Mathematical
Inequalities
& Applications
Volume 9, Number 1 (2006), 107–116

ESTIMATES FOR THE MODULII OF THE ZEROS OF A POLYNOMIAL

ABDUL AZIZ AND ALIYA QAYOOM

(*communicated by T. M. Rassias*)

Abstract. In this paper, we prove a more general result concerning the location of the zeros of a polynomial in a ring shaped region from which we deduce an interesting and significant refinement of a classical result of Cauchy. A variety of other results, which in particular include several known extensions and generalizations of Enestrom - Kakeya Theorem, can be established from this result by a fairly uniform procedure.

1. Introduction and statement of results

It is a fundamental problem of algebra to find the solution of algebraic equations. Quite a few results giving bounds for some or all the zeros of a polynomial in terms of its coefficients may be found in [1, 2, 5, 6-8]. More recently, J. L. Diaz-Barrero [3] has obtained an annulus containing all the zeros of a polynomial involving binomial coefficients and Fibonacci's numbers $F_k = F_{k-1} + F_{k-2}$ for $k \geqslant 2$, where $F_0 = 0$ and $F_1 = 1$. In fact, he has proved the following result.

THEOREM A. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ *lie in the annulus* $R = \{z \in \mathbb{C} : r_1 \leqslant |z| \leqslant r_2\}$, *where*

$$r_1 = \frac{3}{2} \min_{1 \leqslant k \leqslant n} \left\{ \frac{2^n F_k C(n,k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \quad \textit{and}$$

$$r_2 = \frac{2}{3} \max_{1 \leqslant k \leqslant n} \left\{ \frac{F_{4n}}{2^n F_k C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}.$$

The main aim of this paper is to prove the following more general result (Theorem 1) which includes not only Theorem A as a special case but also a variety of other interesting results can be established from Theorem 1 by a fairly uniform procedure.

THEOREM 1. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

Mathematics subject classification (2000): 30C15, 26C10.
Key words and phrases: polynomials, zeros, Fibonacci's numbers, Enstrome-Kakeya theorem.

be a non-constant complex polynomial of degree n. If $\lambda_1, \lambda_2, ..., \lambda_n$ *is any set of n real or complex numbers such that*

$$\sum_{k=1}^{n} |\lambda_k| \leqslant 1,$$

then all the zeros of $P(z)$ *lie in the annulus* $R = \{z \in \mathbb{C} : r_1 \leqslant |z| \leqslant r_2\}$*, where*

$$r_1 = \min_{1\leqslant k\leqslant n} \left|\lambda_k \frac{a_0}{a_k}\right|^{1/k} \quad and \quad r_2 = \max_{1\leqslant k\leqslant n} \left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_n}\right|^{1/k}.$$

REMARK 1. If we take in Theorem 1,

$$\lambda_k = \frac{3^k}{2^k}\frac{2^n F_k C(n,k)}{F_{4n}}, \qquad k = 1, 2, ..., n,$$

and make use of the identity [3],

$$\sum_{k=1}^{n} 2^{n-k} 3^k F_k C(n,k) = F_{4n}$$

involving binomial coefficients and Fibonacci's numbers, then

$$\sum_{k=1}^{n} |\lambda_k| = \sum_{k=1}^{n} \lambda_k = 1$$

and we get Theorem A immediately.

If in Theorem 1, we take

$$\lambda_k = \frac{F_k}{F_{n+2}-1}, \quad k = 1, 2, ..., n,$$

and make use of another identity [10, p. 99] concerning Fibonacci's numbers F_k namely

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1,$$

which can be easily established with the help of mathematical induction, we shall obtain the following interesting result.

