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ON THE BEHAVIOR OF THE FIRST EIGENVALUE OF THE SPHERICAL LAPLACIAN OPERATOR ON A SPHERICAL ANNULUS

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Abstract. In this paper, we show that the first Dirichlet eigenvalue of spherical Laplacian operator on a spherical annulus with fixed area and outer disk is decreasing while the inner disk moving towards the boundary, which is an analogy of [5]. Moreover, with [7], we conclude that: among all annuli on S^2 with fixed area, the sphereical band which is symmetric to the equator has the largest first Dirichlet eigenvalue.

1. Introduction

The study of eigenvalues for Laplacian operator problems is quite interesting. Many researchers have studied this subject by using variational principle and integral inequalities, for examples, Polya ([4]), Hersch ([2], [3]), Chie-Ping Chu ([1]), and many other mathematicians. But, in this paper, we are going to study some eigenvalue problems of spherical Laplacian operator by alternative approach. Instead of variational principle, we will use shape derivative to study our problems. To describe our problems explicitly, we need some notations. Let S^2 denote the unit sphere in $\mathbb{R}^3$, and

$$X(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\cos\phi),\ 0 \leqslant \phi \leqslant \pi, 0 \leqslant \theta < 2\pi,$$

the Euler coordinate for S^2. We use

$$T_\xi := \begin{pmatrix} \cos\xi & 0 & \sin\xi \\ 0 & 1 & 0 \\ -\sin\xi & 0 & \cos\xi \end{pmatrix} \tag{1.1}$$

to denote the rotation from the $\mathscr{Z}$ axis to the $\mathscr{X}$ axis with angle ξ and $C(\phi_0) = \{X(\theta,\phi) | 0 \leqslant \phi \leqslant \phi_0,\ 0 \leqslant \theta \leqslant 2\pi\}$ be a spherical cap. For $0 < \phi_0 < \phi_1$, we denote the spherical band

$$B(\phi_0,\phi_1) = \text{ the closure of } \{C(\phi_1) \setminus C(\phi_0)\},$$

and

$$C^\xi(\phi_0) = T_\xi(C(\phi_0)),$$
$$B(\phi_0,\phi_1;\xi) = \text{ the closure of } \{C(\phi_1) \setminus C^\xi(\phi_0)\},$$

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for $0 \leqslant \xi \leqslant \phi_1 - \phi_0$.

We will concentrate on the study of the following eigenvalue value problem

$$\begin{cases} \triangle_{S^2} u + \lambda u & = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} & = 0. \end{cases} \tag{1.2}$$

where $\Omega = B(\phi_0, \phi_1; \xi)$ for $0 < \xi < \phi_1 - \phi_0$. Also note that the Laplacian operator Δ_{S^2} on S^2 can be written as

$$\Delta_{S^2} u(\phi, \theta) = \frac{1}{\sin\phi}[\frac{\partial}{\partial\phi}(\sin\phi u_\phi) + \frac{\partial}{\partial\theta}(\frac{u_\theta}{\sin\phi})]. \tag{1.3}$$

2. The behavior of the first Dirichlet eigenvalue on a spherical annulus

In [5], the authors studied the behavior of the first eigenvalue $\lambda_1(h)$ of

$$\begin{cases} \triangle u + \lambda u & = 0 \text{ in } A(r;h) \\ u|_{\partial A(r;h)} & = 0 \end{cases} \tag{2.1}$$

where $0 < r < 1$, $A(r;h) = D_1(0) \setminus D_r(h)$ and $D_r(h) = \{(x,y) \in \mathbb{R}^2 | (x-h)^2 + y^2 \leqslant r^2\}$. They proved that $\lambda_1(h)$ is decreasing in $(0, 1-r)$, i.e., $\lambda_1(h)$ is decreasing as the inner disk $D(r,h)$ of $A(r;h)$ moving towards to the boundary.

In the next, we are going to show an analogy of this fact on S^2. Let $\lambda_1(\xi)$ denote the first eigenvalue of

$$\begin{cases} \triangle_{S^2} u + \lambda u = & 0 \text{ in } B(\phi_0, \phi_1; \xi), \\ u_{|\partial B(\phi_0,\phi_1;\xi)} = & 0, \end{cases} \tag{2.2}$$

where $B(\phi_0, \phi_1; \xi)$ is as that defined in section 1.

