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ON APPROXIMATE DERIVATIONS

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(*communicated by Th. M. Rassias*)

Abstract. Let $\mathscr{A}_1$ be a subalgebra of a Banach algebra $\mathscr{A}$ and let $f : \mathscr{A}_1 \to \mathscr{A}$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta \quad \text{and} \quad \|f(x \cdot y) - x \cdot f(y) - f(x) \cdot y\| \leqslant \varepsilon,$$

for all $x, y \in \mathscr{A}_1$ and for some constants $\delta, \varepsilon \geqslant 0$. Then we prove that there exists a unique derivation $d : \mathscr{A}_1 \to \mathscr{A}$ such that

$$\|f(x) - d(x)\| \leqslant \delta, \quad x \in \mathscr{A}_1$$

and

$$x \cdot (f(y) - d(y)) = 0, \quad x, y \in \mathscr{A}_1.$$

Moreover, we also prove the Rassias type stability result for derivations.

Let $\mathscr{A}$ be an algebra and let $\mathscr{A}_1$ be a subalgebra of $\mathscr{A}$. A function $d : \mathscr{A}_1 \to \mathscr{A}$ is called a *derivation* if and only if it satisfies the following functional equations

$$d(x+y) = d(x) + d(y), \quad x, y \in \mathscr{A}_1; \tag{1}$$

$$d(x \cdot y) = x \cdot d(y) + d(x) \cdot y, \quad x, y \in \mathscr{A}_1. \tag{2}$$

The aim of the present paper is to examine the stability problem of derivations. For the theory of the stability of functional equations see Hyers, Isac and Rassias [4]. On approximately derivations Šemrl in [8] proved the following.

THEOREM 1. (P. Šemrl) *Let X be an infinite dimensional Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. Assume that $\mathscr{B}$ is a standard subalgebra of $B(X)$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{3}$$

Suppose that $d : \mathscr{B} \to B(X)$ is a mapping such that

$$\|d(x \cdot y) - x \cdot d(y) - d(x) \cdot y\| \leqslant \varphi(\|x\| \cdot \|y\|),$$

for all $x, y \in \mathscr{B}$. Then there exists $z \in B(X)$ such that

$$d(x \cdot y) = x \cdot z - z \cdot x \quad x \in \mathscr{B}.$$

So, d is a derivation.

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Generally this superstability result is not true. The counter-example reads as follows: take a function f defined on the subalgebra

$$\mathscr{A}_1 = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{R} \right\}$$

of the algebra $\mathscr{A} = M_{2\times 2}$ of all 2×2-matrices given by the formula:

$$f\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in \mathscr{A}_1.$$

Then

$$f\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right) - f\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) - f\left(\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$f\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right) - \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \cdot f\left(\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right) - f\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

for all $x, y \in \mathbb{R}$. Therefore, the function f satisfies assumptions of Šemrl's theorem (with a constant function φ) but f is not a derivation.

For arbitrary Banach algebras we have the following stability result.

THEOREM 2. *Let $\mathscr{A}_1$ be a subalgebra of a Banach algebra $\mathscr{A}$. Assume that $f : \mathscr{A}_1 \to \mathscr{A}$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta, \quad x, y \in \mathscr{A}_1 \tag{4}$$

and

$$\|f(x \cdot y) - x \cdot f(y) - f(x) \cdot y\| \leqslant \varepsilon, \quad x, y \in \mathscr{A}_1, \tag{5}$$

for some constants $\delta, \varepsilon \geqslant 0$. Then there exists a unique derivation $d : \mathscr{A}_1 \to \mathscr{A}$ such that

$$\|f(x) - d(x)\| \leqslant \delta, \quad x \in \mathscr{A}_1. \tag{6}$$

Moreover,

$$x \cdot (f(y) - d(y)) = 0, \quad x, y \in \mathscr{A}_1. \tag{7}$$

Proof. By condition (4), the Hyers theorem (see [3] or Hyers, Isac and Rassias [4]) shows that there exists an additive function $d : \mathscr{A}_1 \to \mathscr{A}$ such that

$$\|f(x) - d(x)\| \leqslant \delta, \quad x \in \mathscr{A}_1. \tag{8}$$

Now we only need to show that d satisfies (2). Our inequality (8) implies that

$$\|f(nx) - d(nx)\| \leqslant \varepsilon, \quad x \in \mathscr{A}_1, \quad n \in \mathbb{N}.$$