COROLLARY 1. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non-constant complex polynomial of degree n. Then all the zeros of $P(z)$ *lie in the annulus* $R = \{z \in \mathbb{C} : r_1 \leqslant |z| \leqslant r_2\}$*, where*

$$r_1 = \min_{1\leqslant k\leqslant n} \left\{\frac{F_k}{F_{n+2}-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \quad and \quad r_2 = \max_{1\leqslant k\leqslant n} \left\{\frac{F_{n+2}-1}{F_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}.$$

Let

$$R = \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right|.$$

In Theorem 1, we take

$$\lambda_k = \frac{1}{R} \left\{ \frac{a_{n-k}}{a_n} \right\}$$

then

$$\sum_{k=1}^{n} |\lambda_k| = \frac{1}{R} \sum_{k=1}^{n} \left\{ \frac{a_{n-k}}{a_n} \right\} = 1$$

and it can be easily seen that

$$r_2 = \max_{1 \leqslant k \leqslant n} R^{1/k} = \max \left\{ R, R^{1/n} \right\}.$$

Hence we get the following result.

COROLLARY 2. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non-constant polynomial of degree n then all the zeros of $P(z)$ *lie in the disk*

$$|z| \leqslant \max \left\{ R, R^{1/n} \right\}$$

where

$$R = \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right|.$$

Next we use Theorem 1 to prove the following interesting and significant refinement of a classical result of Cauchy [5, p. 123] which states that all the zeros of a polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0, \tag{1}$$

lie in the disk

$$|z| < 1 + M \quad \text{where} \quad M = \max_{1 \leqslant k \leqslant n} \left| \frac{a_{n-k}}{a_n} \right|.$$

THEOREM 2. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a non-constant polynomial of degree n, then all the zeros of $P(z)$ *lie in the disk*

$$|z| \leqslant \{ (1+M)^n - 1 \}^{1/n} \quad \textit{where} \quad M = \max\nolimits_{1 \leqslant k \leqslant n} \left| \frac{a_{n-k}}{a_n} \right|.$$

REMARK 2. Using

$$\{\ (1+M)^n-1\ \}^{1/n}<1+M,$$

in Theorem 2, we immediately get the above mentioned classical result of Cauchy.

REMARK 3. Let r be the positive root of the equation

$$|a_0|+|a_1|z+\cdots+|a_{n-1}|z^{n-1}-|a_n|z^n=0, \quad a_n\neq 0, \tag{2}$$

then clearly

$$\left|\frac{a_0}{a_n}\right|\frac{1}{r^n}+\left|\frac{a_1}{a_n}\right|\frac{1}{r^{n-1}}+\cdots+\left|\frac{a_{n-1}}{a_n}\right|\frac{1}{r}=1. \tag{3}$$

If we choose

$$\lambda_k=\left(\frac{a_{n-k}}{a_n}\right)\frac{1}{r^k}, \quad k=1,2,...,n,$$

then by (3), we have

$$\sum_{k=1}^{n}|\lambda_k|=\sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|\frac{1}{r^k}=1 \quad \text{and} \quad r_2=\max_{1\leqslant k\leqslant n}\left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_n}\right|^{1/k}=r.$$

Hence from Theorem 1, it follows that all the zeros of $P(z)$ defined by (1) lie in the closed disk $|z|\leqslant r$ where r is the positive root of the equation defined by (2). This is another classical result also due to Cauchy [5, p.122].

Many other interesting results can be easily deduced from Theorem 1. Here we mention a few of these. First we state the following result which is obtained by applying Theorem 1 to the $(n+1)$ th degree polynomial

$$P(z)=(z-t)P(z)$$

where $P(z)$ is defined by (1) and t is any real or a complex number.

THEOREM 3. *Let*

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,$$

be a non constant complex polynomial of degree n. *If* $\lambda_1,\lambda_2,...,\lambda_n,\lambda_{n+1}$ *is any set of* $(n+1)$ *real or complex numbers such that*

$$\sum_{k=1}^{n}|\lambda_k|\leqslant 1,$$

then for any real or a complex number t, *all the zeros of* $P(z)$ *lie in the annulus*

$$R=\{z\in\mathbb{C}:r_1\leqslant|z|\leqslant r_2\}$$

where

$$r_1=\min_{1\leqslant k\leqslant n+1}\left|\lambda_k\frac{ta_0}{(a_{k-1}-ta_k)}\right|^{1/k}, \quad (a_{n+1}=0)$$

and

$$r_2=\max_{1\leqslant k\leqslant n+1}\left|\frac{1}{\lambda_k}\frac{(a_{n-k}-ta_{n-k+1})}{a_n}\right|^{1/k}, \quad (a_{-1}=0).$$

The following corollary immediately follows from Theorem 3 by taking t = 1.

