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SOME CLASSES OF P–ANALYTIC FUNCTIONS DEFINED BY CERTAIN INTEGRAL OPERATOR

KHALIDA INAYAT NOOR

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Abstract. Let $\mathscr{A}(p), p \in N$, be the class of functions $f(z) = z^p + a_{p+1}z^{p+1} + \dots$, analytic in the open unit disc E. For $n = 0, 1, 2, \dots,\ n > -p$, a certain integral operator $I_{n+p-1} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$ is defined as $I_{n+p-1}f = f^{(-1)}_{n+p-1} \star f$ such that $(f^{(-1)}_{n+p-1} \star f_{n+p-1})(z) = \frac{z^p}{(1-z)^p}$, where $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and $\star$ denotes convolution or Hadamard product. Using this integral operator, a new subclass $R_k(n, p, \alpha)$ of $\mathscr{A}(p)$, $0 \leqslant \alpha < p$ is introduced in E and some interesting properties of this class are investigated.

1. Introduction

Let $\mathscr{A}(p)$ denote the class of functions

$$f(z) = z^p + \sum_{k=2}^{\infty} a_{p+k} z^{p+k}, \quad (p \in N = \{1, 2, \dots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $E = \{z : |z| < 1\}$. A function $f \in \mathscr{A}(p)$ is said to belong to the class $S(p, \alpha)$ of p-valently starlike functions of order $\alpha (0 \leqslant \alpha < p)$ if it satisfies, for $z \in E$, the conditions

$$Re \frac{zf'(z)}{f(z)} > \alpha \quad \text{and} \quad \int_0^{2\pi} Re\{\frac{zf'(z)}{f(z)}\} d\theta = 2p\pi. \tag{1.2}$$

The class $S(p, \alpha)$ was introduced by Goodman [1] and studied in [5] and others.

The class $\mathscr{A}(p)$ is closed under the Hadamard product or convolution

$$(f_1 \star f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k},$$

where

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k}, \quad (j = 1, 2).$$

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Denote by $D^{n+p-1} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p),$ the operator defined by

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} \star f(z), \quad (n > -p).$$

The symbol D^{n+p-1} is called the Ruscheweyh derivative of $(n+p-1)th$ order.

Let $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}} \quad (n > -p)$ and let $f_{n+p-1}^{(-1)}$ be defined such that

$$(f_{n+p-1} \star f_{n+p-1}^{(-1)})(z) = \frac{z^p}{(1-z)^p}. \tag{1.3}$$

Analogous to symbol $D^{n+p-1},$ we here define an integral operator $I_{n+p-1} : \mathscr{A}(p) \longrightarrow \mathscr{A}(p)$ as follows:

$$\begin{aligned} I_{n+p-1}f(z) &= (f_{n+p-1}^{(-1)} \star f)(z) \\ &= \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} \star f(z), \quad (n > -p) \end{aligned} \tag{1.4}$$

We note that $I_0 f = zf'$ and $I_1 f = f,$ see Noor [3].

From (1.3) and (1.4), we obtain the following identity for the operator I_{n+p-1} :

$$(n+1)I_{n+p-1}f - (n+p-1)I_{n+p}f = z(I_{n+p}f)'. \tag{1.5}$$

It is clear, from (1.5), that

$$\frac{z(I_{n+p}f)'}{I_{n+p}f} = (n+1)\frac{I_{n+p-1}f}{I_{n+p}f} - (n-p+1),$$

and

$$Re\frac{z(I_{n+p}f)'}{I_{n+p}f} > 0 \quad \text{and} \quad Re\frac{I_{n+p-1}f}{I_{n+p}f} > \frac{n+p-1}{n+1}$$

are equivalent.

Let $P_k(\alpha)$ be the class of functions $h(z)$ analytic in the unit disc satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left|\frac{Reh(z) - \alpha}{p - \alpha}\right| d\theta \leqslant k\pi, \tag{1.6}$$

where $z = re^{i\theta}, k \geqslant 2$ and $0 \leqslant \alpha < p.$ This class has been introduced in [4] for $p = 1.$ We note that $P_k(0) = P_k,$ see [6] and $P_2(\alpha) = P(\alpha),$ the class of analytic functions with positive real part greater than α and $P_2(0) = P,$ is the class of functions with positive real part.