By the additivity of d it is easy to see that then

$$\left\|\frac{1}{n} f(nx) - d(x)\right\| \leqslant \frac{1}{n}\varepsilon, \quad x \in \mathscr{A}_1, \quad n \in \mathbb{N},$$

which means that

$$d(x) = \lim_{n\to\infty} \frac{1}{n} f(nx), \quad x \in \mathscr{A}_1. \tag{9}$$

Condition (5) implies that the function $r : \mathscr{A}_1 \times \mathscr{A}_1 \to \mathscr{A}$ defined by

$$r(x,y) = f(x\cdot y) - x\cdot f(y) - f(x)\cdot y, \quad x,\ y \in \mathscr{A}_1 \tag{10}$$

is bounded. Hence,

$$\lim_{n\to\infty} \frac{r(nx,y)}{n} = 0, \quad x,y \in \mathscr{A}_1. \tag{11}$$

Now, applying (9) we get

$$d(x\cdot y) = x\cdot f(y) + d(x)y, \quad x,y \in \mathscr{A}_1. \tag{12}$$

Indeed,

$$\begin{aligned} d(x\cdot y) &= \lim_{n\to\infty} \frac{f(n(x\cdot y))}{n} = \lim_{n\to\infty} \frac{f((nx)\cdot y)}{n} \\ &= \lim_{n\to\infty} \frac{nx\cdot f(y) + f(nx)\cdot y + r(nx,y)}{n} \\ &= \lim_{n\to\infty} \left(x\cdot f(y) + \frac{f(nx)}{n}\cdot y + \frac{r(nx,y)}{n} \right) \\ &= x\cdot f(y) + d(x)\cdot y, \quad x,y \in \mathscr{A}_1. \end{aligned}$$

Let $x,y \in \mathscr{A}_1$ and $n \in \mathbb{N}$ be fixed. Then, using (12) and the additivity of d, we have

$$\begin{aligned} x\cdot f(ny) + nd(x)\cdot y &= x\cdot f(ny) + d(x)\cdot ny = d(x\cdot ny) \\ &= d(nx\cdot y) = nx\cdot f(y) + d(nx)\cdot y \\ &= nx\cdot f(y) + nd(x)\cdot y. \end{aligned}$$

Therefore,

$$x\cdot f(y) = x\cdot \frac{f(ny)}{n}, \quad x,y \in \mathscr{A}_1,\ n \in \mathbb{N}.$$

Sending n to infinity, by (9), we see that

$$x\cdot f(y) = x\cdot d(y), \quad x,y \in \mathscr{A}_1. \tag{13}$$

Combining this formula with equation (12) we have that d satisfies (2) which is the desired conclusion.

To prove the uniqueness property of d, assume that d^* is another derivation fulfilling $\|f(x) - d^*(x)\| \leqslant \delta$, $x \in \mathscr{A}_1$. Since both d and d^* are additive we deduce that

$$n\|d(x) - d^*(x)\| = \|d(nx) - d^*(nx)\| \leqslant 2\delta$$

so that

$$\|d(x) - d^*(x)\| \leqslant \frac{2\delta}{n},$$

for all $x \in \mathscr{A}_1$ and $n \in \mathbb{N}$. Letting n to infinity we find that

$$d(x) = d^*(x), \quad x \in \mathscr{A}_1.$$

Moreover, identity (13) leads to (7), which completes the proof.