COROLLARY 3. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

of degree $n \geqslant 1$. *If* $\lambda_1, \lambda_2, ..., \lambda_n, \lambda_{n+1}$ *is any set of* $(n+1)$ *real or complex numbers such that*

$$\sum_{k=1}^{n+1} |\lambda_k| \leqslant 1,$$

then all the zeros of $P(z)$ *lie in the annulus*

$$R = \{z \in \mathbb{C} : r_1 \leqslant |z| \leqslant r_2\}$$

where

$$r_1 = \min_{1 \leqslant k \leqslant n+1} \left| \lambda_k \frac{a_0}{(a_k - a_{k-1})} \right|^{1/k}, \quad (a_{n+1} = 0)$$

and

$$r_2 = \max_{1 \leqslant k \leqslant n+1} \left| \frac{1}{\lambda_k} \frac{(a_{n-k+1} - a_{n-k})}{a_n} \right|^{1/k}, \quad (a_{-1} = 0).$$

Let

$$L = \sum_{k=1}^{n+1} \left| \frac{a_{n-k+1} - a_{n-k}}{a_n} \right|, \quad (a_{-1} = 0),$$

then

$$L \geqslant \left| \sum_{k=1}^{n+1} \frac{a_{n-k+1} - a_{n-k}}{a_n} \right| = \frac{1}{|a_n|} |a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0 + a_0| = \frac{|a_n|}{|a_n|} = 1. \tag{4}$$

Choosing

$$\lambda_k = \frac{1}{L} \left\{ \frac{a_{n-k+1} - a_{n-k}}{a_n} \right\}, \quad k = 1, 2, ..., n,$$

so that

$$\sum_{k=1}^{n+1} |\lambda_k| = \frac{1}{L} \sum_{k=1}^{n+1} \left| \frac{a_{n-k+1} - a_{n-k}}{a_n} \right| = 1$$

and noting by (4) that

$$r_2 = \max_{1 \leqslant k \leqslant n+1} \{L\}^{1/k} = L,$$

the following result is an immediate consequence of Corollary 3.

COROLLARY 4. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree $n \geqslant 1$. *Then all the zeros of* $P(z)$ *lie in the closed disk* $|z| \leqslant L$, *where*

$$L = \sum_{k=1}^{n+1} \left| \frac{a_{n-k+1} - a_{n-k}}{a_n} \right|, \quad (a_{-1} = 0).$$

Again if we take

$$R = \sum_{k=1}^{n+1} \left| \frac{a_k - a_{k-1}}{a_0} \right|, \quad (a_{n+1} = 0),$$

where we may assume $a_0 \neq 0$, then clearly as before $R \geqslant 1$. Choosing now

$$\lambda_k = \frac{1}{R} \left\{ \frac{a_k - a_{k-1}}{a_0} \right\}, \quad k = 1, 2, ..., n+1,$$

which gives

$$\sum_{k=1}^{n+1} |\lambda_k| = \frac{1}{R} \sum_{k=1}^{n+1} \left| \frac{a_k - a_{k-1}}{a_0} \right| = 1.$$

Since

$$r_1 = \min_{1 \leqslant k \leqslant n+1} \left| \lambda_k \frac{a_0}{(a_k - a_{k-1})} \right|^{1/k} = \min_{1 \leqslant k \leqslant n+1} \left\{ \frac{1}{R} \right\}^{1/k} = \frac{1}{R},$$

from Corollary 3, we immediately get the following interesting result.

