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NEW ITERATIVE APPROXIMATION FOR A SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS WITH SET–VALUED MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new system of generalized nonlinear variational inclusions involving generalized m-accretive mappings. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang [10], we also prove the existence theorems of the solution and convergence theorems of the generalized Mann iterative procedures with mixed errors for this system of variational inclusions in q-uniformly smooth Banach spaces.

1. Introduction and preliminaries

Variational inclusion is an important generalization of variational inequality, which has been studied extensively by many authors (see, for example, [1]-[9], [13], [15], [18], [19] and the references therein). In 1995, Liu [14] introduced and studied the Ishikawa iteration process and Mann iteration process with errors and proved that if X is a uniformly smooth Banach space and $T : X \to X$ is Lipschitzian and strongly accretive, then the Ishikawa iteration process and Mann iteration process with errors converge strongly to the unique solution of the equation $Tx = f$, respectively. Furthermore, Huang [8] introduced a new class of generalized nonlinear implicit quasi-variational inclusions and proved its equivalent with a class of fixed point problems by making use of the properties of maximal monotone. The author also proved the existence of solutions for this generalized nonlinear implicit quasi-variational inclusions and the convergence of iterative sequences generated by the perturbed algorithms.

On the other hand, in 2001, Huang and Fang [10] introduced the concept of a generalized m-accretive mapping, which is a generalization of an m-accretive mapping and gave the definition of the resolvent operator for the generalized m-accretive mapping in Banach space. Recently, Huang, Fang and Deng [11] and Huang [12] introduced and studied some new classes of nonlinear variational inclusions involving generalized m-accretive mappings in Banach spaces. By using the resolvent operator technique in [10],

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they constructed some new iterative algorithms for solving the nonlinear variational inclusions involving generalized m-accretive mappings. They also proved the existence of solution for nonlinear variational inclusions involving generalized m-accretive mappings and convergence of sequences generated by the algorithms.

Motivated and inspired by the recent works of [9], [11], [12], [15], in this paper, we introduce and study a new system of generalized nonlinear variational inclusions involving generalized m-accretive mappings. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang [10], we construct some new iterative algorithms for approximating the solution for this system of variational inclusions. We also give the convergence analysis of the generalized Mann iterative procedures with mixed errors generated by the algorithms in q-uniformly smooth Banach spaces. Our results improve and generalize the corresponding results of [2], [6], [15] and [19].

Throughout this paper, we suppose that X is a real Banach space endowed with dual space X^* and the dual pair $\langle \cdot, \cdot \rangle$ between X and X^*. Let 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex (see, for example, [17]). If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. In what follows we shall denote the single-valued generalized duality mapping by j_q.

Let $S, A, p : X \to X$ and $N_1, N_2, \eta_1, \eta_2 : X \times X \to X$ be single-valued mappings, $T, B : X \to 2^X$ be two set-valued mappings and M_1, M_2 be two generalized m-accretive mappings. For any given $f, g \in X$, we consider the following problem:

Find $x, y \in X$ such that $p(x) \in dom(M_1)$ and

$$\begin{cases} y - x - \lambda_1(N_1(S(y), v) - f) \in \lambda_1 M_1(p(x)), & \forall v \in T(y), \\ x - y - \lambda_2(N_2(A(x), u) - g) \in \lambda_2 M_2(y), & \forall u \in B(x), \end{cases} \tag{1.1}$$

where λ_1, λ_2 are two constants. Problem (1.1) is called the system of generalized nonlinear variational inclusions involving Generalized m-accretive mappings in Banach spaces.

If $p \equiv I$, the identity mapping, then problem (1.1) is equivalent to finding $x, y \in X$ such that

$$\begin{cases} y - x - \lambda_1(N_1(S(y), v) - f) \in \lambda_1 M_1(x), & \forall v \in T(y) \\ x - y - \lambda_2(N_2(A(x), u) - g) \in \lambda_2 M_2(y), & \forall u \in B(x). \end{cases} \tag{1.2}$$

If $X = X^* = H$ is a Hilbert space and $M_i = \partial\phi_i$, where $\partial\phi_i$ denote the subdifferential of a proper convex lower semi-continuous function ϕ_i on H, $i = 1, 2$, then problem (1.1) becomes the following problem:

Find $x,y \in X$ such that $p(x) \in dom(\partial\phi_1)$ and

$$\begin{cases} \langle \lambda_1(N_1(S(y),v) - f) + x - y, \eta_1(z,p(x))\rangle \geqslant \lambda_1\phi_1(p(x)) - \lambda_1\phi_1(z), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in T(y), z \in X, \\ \langle \lambda_2(N_2(A(x),u) - g) + y - x, \eta_2(z,y)\rangle \quad \geqslant \lambda_2\phi_2(y) - \lambda_2\phi_2(z), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall u \in B(x), z \in X. \end{cases} \tag{1.3}$$

which is called the system of generalized nonlinear variational inclusions for set-valued mappings in Hilbert spaces.