A generalization of Hyers's theorem given by Isac and Rassias (see [5] or [4]) shows that if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \tag{14}$$

$$\psi(ts) \leqslant \psi(t)\psi(s), \quad t,s > 0, \tag{15}$$

$$\psi(t) < t, \quad t > 1, \tag{16}$$

δ is a positive number and $f : \mathscr{A}_1 \to \mathscr{A}$ fulfills the inequality

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta(\psi(\|x\|) + \psi(\|y\|)), \quad x,y \in \mathscr{A}_1,$$

then there exists a unique additive mapping $d : \mathscr{A}_1 \to \mathscr{A}$ and a constant $k \in \mathbb{R}$ such that

$$\|f(x) - d(x)\| \leqslant k\delta\psi(\|x\|), \quad x \in \mathscr{A}_1.$$

The classical example of the function ψ fulfilling (14), (15) and (16) is a map $\psi(t) = t^q$, $t \in \mathbb{R}_+$, where $q < 1$. The example of the function φ satisfying (3) is $\varphi(t) = t^p$, $t \in \mathbb{R}_+$, where $p < 1$.

For derivation we have the following generalization of Theorem 2.

THEOREM 3. *Let* $\mathscr{A}_1$ *be a subalgebra of a Banach algebra* $\mathscr{A}$, δ *be a positive number,* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function with properties* (14), (15), (16) *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *fulfills* (3). *Assume that* $f : \mathscr{A}_1 \to \mathscr{A}$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta(\psi(\|x\|) + \psi(\|y\|)), \quad x,y \in \mathscr{A}_1 \tag{17}$$

and

$$\|f(x\cdot y) - x\cdot f(y) - f(x)\cdot y\| \leqslant \varphi(\|x\|\cdot\|y\|), \quad x,y \in \mathscr{A}_1. \tag{18}$$

Then there exists a unique derivation $d : \mathscr{A}_1 \to \mathscr{A}$ *and a constant* $k \in \mathbb{R}$ *such that*

$$\|f(x) - d(x)\| \leqslant k\delta\psi(\|x\|), \quad x \in \mathscr{A}_1. \tag{19}$$

Moreover, condition (7) *holds true.*

Proof. The proof is similar to that of Theorem 2. We start with the Isac - Rassias theorem. Then we have the existence of an additive function $d : \mathscr{A}_1 \to \mathscr{A}$ such that

$$\|f(x) - d(x)\| \leqslant k\delta\psi(\|x\|), \quad x \in \mathscr{A}_1, \tag{20}$$

for some real constant k. Hence,

$$\|f(nx) - d(nx)\| \leqslant k\delta\psi(\|nx\|), \quad x \in \mathscr{A}_1, n \in \mathbb{N}$$

and by the additivity of d it is easy to observe that then

$$\|\frac{1}{n}f(nx) - d(x)\| \leqslant k\delta\frac{\psi(n\|x\|)}{n}, \quad x \in \mathscr{A}_1, n \in \mathbb{N},$$

which jointly with (14) leads to

$$d(x) = \lim_{n\to\infty} \frac{1}{n} f(nx), \quad x \in \mathscr{A}_1. \tag{21}$$

Let r be a function defined by (10). Then, using inequality (18) and assumption (3), we obtain that the function r satisfies condition (11) and the further part of the proof is the same as in the proof of Theorem 2.

If we want to extend our result to the case of $p, q > 1$, then we can adopt the method presented by Gajda in [2] to obtain the Isac - Rassias result for the function ψ fulfilling

$$\lim_{t\to 0} \frac{\psi(t)}{t} = 0, \tag{22}$$

$$\psi(ts) \leqslant \psi(t)\psi(s), \quad t, s > 0, \tag{23}$$

$$\psi(t) < t, \quad t \in (0,1), \tag{24}$$

After this modification we get the following version of Theorem 3.