COROLLARY 5. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree $n \geqslant 1$. *Then* $P(z)$ *does not vanish in the disk*

$$|z| < \frac{1}{R} = \frac{|a_0|}{\sum_{k=1}^{n+1} |a_k - a_{k-1}|}.$$

If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n such that

$$a_n \geqslant a_{n-1} \geqslant ... \geqslant a_1 \geqslant a_0 \geqslant 0,$$

then clearly

$$L = \sum_{k=1}^{n+1} \left| \frac{a_{n-k+1} - a_{n-k}}{a_n} \right| = \sum_{k=1}^{n+1} \frac{(a_{n-k+1} - a_{n-k})}{a_n} = \frac{a_n}{a_n} = 1, \quad (a_{-1} = 0)$$

and from Corollary 4, it follows that all the zeros of $P(z)$ lie in the closed unit disk $|z| \leqslant 1$. This is a famous result known as Enestrome-Kakeya Theorem [5, p.136].

If we assume that the coefficients of the polynomial $P(z)$ are monotonic but not necessarily non-negative, that is, if

$$0 < a_n \geqslant a_{n-1} \geqslant ... \geqslant a_1 \geqslant a_0,$$

then it can be easily seen that

$$L = \frac{a_n - a_0 + |a_0|}{|a_n|} \quad \text{and} \quad \frac{1}{R} = \frac{|a_0|}{2a_n - a_0}.$$

In the first case, an application of Corollary 4 shows that all the zeros of $P(z)$ lie in the closed disk

$$|z| \leqslant \frac{1}{|a_n|}\left\{a_n - a_0 + |a_0|\right\},$$

which is an extension of Enestrome-Kakeya Theorem due to Joyal. Labelle and Rahman [4] whereas, in the second case Corollary 5 allows us to conclude that the polynomial $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{2a_n - a_0}.$$

This result was earlier proved by A. Aziz and Q. G. Mohammad [1] for polynomials with non-negative coefficients. A number of other interesting results which include several known extensions and generalizations of Enestrome-Kakeya Theorem can be established form above theorems and corollaries by a fairly uniform procedure. Here, in this paper, we shall finally deduce from Theorem 1 a well known result due to Walsh [11] and thereby present an independent proof of this result according to which all the zeros of a polynomial

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

be a polynomial of degree n lie in the disk

$$|z| \leqslant R = \sum_{j=1}^{n} \left|\frac{a_{n-j}}{a_n}\right|^{1/j}. \tag{5}$$

To deduce Walsh's result from Theorem 1, we take

$$\lambda_k = \left(\frac{a_{n-k}}{a_n}\right)\frac{1}{R^k}, \qquad k = 1, 2, ..., n.$$

Then

$$\sum_{k=1}^{n} |\lambda_k|^{1/k} = \sum_{k=1}^{n} \frac{1}{R}\left|\frac{a_{n-k}}{a_n}\right|^{1/k} = \frac{1}{R}\sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|^{1/k} = \frac{R}{R} = 1. \tag{6}$$

Since

$$\left|\frac{a_{n-k}}{a_n}\right|^{1/k} \leqslant \sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|^{1/k} = R,$$

for all $k = 1, 2, \ldots, n,$ we have

$$\left|\frac{a_{n-k}}{a_n}\right| \leqslant R^k, \qquad k = 1, 2, ..., n.$$

This gives

$$0 \leqslant |\lambda_k| \leqslant 1, \qquad k = 1, 2, ..., n,$$

which implies

$$|\lambda_k| \leqslant |\lambda_k|^{1/k}, \qquad k = 1, 2, ..., n.$$

Hence by (6),

$$\sum_{k=1}^{n} |\lambda_k| \leqslant \sum_{k=1}^{n} |\lambda_k|^{1/k} = 1.$$

Using Theorem 1, it follows that all the zeros of $P(z)$ lie in the disk

$$|z| \leqslant \max_{1\leqslant k\leqslant n} \left| \frac{1}{\lambda_k} \frac{a_{n-k}}{a_n} \right|^{1/k} = R,$$

which establishes (5).