If $T,B : X \to X$ are two single-valued mappings, $\phi_1 = \phi_2 = \phi$ and for all $u,v \in X$, $\eta_1(u,v) = \eta_2(u,v) = u - v$ and $N_1(u,v) = N_2(u,v) = u - v$, then problem (1.3) becomes the following system of nonlinear variational inequalities:

Find $x,y \in X$ such that $p(x) \in dom(\partial\phi)$ and

$$\begin{cases} \langle \lambda_1(S(y) - T(y) - f) + x - y, z - p(x)\rangle \geqslant \lambda_1\phi(p(x)) - \lambda_1\phi(z), \quad \forall z \in X, \\ \langle \lambda_2(A(x) - B(x) - g) + y - x, z - y\rangle \geqslant \lambda_2\phi(y) - \lambda_2\phi(z), \quad \forall z \in X. \end{cases} \tag{1.4}$$

Problem (1.4) has been studied by Nie et al. [15] when $p = I$, $A = S$ and $B = T$.

REMARK 1.1. For appropriate and suitable choices of f, g, p, N_i, M_i for $i = 1,2$ and for the space X, a number of known classes of variational inequality, quasivariational inequality, complementarity and quasicomplementarity problems were studied previously by many authors, see, for example, [2], [4]-[6], [15], [19] and the references therein.

In this paper, we will use the following definitions and lemmas.

DEFINITION 1.1. The mapping $g : X \to X$ is said to be

(1) α-strongly accretive if there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle g(x) - g(y), j_2(x-y)\rangle \geqslant \alpha\|x-y\|^2, \quad \forall x,y \in X,$$

where $\alpha > 0$ is a constant;

(2) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|g(x) - g(y)\| \leqslant \beta\|x-y\|, \quad \forall x,y \in X.$$

DEFINITION 1.2. Let $A : X \to X$ be a single-valued mapping. A mapping $N : X \times X \to X$ is said to be

(1) σ-strongly accretive with respect to A in the first argument if there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle N(A(x),\cdot) - N(A(y),\cdot), j_2(x-y)\rangle \geqslant \sigma\|x-y\|^2, \quad \forall x,y \in X,$$

where $\sigma > 0$ is a constant;

(2) ϵ-Lipschitz continuous in the first argument if there exists a constant $\epsilon > 0$ such that

$$\|N(x,\cdot) - N(y,\cdot)\| \leqslant \epsilon\|x-y\|, \quad \forall x,y \in X.$$

Similarly, we can define the γ-Lipschitz continuity in the second argument of $N(\cdot,\cdot)$.

DEFINITION 1.3. ([10]) Let $\eta : X \times X \to X^*$ be a single-valued mapping and $A : X \to 2^X$ be a multi-valued mapping. Then A is said to be

(1) η-accretive if

$$\langle u - v, \eta(x, y)\rangle \geqslant 0, \quad \forall x, y \in X, u \in A(x), \text{ and } v \in A(y);$$

(2) strictly η-accretive if

$$\langle u - v, \eta(x, y)\rangle \geqslant 0, \quad \forall x, y \in X, u \in A(x), \text{ and } v \in A(y),$$

and equality holds if and only if $x = y$;

(3) strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geqslant r\|x - y\|^2, \quad \forall x, y \in X, u \in A(x), \text{ and } v \in A(y);$$

(4) generalized m-accretive if M is η-accretive and $(I + \lambda M)(X) = X$ for all (equivalently, for some) $\lambda > 0$.

REMARK 1.2. If $X = X^* = H$ is a Hilbert space, then (i)-(iv) of Definition 1.2. reduce to the definition of η-monotonicity, strict η-monotonicity, strong η-monotonicity and maximal η-monotonicity respectively; if X is uniformly smooth and $\eta(x, y) = j_2(x - y) \in J_2(x - y)$, then (i)-(iv) of Definition 1.3. reduce to the definitions of accretivity, strict accretivity, strong accretivity and m-accretivity in uniformly smooth Banach spaces, respectively (see [10], [11]).