LEMMA 2.1. *$\dot{\lambda}_1(\xi) = -\int_{\partial C^\xi(\phi_0)} (\frac{\partial u_1}{\partial n})^2 \cos\tilde{\theta} ds$, where $\dot{\lambda}_1(\xi) = \frac{d\lambda_1(\xi)}{d\xi}$, $u_1(\cdot;\xi)$ is the normalized positive eigenfunction corresponding $\lambda_1(\xi)$, n is the outer normal vector along $\partial B(\phi_0, \phi_1; \xi)$ and $\tilde{X}(\tilde{\theta}, \tilde{\phi})$ is a choice of Euler coordinate such that the north pole of $\tilde{X}(\tilde{\theta}, \tilde{\phi})$ locates at the center of $C^\xi(\phi_0)$ and $B(\phi_0, \phi_1; \xi)$ is symmetric to $\{\tilde{X}(0, \phi) | 0 \leqslant \phi \leqslant \pi\}$.*

Proof. For the differentiability, the readers can refer to [6]. Let $X(\theta, \phi)$ be the original Euler coordinate. Denote $\dot{u}_1(\theta, \phi; \xi) = \dfrac{\partial u_1(\theta, \phi; \xi)}{\partial \xi}$ and differentiate the equation

$$\begin{cases} \triangle_{S^2} u_1 + \lambda_1 u & = 0 \text{ in } B(\phi_0, \phi_1; \xi), \\ u|_{\partial B(\phi_0,\phi_1;\xi)} & = 0, \end{cases} \tag{2.3}$$

with respect to ξ, then we have

$$\begin{cases} \triangle_{S^2} \dot{u}_1 + \lambda_1 \dot{u}_1 & = -\dot{\lambda}_1 u \text{ in } B(\phi_0, \phi_1; \xi), \\ \dot{u}_1|_{\partial C(\phi_1)} & = 0. \end{cases} \tag{2.4}$$

(2.3) and (2.4) lead to

$$\begin{aligned}\dot{\lambda}_1(\xi) &= \iint_{B(\phi_0,\phi_1;\xi)} (\dot{u}_1 \triangle_{S^2} u_1 - u_1 \triangle_{S^2} \dot{u}_1)\, dA \\ &= \int_{\partial B(\phi_0,\phi_1;\xi)} (\dot{u}_1 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial \dot{u}_1}{\partial n})\, ds \\ &= \int_{\partial C^{\xi}(\phi_0)} \dot{u}_1 \frac{\partial u_1}{\partial n}\, ds. \end{aligned} \tag{2.5}$$

It remains to compute $\dot{u}_1$ on $C^{\xi}(\phi_1)$. For simplicity, we may just compute $\dot{\lambda}_1(0)$. Note that for $\xi = 0$, $\tilde{X}(\tilde{\theta},\tilde{\phi}) = X(\theta,\phi)$ and the first normalized positive eigenfunction u_1 for 2.3 is parameterized by (θ,ϕ,ξ). For $\xi > 0$, We denote $(\hat{\theta},\hat{\phi}) = X^{-1}T_{-\xi}X$ and $w(\hat{\theta},\hat{\phi};\xi) = u_1(\theta,\phi;\xi)$, then it is easy to see that $\frac{\partial u_1}{\partial n} = \frac{-\partial w}{\partial \hat{\phi}}$ on $\partial C^{\xi}(\phi_0)$. Also note that the $\hat{\phi}(\theta,\phi;\xi) = \phi_0$ if $X(\theta,\phi)$ locates on $\partial C^{\xi}(\phi_0)$. Since $w(\hat{\theta},\hat{\phi};\xi) = 0$ for $\hat{\phi} = \phi_0$, we obtain

$$w_{\hat{\theta}} \frac{\partial \hat{\theta}}{\partial \xi} + w_{\hat{\phi}} \frac{\partial \hat{\phi}}{\partial \xi} + w_{\xi} = 0 \tag{2.6}$$

on $C^{\xi}(\phi_0)$. Moreover, with

$$\begin{aligned} T_{-\xi} \begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix} &= \begin{pmatrix} \cos\xi & 0 & -sin\xi \\ 0 & 1 & 0 \\ \sin\xi & 0 & \cos\xi \end{pmatrix} \begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} cos\xi\cos\theta\sin\phi - \sin\xi\cos\phi \\ \sin\theta\sin\phi \\ \sin\xi\cos\theta\sin\phi + \cos\xi\cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\hat{\theta}\sin\hat{\phi} \\ \sin\hat{\theta}\sin\hat{\phi} \\ \cos\hat{\phi} \end{pmatrix} \end{aligned} \tag{2.7}$$

differentiate the third component of (2.7) with respect to ξ, we can derive

$$-\frac{\partial \hat{\phi}}{\partial \xi} \sin\hat{\phi} = \cos\xi\cos\theta\sin\phi - \sin\xi\cos\phi, \tag{2.8}$$

take $\xi = 0$, then $\hat{\phi} = \phi$, we have $\frac{\partial \hat{\phi}}{\partial \xi}|_{\xi=0} = -\cos\theta$. Note that $w_{\hat{\theta}}(\hat{\theta},\phi_0;\xi) = 0$ and plug it into 2.6, we obtain $w_{\xi}(\cdot;0) = \dot{u}_1(\cdot;0) = -\frac{\partial u_1}{\partial n}\cos\theta$ on $\partial C(\phi_0)$. Hence, by (2.5)

$$\dot{\lambda}_1(0) = -\int_{\partial C(\phi_0)} (\frac{\partial u_1}{\partial n})^2 \cos\theta\, ds,$$

and by the same argument, we can show

$$\dot{\lambda}_1(\xi) = -\int_{\partial C^{\xi}(\phi_0)} (\frac{\partial u_1}{\partial n})^2 \cos\tilde{\theta}\, ds,$$

□

With the proceeding lemma, we have

THEOREM 2.2. $\lambda_1(\xi)$ *is decreasing for* $0 < \xi < \phi_1 - \phi_0$.