2. Proof of the theorems

Proof of Theorem 1. We first show that all the zeros of $P(z)$ lie in

$$|z| \leqslant r_2 = \max_{1\leqslant k\leqslant n} \left| \frac{1}{\lambda_k} \frac{a_{n-k}}{a_n} \right|^{1/k}. \tag{7}$$

From (7) it follows that

$$\left| \frac{a_{n-k}}{a_n} \right| \leqslant |\lambda_k| r_2^k, \qquad k = 1, 2, ..., n,$$

and hence

$$\sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_2^k} \leqslant \sum_{k=1}^{n} |\lambda_k|. \tag{8}$$

Now for $|z| > r_2$, we have

$$\begin{aligned} |P(z)| &= |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\ &\geqslant |a_n||z|^n \left\{ 1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{|z|^k} \right\} \\ &> |a_n||z|^n \left\{ 1 - \sum_{k=1}^{n} \left| \frac{a_{n-k}}{a_n} \right| \frac{1}{r_2^k} \right\}. \end{aligned}$$

Using (8) and noting that by hypothesis,

$$\sum_{k=1}^{n} |\lambda_k| \leqslant 1,$$

we obtain for $|z| > r_2$,

$$|P(z)| > |a_n||z|^n \left\{ 1 - \sum_{k=1}^{n} |\lambda_k| \right\} \geqslant 0.$$

Thus $|P(z)| > 0$ for $|z| > r_2$. Consequently all the zeros of $P(z)$ lie in $|z| \leqslant r_2$ and this proves the second part of Theorem 1.

To prove the first part of this theorem, we shall use the second part. If $a_0 = 0$, then clearly

$$r_1 = \min_{1\leqslant k\leqslant n}\left|\lambda_k \frac{a_0}{a_k}\right| = 0$$

and there is nothing to prove. So we assume that $a_0 \neq 0$. Consider the polynomial

$$Q(z) = z^n P\left(\frac{1}{z}\right) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n.$$

By the second part of the theorem, all the zeros of the polynomial $Q(z)$ lie in

$$\begin{aligned}|z| &\leqslant \max_{1\leqslant k\leqslant n}\left|\frac{1}{\lambda_k}\frac{a_k}{a_0}\right|^{1/k} = \max_{1\leqslant k\leqslant n}\left|\frac{1}{\lambda_k \frac{a_0}{a_k}}\right|^{1/k} \\ &= \max_{1\leqslant k\leqslant n}\left\{\frac{1}{\left|\lambda_k \frac{a_0}{a_k}\right|^{1/k}}\right\} = \frac{1}{\min_{1\leqslant k\leqslant n}\left|\lambda_k \frac{a_0}{a_k}\right|^{1/k}} = \frac{1}{r_1}.\end{aligned}$$

Replacing z by $\frac{1}{z}$ and observing that $P(z) = z^n Q\left(\frac{1}{z}\right)$, we conclude that all the zeros of $P(z)$ lie in

$$|z| \geqslant r_1 = \min_{1\leqslant k\leqslant n}\left|\lambda_k \frac{a_0}{a_k}\right|^{1/k}. \tag{9}$$

The desired result follows by combining (7) and (9).

Proof of Theorem 2. Since

$$M = \max_{1\leqslant k\leqslant n}\left|\frac{a_{n-k}}{a_n}\right|,$$

we have

$$\left|\frac{a_{n-k}}{a_n}\right| \leqslant M, \qquad k = 1, 2, ..., n. \tag{10}$$

We take

$$\lambda_k = \left\{\frac{(1+M)^n}{(1+M)^n - 1}\right\}\left\{\frac{a_{n-k}}{a_n(1+M)^k}\right\}, \qquad k = 1, 2, ..., n. \tag{11}$$

Then with the help of (10), we get

$$\sum_{k=1}^{n}|\lambda_k| = \frac{(1+M)^n}{(1+M)^n - 1}\sum_{k=1}^{n}\left|\frac{a_{n-k}}{a_n}\right|\frac{1}{(1+M)^k} \leqslant \left\{\frac{(1+M)^n}{(1+M)^n - 1}\right\}\sum_{k=1}^{n}\frac{M}{(1+M)^k}. \tag{12}$$