We can also represent $h \in P_k(\alpha)$ as

$$h(z) = (\frac{k}{4} + \frac{1}{2})h_1(z) - (\frac{k}{4} - \frac{1}{2})h_2(z), \tag{1.7}$$

where $h_i \in P(\alpha), \ i = 1, 2,$ and $z \in E.$

REMARK 1.1. We can use the hypergeometric functions to define $I_{n+p-1}f$ as follows: since $(1-z)^{-a} = {}_2F_1(a,1;1,z)$, where ${}_2F_1$ is the hypergeometric function, we have, for $a > 1$

$$\left[\frac{1}{(1-z)^a}\right]^{(-1)} ={}_2 F_1(1,1;a,z) = (a-1)\int_0^1 (1-t)^{a-2}\frac{dt}{1-tz}.$$

Therefore

$$I_{n+p-1}f = [z^p {}_2F_1(1,p;n+p-1,z)] \star f \quad (n > -p)$$

We now define the following.

DEFINITION 1.1. Let $f \in \mathscr{A}(p)$. Then $f \in R_k(n,p,\alpha)$ if and only if $\frac{zf'}{f} \in P_k(\alpha)$ for $z \in E,\ 0 \leqslant \alpha < p,\ p \in N$.

For $k = 2$ we obtain the class $R_2(p,\alpha) = S(p,\alpha)$ defined by (1.2).

DEFINITION 1.2. Let $f \in \mathscr{A}(p)$. Then $f \in R_k(n,p,\alpha)$ if and only if $I_{n+p-1}f \in R_k(p,\alpha)$ for $z \in E,\ 0 \leqslant p,\ p \in N, n > -p$.

It is clear that $R_k(0,1,\alpha) = V_k(\alpha)$, the class of functions of bounded boundary rotation of order α.

2. Preliminary results

LEMMA 2.1. [2] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let Φ be a complex-valued function satisfying the conditions:*

(i) $\Phi(u,v)$ is continuous in a domain $D \subset \mathbf{C}^2$,

(ii) $(1,0) \in D$ and $\Phi(1,0) > 0$.

(iii) $Re\Phi(iu_2,v_1) \leqslant 0$, whenever $(iu_2,v_1) \in D$ and $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{m=2}^{\infty} c_m z^m$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $Re(h(z), zh'(z)) > 0$ for $z \in E$, then $Reh(z) > 0$ in E.

LEMMA 2.2. *Let p be an analytic function in E with $p(0) = 1$ and $Rep(z) > 0$, $z \in E$. Then, for $s > 0$ and $\mu \neq -1$ (complex),*

$$Re\{p(z) + \frac{szp'(z)}{p(z)+\mu}\} > 0, \quad for \quad |z| < r_0,$$

where r_0 is given by

$$r_0 = \frac{|\mu+1|}{\sqrt{A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}}},$$

$$A = 2(s+1)^2 + |\mu|^2 - 1 \qquad \textit{and this radius is best possible.}$$

Lemma 2.2 is due to Ruscheweyh and Singh [7].

3. Main results

THEOREM 3.1. *Let for* $z \in E$, $f \in R_k(n,p,0)$. *Then* $f \in R_k(n+1,p,\alpha)$ *in* E, *where*

$$\alpha = \frac{2p}{(2n+2p-1)+\sqrt{4p^2+4p(2n+1)+(2n-1)^2}}. \tag{3.1}$$

Proof. Set

$$\frac{z(I_{n+p}f(z))'}{I_{n+p}f(z)} = H(z) = (p-\alpha)h(z)+\alpha,$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \dots$

Using identity (1.5) for $z \in E$ we obtain

$$\frac{z(I_{n+p-1}f(z))'}{I_{n+p-1}f(z)} = \left\{H(z) + \frac{zH'(z)}{H(z)+n+p-1}\right\} \in P_k(0) \quad \text{in} \quad E.$$

