtain the further results described as Theorem 5 and show a simple proof of Theorem 5.

THEOREM 5. *If* $0 < x \leqslant r \leqslant \pi/2$, *then*

$$\frac{\sin r}{r} + \frac{\sin r - r\cos r}{2r^3}(r^2 - x^2) \leqslant \frac{\sin x}{x} \leqslant \frac{\sin r}{r} + \frac{r - \sin r}{r^3}(r^2 - x^2). \tag{5}$$

Furthermore, $\frac{\sin r - r\cos r}{2r^3}$ *and* $\frac{r - \sin r}{r^3}$ *are best constants in* (5).

Clearly, Theorem 5 is the generalization of Theorem 2 or Theorem 3.

In the other hand, [4] improved Yang Le inequality as the applications of the left inequality of (3) and the right inequality of (1), and obtained the following results.

THEOREM 6. *If* $A_i > 0 (i = 1, 2, \cdots, n), \sum_{i=1}^n A_i \leqslant \pi, 0 \leqslant \lambda \leqslant 1$. *Let* $n \geqslant 2$ *be a natural number. Then*

$$C_n^2 \lambda^2 (3 - \lambda^2)^2 \cos^2 \frac{\lambda}{2}\pi \leqslant (n-1)\sum_{k=1}^n \cos^2 \lambda A_k - 2\cos\lambda\pi \sum_{1 \leqslant i < j \leqslant n} \cos\lambda A_i \cos\lambda A_j \leqslant C_n^2 \lambda^2 \pi^2. \tag{6}$$

Using Theorem 5, we sharpen Yang Le inequality as follows

THEOREM 7. *If* $A_i > 0 (i = 1, 2, \cdots, n), \sum_{i=1}^n A_i \leqslant \pi, 0 \leqslant \lambda \leqslant 1$, *and* $0 < r \leqslant \pi/2$. *Let* $n \geqslant 2$ *be a natural number. Then*

$$\begin{aligned} &4C_n^2 [\frac{\sin r}{r} \frac{\lambda}{2}\pi + \frac{\sin r - r\cos r}{2r^3}(r^2 \frac{\lambda}{2}\pi - \frac{\lambda^3}{8}\pi^3)]^2 \cos^2 \frac{\lambda}{2}\pi \\ &\leqslant (n-1)\sum_{k=1}^n \cos^2 \lambda A_k - 2\cos\lambda\pi \sum_{1 \leqslant i < j \leqslant n} \cos\lambda A_i \cos\lambda A_j \\ &\leqslant 4C_n^2 [\frac{\sin r}{r} \frac{\lambda}{2}\pi + \frac{r - \sin r}{r^3}(r^2 \frac{\lambda}{2}\pi - \frac{\lambda^3}{8}\pi^3)]^2. \end{aligned} \tag{7}$$

2. One Lemma

LEMMA 1. ([5, 6]) *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g' \neq 0$ on (a,b). If f'/g' is decreasing on (a,b), then the functions*

$$\frac{f(x)-f(b)}{g(x)-g(b)}$$

and

$$\frac{f(x)-f(a)}{g(x)-g(a)}$$

are also decreasing on (a,b).

3. A simple proof of Theorem 5

Now, let $f_1(x) = \frac{\sin x}{x}, f_2(x) = -x^2, f_3(x) = \sin x - x\cos x, f_4(x) = x^3$, and $x \in (0,r]$, where $r \leqslant \pi/2$. Then we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{2}\frac{\sin x - x\cos x}{x^3} = \frac{1}{2}\frac{f_3(x)}{f_4(x)},$$
$$\frac{f_3'(x)}{f_4'(x)} = \frac{1}{3}\frac{\sin x}{x} = \frac{1}{3}f_1(x).$$

Since $f_1(x) = \frac{\sin x}{x}$ is decreasing on $(0,r)$ or $\frac{f_3'(x)}{f_4'(x)}$ is decreasing on $(0,r)$, then $\frac{f_3(x)}{f_4(x)} = \frac{f_3(x)-f_3(0)}{f_4(x)-f_4(0)}$ is decreasing on $(0,r)$ by Lemma 1. So $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $(0,r)$, and $h(x) = \frac{\frac{\sin x}{x}-\frac{\sin r}{r}}{r^2-x^2} = \frac{f_1(x)-f_1(r)}{f_2(x)-f_2(r)}$ is decreasing on $(0,r)$ by Lemma 1.