THEOREM 4. *Let* $\mathscr{A}_1$ *be a subalgebra of a Banach algebra* $\mathscr{A}$, δ *be a positive number,* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a function with properties* (22), (23), (24) *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *fulfills the following condition*

$$\lim_{t\to 0} \frac{\varphi(t)}{t} = 0. \tag{25}$$

Assume that $f : \mathscr{A}_1 \to \mathscr{A}$ *satisfies* (17) *and* (18). *Then there exists a unique derivation* $d : \mathscr{A}_1 \to \mathscr{A}$ *and a constant* $k \in \mathbb{R}$ *satisfying* (19) *and* (7).

The proof of this fact is analogous to the presented. We must only observe that if d is an additive function fulfilling (20), which existence guarantees the modification of the Isac - Rassias theorem, then condition (22) implies

$$d(x) = \lim_{n\to\infty} nf\left(\frac{1}{n}x\right), \quad x \in \mathscr{A}_1$$

(instead of (21)) and a function r defined by (10) satisfies

$$\lim_{n\to\infty} nr\left(\frac{1}{n}x, y\right) = 0, \quad x, y \in \mathscr{A}_1$$

(instead of (11)).

In the next part of this paper we assume that the algebra $\mathscr{A}$ contains the identity. As a simply consequence of Theorem 2 we have the following superstability result.

COROLLARY 1. *Let* $\mathscr{A}$ *be a Banach algebra with the identity* e *and let* $\mathscr{A}_1$ *be a subalgebra of* $\mathscr{A}$ *such that* $e \in \mathscr{A}_1$. *If* $\delta, \varepsilon \geqslant 0$ *and a function* $f : \mathscr{A}_1 \to \mathscr{A}$ *satisfies* (4) *and* (5), *then* f *is a derivation.*

Proof. Condition (7) and the fact that $e \in \mathscr{A}_1$ implies $f = d$, which ends the proof.

In the case of algebras with the identity we can also proved the superstability of equation (2).

THEOREM 5. *Let $\mathscr{A}$ be a normed algebra with the identity e and the norm satisfying the inequality*

$$\|x\cdot y\| \leqslant \|x\|\|y\|, \quad x,y\in\mathscr{A} \tag{26}$$

and let $\mathscr{A}_1$ be a subalgebra of $\mathscr{A}$ such that $e\in\mathscr{A}_1$. If $\varepsilon\geqslant 0$ and a function $f:\mathscr{A}_1\to\mathscr{A}$ satisfies (5), *then f fulfills* (2).

Proof. By (5), we have

$$\|f(nx\cdot y)-nx\cdot f(y)-f(nx)\cdot y\|\leqslant\varepsilon, \quad x,y\in\mathscr{A}_1, n\in\mathbb{N}.$$

Hence,

$$\|\frac{f(nx\cdot y)}{n}-x\cdot f(y)-\frac{f(nx)\cdot y}{n}\|\leqslant\frac{\varepsilon}{n}, \quad x,y\in\mathscr{A}_1, n\in\mathbb{N},$$

which leads to

$$\lim_{n\to\infty}\left(\frac{f(nx\cdot y)}{n}-\frac{f(nx)\cdot y}{n}\right)=x\cdot f(y), \quad x,y\in\mathscr{A}_1. \tag{27}$$

Similarly, putting $y=ny$ in (5) and dividing the result by n we get

$$\|\frac{f(x\cdot ny)}{n}-\frac{x\cdot f(ny)}{n}-f(x)\cdot y\|\leqslant\frac{\varepsilon}{n}, \quad x,y\in\mathscr{A}_1, n\in\mathbb{N},$$

which means that

$$\lim_{n\to\infty}\left(\frac{f(x\cdot ny)}{n}-\frac{x\cdot f(ny)}{n}\right)=f(x)\cdot y, \quad x,y\in\mathscr{A}_1. \tag{28}$$