Let

$$\Psi(z) = \sum_{j=1}^{\infty} \frac{n+p-1+j}{n+p}$$
$$= \frac{p}{p+n}\frac{z^p}{1-z} + \frac{n}{p+n}\frac{z^p}{(1-z)2},$$

and

$$H(z) = (\frac{k}{4}+\frac{1}{2}\{(p-\alpha)h_1(z)+\alpha\} - (\frac{k}{4}-\frac{1}{2}\{(p-\alpha)h_2(z)+\alpha\}.$$

We want to show that $H \in P_k(\alpha)$, where α is given by (3.1) or equivalently $h_i \in P$, $i = 1,2$.

Now

$$H \star \frac{\Psi(z)}{z^p} = H + \frac{zH'}{H+n+p-1}$$
$$= (\frac{k}{4}+\frac{1}{2})\left\{(p-\alpha)[h_1 \star \frac{\Psi(z)}{z^p}]+\alpha\right\} - (\frac{k}{4}-\frac{1}{2})\left\{(p-\alpha)[h_2 \star \frac{\Psi(z)}{z^p}]+\alpha\right\}.$$

Therefore it follows that, $i = 1,2$,

$$Re\left[(p-\alpha)h_i(z)+\alpha+\frac{(p-\alpha)zh_i'(z)}{(p-\alpha)h_i(z)+(n+p+\alpha-1)}\right] > 0, \quad z \in E. \tag{3.2}$$

We form the functional $\Phi(u,v)$ by choosing $u = h_i(z)$ and $v = zh_i'(z)$ in (3.2). Thus

$$\Phi(u,v) = (p-\alpha)+\alpha+\frac{(p-\alpha)v}{(p-\alpha)u+(\alpha+n+p-1)}.$$

The first two conditions of Lemma 2.1 are clearly satisfied as $\Phi(u,v)$ is continuous in $D = \mathbf{C} - (-\frac{\alpha+n+p-1)}{p-\alpha}) \times \mathbf{C}$, $(1,0) \in D$ and $Re\{\Phi(1,0)\} > 0$. We verify the third condition as follows.

$$
\begin{aligned}
Re\Phi(iu_2, v_1) &= \alpha + Re\left[\frac{(p-\alpha)v_1}{(p-\alpha)iu_2 + (\alpha+n+p-1)}\right] \\
&= \alpha + \frac{(p-\alpha)(\alpha+n+p-1)v_1}{(\alpha+n+p-1)^2 + (p-\alpha)^2 u_2^2}.
\end{aligned}
$$

By putting $v_1 \leqslant -\frac{(1+u_2^2)}{2}$, we obtain

$$
\begin{aligned}
&Re\Phi(iu_2, v_1) \\
&\quad \leqslant \alpha - \frac{1}{2}\frac{(p-\alpha)(\alpha+n+p-1)(1+u_2^2)}{(\alpha+n+p-1)^2 + (p-\alpha)^2 u_2^2} \\
&\quad = \frac{(\alpha+n+p-1)\left[2\alpha(\alpha+n+p-1)-(p-\alpha)\right] + (p-\alpha)\left[2\alpha(p-\alpha)-(\alpha+n+p-1)\right]u_2^2}{2\left[(\alpha+n+p-1)^2 + (p-\alpha)^2 u_2^2\right]} \\
&\quad = \frac{A + Bu_2^2}{2C},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= (\alpha+n+p-1)\{2\alpha^2 + (2n+2p-1)\alpha - p\} \\
B &= (p-\alpha)\{2\alpha(p-\alpha) - (\alpha+n+p-1)\} \\
C &= (\alpha+n+p-1)^2 + (p-\alpha)^2 u_2^2 > 0.
\end{aligned}
$$

We note that $Re\phi(iu_2, v_1) \leqslant 0$ if and only if, $A \leqslant 0$ and $B \leqslant 0$. From $A \leqslant 0$, we obtain α as given by (3.1) and $B \leqslant 0$ gives us $0 \leqslant \alpha < 1$.