Proof. It's sufficient to show that $\dot{\lambda}_1(\xi) < 0$ for $0 < \xi < \phi_1 - \phi_0$. Our strategy is as that in [5]. Let $\tilde{X}(\tilde{\theta}, \tilde{\phi})$ is a choice of Euler coordinate for S^2 such that the north pole is located at the center of $C^{\xi}(\phi_0)$ and $B(\phi_0, \phi_1; \xi)$ is symmetric to the geodesic circle

$$\{\tilde{X}(0, \tilde{\phi}) \cup \tilde{X}(\pi, \tilde{\phi}) |\ 0 \leqslant \phi \leqslant \pi\}.$$

Let

$$\Omega^+(\xi) = \{\tilde{X}(\tilde{\theta}, \tilde{\phi})|\ -\pi/2 \leqslant \tilde{\theta} \leqslant \pi/2, \text{ and } \tilde{X}(\tilde{\theta}, \tilde{\phi}) \in B(\phi_0, \phi_1; \xi)\},$$

$$\Omega^-(\xi) = \text{the reflection of } \Omega^+(\xi) \text{ with respect to the geodesic circle}$$
$$\{\tilde{X}(\pi/2, \tilde{\phi}) \cup \tilde{X}(3\pi/2, \tilde{\phi}) | 0 \leqslant \phi \leqslant \pi\},$$

and

$$\Omega(\xi) = \Omega^+(\xi) \cup \Omega^-(\xi).$$

We define a function

$$z(\tilde{\theta}, \tilde{\phi}) = \begin{cases} u_1(\tilde{\theta}, \tilde{\phi}; \xi), \text{ if } -\pi/2 \leqslant \tilde{\theta} \leqslant \pi/2, \\ u_1(\tilde{\theta} - \pi, \tilde{\phi}; \xi), \text{ if } \pi/2 \leqslant 3\pi/2, \end{cases}$$

where $u_1(\tilde{\theta}, \tilde{\phi}; \xi)$ is the normalized and positive first eigenfunction on $B(\phi_0, \phi_1; \xi)$. Then by the maximum principle, we know the function

$$v(\tilde{\theta}, \tilde{\phi}) = u_1(\tilde{\theta}, \tilde{\phi}; \xi) - z(\tilde{\theta}, \tilde{\phi})$$

is positive in $\Omega^-(\xi)$, and next, by the Hopf's Lemma, $\frac{\partial v}{\partial n} < 0$ on $\partial C(\phi_0) \cap \partial \Omega^-(\xi)$, hence

$$\frac{\partial u_1(\tilde{\theta}, \phi_0}{\partial n} < \frac{\partial z(\tilde{\theta}, \phi_0)}{\partial n} < 0$$

on $\partial C(\phi_0) \cap \partial \Omega^-(\xi)$, i.e.,

$$(\frac{\partial u_1(\tilde{\theta}, \phi_0}{\partial n})^2 > (\frac{\partial z(\tilde{\theta}, \phi_0)}{\partial n})^2,$$

on $\partial C(\phi_0) \cap \partial \Omega^-(\xi)$. Since $\int_{\partial C^{\xi}(\phi_0)} (\frac{\partial z}{\partial n})^2 \cos \tilde{\theta}\, ds = 0$, we obtain

$$\dot{\lambda}_1(\xi) = -\int_{\partial C^{\xi}(\phi_0)} (\frac{\partial u_1}{\partial n})^2 \cos \tilde{\theta}\, ds < 0,$$

for $0 < \xi < \phi_1 - \phi_0$. This completes the proof. □

Then, applying the following theorem

THEOREM 2.3. (Theorem 1, [7]) $0 < A < 2$, $0 < \xi \leqslant \cos^{-1}(A/2)$, *and* $f(\xi) = \cos^{-1}(\cos \xi - A)$. *Let* $B(\xi, f(\xi))$ *be a spherical band with area* $2\pi A$ *and* $\mu_1(\xi)$, *defined on* $(0, \pi - \cos^{-1}(1 - A))$, *the corresponding first Dirichlet eigenvalue on* $B(\xi, f(\xi))$. *Then* $\mu_1(\xi)$ *is increasing in* $(0, \cos^{-1}(A/2))$ *and attains its maximum when* $B(\xi, f(\xi))$ *is symmetric to the equator, i.e.,* $\xi = \cos^{-1}(A/2)$.

we conclude that

THEOREM 2.4. *Among all annuli with fixed area $2\pi A$, the spherical band which is symmetric to the equator has the largest first Dirichlet eigenvalue.*

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