Furthermore, $\lim\limits_{x\to 0^+} h(x) = \frac{r-\sin r}{r^3}$ and $\lim\limits_{x\to r^-} h(x) = \frac{\sin r - r\cos r}{2r^3}$, so $\frac{\sin r - r\cos r}{2r^3}$ and $\frac{r-\sin r}{r^3}$ are best constants in (5).

4. The proof of Theorem 6

Let $H_{ij} = cos^2\lambda A_i + cos^2\lambda A_j - 2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j$, we have

$$\sin^2\lambda\pi \leqslant H_{ij} \leqslant 4\sin^2\frac{\lambda}{2}\pi$$

or

$$4\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi \leqslant H_{ij} \leqslant 4\sin^2\frac{\lambda}{2}\pi \tag{8}$$

from [4]. If $1 \leqslant i < j \leqslant n$, taking the sum for all inequalities of (8), then

$$\sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi \leqslant \sum_{1\leqslant i<j\leqslant n} H_{ij} \leqslant \sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\lambda}{2}\pi. \tag{9}$$

In fact,we have

$$\sum_{1\leqslant i<j\leqslant n} H_{ij} = \sum_{1\leqslant i<j\leqslant n} (cos^2\lambda A_i + cos^2\lambda A_j - 2\cos\lambda\pi\cos\lambda A_i\cos\lambda A_j)$$
$$= (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi \sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j \quad (10)$$

and

$$\sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\lambda}{2}\pi\cos^2\frac{\lambda}{2}\pi$$
$$\geqslant 4C_n^2[\frac{\sin r}{r}\frac{\lambda}{2}\pi + \frac{\sin r - r\cos r}{2r^3}(r^2\frac{\lambda}{2}\pi - \frac{\lambda^3}{8}\pi^3)]^2\cos^2\frac{\lambda}{2}\pi, \quad (11)$$

$$\sum_{1\leqslant i<j\leqslant n} 4\sin^2\frac{\lambda}{2}\pi \leqslant 4C_n^2[\frac{\sin r}{r}\frac{\lambda}{2}\pi + \frac{r-\sin r}{r^3}(r^2\frac{\lambda}{2}\pi - \frac{\lambda^3}{8}\pi^3)]^2 \quad (12)$$

by (5). Putting (10), (11) and (12) in (9), we obtain (7).

Finally, let $r = \pi/2$ in (7), then we obtain the following results.

THEOREM 8. *If* $A_i > 0 (i = 1,2,\cdots,n), \sum_{i=1}^{n} A_i \leqslant \pi, 0 \leqslant \lambda \leqslant 1$. *Let* $n \geqslant 2$ *be a natural number. Then*

$$C_n^2\lambda^2(3-\lambda^2)^2\cos^2\frac{\lambda}{2}\pi$$
$$\leqslant (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi\sum_{1\leqslant i<j\leqslant n}\cos\lambda A_i\cos\lambda A_j$$
$$\leqslant 4C_n^2(\lambda^3 + \frac{\lambda(1-\lambda^2)}{2}\pi)^2.$$

REFERENCES

[1] D. S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, 1970.
[2] F. QI, L.-H. CUI AND S.-L. XU, *Some Inequalities Constructed by Tchebysheff's Integral Inequality*, Mathematical Inequalities and Applications, **2**, 4 (1999), 517–528.
[3] L. DEBNATH, C. J. ZHAO, *New Strengthened Jordan's Inequality and Its Applications*, Applied Mathematics Letters, **16**, (2003), 557–560.
[4] C. J. ZHAO, *Generalization and Strengthen of Yang Le Inequality*, Mathematics in Practice and Theory (in Chinese), **30**, 4 (2000), 493–497.
[5] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants,Inequalities,and Quasiconformal Maps*, New York, 1997.
[6] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized Elliptic Integral and Modular Equations*, Pacific J.Math., **192**, (2000), 1–37.

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