Now using Lemma 2.1 we see that $Reh(z) > 0$ for $z \in E$ and hence $Re\frac{z(I_{n+p}f(z))'}{I_{n+p}f(z)} > \alpha$, for $z \in E$ with α given by (3.1). □

Special cases

(i) When $p = 1$ and $k = 2$, we obtain a result proved in [3].

(ii) With $p = 1, n = 0$, and $k = 2$, we have a well-known result that every convex univalent function is starlike univalent of order $\frac{1}{2}$.

(iii) $R_k(n, p, 0) \subset R_k(n+1, p, 0)$, $n = 0, 1, 2, \ldots$.

THEOREM 3.2. *Let $f \in R_k(n+1, p, \alpha)$. Then, g, defined by*

$$
g(z) = \frac{n+1}{z^{n-p+1}} \int_0^z t^{n-p} f(t)\,dt \tag{3.3}
$$

belongs to $R_k(n, p, \alpha)$ and conversely.

Proof. From (3.3), we have

$$
(n+1)f(z) = (n-p+1)g(z) + zg'(z). \tag{3.4}
$$

Using (1.5) and (3.4), we can write

$$
\begin{aligned}
(n+1)I_{n+p}f(z) &= (n-p+1)I_{n+p}g(z) + z(I_{n+p}g(z))' \\
&= (n+1)I_{n+p}g(z).
\end{aligned}
$$

Therefore

$$I_{n+p}f(z) = I_{n+p}g(z),$$

and this proves our result. □

THEOREM 3.3. *Let for* $z \in E$, $f \in R_k(n+1,p,\alpha)$. *Then* $f \in R_k(n,p,\alpha)$ *for* $|z| < R$, *where* R *is given by* (3.5) *and the value of* R *is exact.*

Proof. Let

$$\frac{z(I_{n+p}f(z))'}{I_{n+p}f(z)} = (p-\alpha)H(z) + \alpha, \quad ReH(z) > 0, \quad z \in E.$$

Using (1.5) and proceeding as in Theorem 3.1, we have

$$\begin{aligned}
\frac{1}{(p-\alpha)}\left[\frac{z(I_{n+p-1}f(z))'}{I_{n+p-1}f(z)} - \alpha\right] &= H(z) + \frac{(\frac{1}{p-\alpha})zH'(z)}{H(z) + \frac{\alpha+n+p-1}{p-\alpha}} \\
&= (\frac{k}{4}+\frac{1}{2})\left[h_1(z) + \frac{(\frac{1}{p-\alpha})zh_1'(z)}{h_1(z) + \frac{\alpha+n+p-1}{p-\alpha}}\right] \\
&\quad - (\frac{k}{4}-\frac{1}{2})\left[h_2(z) + \frac{(\frac{1}{p-\alpha})zh_2'(z)}{h_2(z) + \frac{\alpha+n+p-1}{p-\alpha}}\right]
\end{aligned}$$

Using Lemma 2.2, with $\mu = \frac{\alpha+n+p-1}{p-\alpha} (\neq -1)$ and $s = \frac{1}{p-\alpha} > 0$, we see that $f \in R_k(n,p,\alpha)$ for $|z| < R$, with

$$\begin{aligned}
R &= \frac{|\mu+1|}{\sqrt{A + (A^2 - |\mu^2 - 1|^2)^{\frac{1}{2}}}} \\
A &= 2(s+1)^2 + |\mu|^2 - 1,
\end{aligned} \tag{3.5}$$

and this radius is exact. □

As a special case, we note that, for $p = 1, n = 0, \alpha = 0$, and $k = 2$, we obtain a well-known result that radius of convexity R of a starlike (univalent) function is

$$R = \frac{1}{\sqrt{7+\sqrt{48}}} \simeq 0.268 \simeq 2 - \sqrt{3}.$$

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Mathematics Department
COMSATS Institute of Information Technology
Islamabad
Pakistan
e-mail: khalidanoor@hotmail.com
e-mail: khalidainayat@comsats.edu.pk