Let $x,y\in\mathscr{A}_1$ and $n\in\mathbb{N}$ be fixed. Then

$$\begin{aligned}
0 &\leqslant \|f(x\cdot y)-x\cdot f(y)-f(x)\cdot y\| \\
&\leqslant \|f(x\cdot y)-\frac{f(ne\cdot x\cdot y)}{n}+\frac{f(ne)}{n}\cdot x\cdot y-x\cdot f(y)+\frac{f(nx\cdot y)}{n} \\
&\qquad -\frac{f(nx)}{n}\cdot y-f(x)\cdot y+\frac{f(nx\cdot y)}{n}-x\cdot\frac{f(ny)}{n}\| \\
&\quad +\|\frac{f(ne\cdot x\cdot y)}{n}-\frac{f(ne)}{n}\cdot x\cdot y-\frac{f(nx\cdot y)}{n}+\frac{f(nx)}{n}\cdot y-\frac{f(nx\cdot y)}{n}+x\cdot\frac{f(ny)}{n}\| \\
&= \|f(x\cdot y)-\frac{f(ne\cdot x\cdot y)}{n}+\frac{f(ne)}{n}\cdot x\cdot y-x\cdot f(y)+\frac{f(nx\cdot y)}{n} \\
&\qquad -\frac{f(nx)}{n}\cdot y-f(x)\cdot y+\frac{f(nx\cdot y)}{n}-x\cdot\frac{f(ny)}{n}\| \\
&\quad +\frac{1}{n}\|f(nx)\cdot y-f(ne)\cdot x\cdot y+x\cdot f(ny)-f(nx\cdot y)\| \\
&= \|f(x\cdot y)-\frac{f(ne\cdot x\cdot y)}{n}+\frac{f(ne)}{n}\cdot x\cdot y-x\cdot f(y)+\frac{f(nx\cdot y)}{n} \\
&\qquad -\frac{f(nx)}{n}\cdot y-f(x)\cdot y+\frac{f(nx\cdot y)}{n}-x\cdot\frac{f(ny)}{n}\| \\
&\quad +\frac{1}{n}\|(f(ne\cdot x)-ne\cdot f(x)-f(ne)\cdot x)\cdot y+nf(x)\cdot y+x\cdot f(ny)-f(nx\cdot y)\|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \|f(x\cdot y) - \frac{f(ne\cdot x\cdot y)}{n} + \frac{f(ne)}{n}\cdot x\cdot y - x\cdot f(y) + \frac{f(nx\cdot y)}{n} \\
&\qquad - \frac{f(nx)}{n}\cdot y - f(x)\cdot y + \frac{f(nx\cdot y)}{n} - x\cdot\frac{f(ny)}{n}\| \\
&\quad + \frac{1}{n}\|f(ne\cdot x) - ne\cdot f(x) - f(ne)\cdot x\|\cdot\|y\| \\
&\qquad + \frac{1}{n}\|f(x)\cdot ny + x\cdot f(ny) - f(x\cdot ny)\| \\
&\leqslant \|f(x\cdot y) - \left(\frac{f(ne\cdot x\cdot y)}{n} - \frac{f(ne)\cdot x\cdot y}{n}\right) - x\cdot f(y) + \left(\frac{f(nx\cdot y)}{n} - \frac{f(nx)\cdot y}{n}\right) \\
&\qquad - f(x)\cdot y + \left(\frac{f(nx\cdot y)}{n} - \frac{x\cdot f(ny)}{n}\right)\| + \frac{1}{n}\varepsilon\|y\| + \frac{1}{n}\varepsilon.
\end{aligned}$$

Applying (27) and (28) we observe that the right side of the last inequality tends to 0 when n tends to infinity. So, the function f satisfies equation (2), which ends the proof.

REMARK 1. If $\mathscr{A}$ is a Banach algebra, then condition (26) can be assumed.

REMARK 2. As in the previous part of our paper we can generalize our results replacing constants δ and ε by functions fulfilling corresponding conditions.

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