Now

$$\begin{aligned}\sum_{k=1}^{n}\frac{M}{(1+M)^k} &= \frac{M}{(1+M)}\left\{1 + \frac{1}{(1+M)} + \cdots + \frac{1}{(1+M)^{n-1}}\right\} \\ &= \frac{M}{(1+M)}\left\{\frac{1 - \frac{1}{(1+M)^n}}{1 - \frac{1}{(1+M)}}\right\} = \frac{(1+M)^n - 1}{(1+M)^n}.\end{aligned}$$

Using this in (12), we see that

$$\sum_{k=1}^{n} |\lambda_k| \leqslant 1.$$

Applying Theorem 1 with λ_k defined by (11), it follows that all the zeros of $P(z)$ lie in the disk

$$\begin{aligned}|z| \leqslant r_2 &= \max_{1\leqslant k\leqslant n} \left|\frac{1}{\lambda_k}\frac{a_{n-k}}{a_n}\right|^{1/k} = \max_{1\leqslant k\leqslant n}\left\{\frac{(1+M)^n-1}{(1+M)^n}\right\}^{1/k}(1+M)\\ &= (1+M)\max_{1\leqslant k\leqslant n}\left\{1-\frac{1}{(1+M)^n}\right\}^{1/k} = (1+M)\left\{1-\frac{1}{(1+M)^n}\right\}^{1/n}\\ &= \left\{(1+M)^n-1\right\}^{1/n},\end{aligned}$$

and this completes the proof of Theorem 2.

REFERENCES

[1] A. AZIZ, Q. G. MOHAMMAD, *Zero-free Regions for Polynomials and some Generalizations of Enestrome-Kakeya Theorem*, Canad. Math.Bull., **27**, 3, (1984), 265–272.
[2] P. BORWEIN, T. ERDELYI, *Polynomial and Polynomial Inequalities*, Springer-Verlag, Beijing, 1997, pp. 16–17.
[3] J. L. DIAZ BARRERO, *An Annulus for the Zero's of Polynomials*, J. Math. Anal. Appl.. **273**, (2002), 349–352.
[4] A. JOYAL, G. LABELLE AND Q. I. RAHMAN, *On the Location of the Zero's of Polynomials*, Canadian Math, Bull., **10**, (1967), 53–63.
[5] M. MARDEN, *Geometry of Polynomials*, IInd Ed., Math, Surveys, No.3, Amer.Math.Soc. Providence, R.I, 1966.
[6] G. V. MILOVANOVIĆ, D. S. MITRINOVIĆ AND TH. M. RASSIAS, *Topics in Polynomials; Extremal Problems, Inequalities Zeros*, World Scientific, Singapore, New Jersey, London, Hong-Kong, 1994.
[7] G. V. MILOVANOVIĆ, T. M RASSIAS, *Inequalities for Polynomial Zeros* in T.M.Rassias [Ed], Survey on Classical Inequalities, Kluwer Academic, Netherlands, 2000, pp. 165–202.
[8] G. V. MILOVANOVIĆ, T. M RASSIAS, *Distribution of Zeros and Inequalities for zeros of algebraic polynomials* in T. M Rassias, [Ed], Functional Equations and Inequalities, Kluwer Academic, Dordrecht, 2000, pp. 171–204.
[9] Q. I. RAHAMN, G. SCHMEISSER, *Analytic Theory of Polynomials*, Clarendon Press-Oxford 2002, pp. 243–270.
[10] I. NIVERI, H. S. ZUCKERMAN, *An Introduction to the Theory of Numbers*, Third Ed., Wiley Easter Limited, 1972.
[11] J. L. WALSH, *An Inequality for the Roots of an Algebraic Equation*, Ann. of Math. **25**, (1924), 749–754.

(Received June 3, 2005)

Abdul Aziz
Post Graduate Department of Mathematics
University of Kashmir
Hazratbal
Srinagar-190006
Kashmir
India
e-mail: aaulauzeem@rediffmail.com