DEFINITION 1.4. The mapping $\eta : X \times X \to X^*$ is said to be

(1) $\delta-$ strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle x - y, \eta(x, y)\rangle \geqslant \delta\|x - y\|^2, \quad \forall x, y \in X;$$

(2) $\tau-$ Lipschitz continuous if there exists a constnt $\tau > 0$ such that

$$\|\eta(x, y)\| \leqslant \tau\|x - y\|, \quad \forall x, y \in X.$$

DEFINITION 1.5. The mapping $T : X \to CB(X)$ is said to be ς-H-Lipschitz continuous if there exists a constant $\varsigma > 0$ such that

$$H(T(x), T(y)) \leqslant \varsigma\|x - y\|, \quad \forall x, y \in X,$$

where $CB(X)$ denotes the family of all the nonempty bounded closed sets of X and $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

The modules of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}\|x + y\| + \|x - y\| - 1 : \|x\| \leqslant 1, \|y\| \leqslant t\}.$$

A Banach space X is called uniformly smooth if $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$ and X is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X \leqslant ct^q$, where $q > 1$ is a real number.

It is well known that Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [17] proved the following result:

LEMMA 1.1. *Let $q > 1$ be a given real number and X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$, $j_q(x) \in J_q(x)$, there holds the following inequality*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

LEMMA 1.2. ([10], [11]) *Let $\eta : X\times X \to X^*$ be strict monotone and $A : X \to 2^X$ be a generalized m-accretive mapping. Then the following results hold:*

(1) $\langle u-v, \eta(x,y)\rangle \geqslant 0$, $\forall (y,v) \in Graph(A)$ implies $(x,u) \in Graph(A)$, where $Graph(A) = \{(x,u) \in X \times X : u \in A(x)\}$;

(2) for any $\rho > 0$, inverse mapping $(I+\rho A)^{-1}$ is single-valued.

DEFINITION 1.6. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is defined as follows:

$$J_A^\rho(z) = (I+\rho A)^{-1}(z), \quad \forall z \in X,$$

where $\rho > 0$ is a constant and $\eta : X \times X \to X^*$ is a strictly monotone mapping.

LEMMA 1.3. ([10], [11]) *Let $\eta : X\times X \to X^*$ be strongly monotone and Lipschitz continuous with constants $\delta > 0$ and $\tau > 0$, respectively. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is Lipschitz continuous with constant τ/δ, i.e.,*

$$\|J_A^\rho(x) - J_A^\rho(y)\| \leqslant \frac{\tau}{\delta}\|x-y\|, \quad \forall x, y \in X.$$

2. Existence theorems

In this section, we shall give the existence theorems of problems $(1.1)-(1.3)$, respectively.

LEMMA 2.1. *For given x, y in X, (x,y) is a solution of problem (1.1) if and only if*

$$\begin{cases} p(x) \in J_{M_1}^{\lambda_1}(p(x)+y-x-\lambda_1(N_1(S(y),T(y))-f)), \\ y \in J_{M_2}^{\lambda_2}(x-\lambda_2(N_2(A(x),B(x))-g)). \end{cases} \tag{2.1}$$

Proof. The proof directly follows from the definition of $J_{M_i}^{\lambda_i}$ for $i = 1,2$ and so it is omitted.

LEMMA 2.2. ([16]) *Let (X,d) be a complete metric space. Suppose that $Q : X \to CB(X)$ satisfies*

$$H(Q(x),Q(y)) \leqslant td(x,y), \quad \forall x, y \in X,$$

where $t \in (0,1)$ is a constant. Then the mapping Q has a fixed point in X.

THEOREM 2.1. *Let X be a q-uniformly smooth Banach space, mappings $S,A : X \to X$ be μ-Lipschitz continuous and ξ-Lipschitz continuous, respectively. $T,B : X \to CB(X)$ be ν-H-Lipschitz continuous and ζ-H-Lipschitz continuous, respectively. Let $p : X \to X$ be α-strongly monotone and β-Lipschitz continuous, $N_1 : X \times X \to X$ be ϵ_1-Lipschitz continuous and σ_1-strongly accretive with respect to S in the first argument, $N_2 : X \times X \to X$ be ϵ_2-Lipschitz continuous and σ_2-strongly accretive with respect to A in the first argument, N_i be γ_i-Lipschitz continuous in the second argument, $\eta_i : X \times X \to X^*$ be τ_i-Lipschitz continuous and δ_i-strongly monotone and $M_i : X \to 2^X$ be generalized m-accretive, for $i = 1,2$. Suppose that the following conditions hold:*

$$\begin{cases} h = (1 + \dfrac{\tau_1}{\delta_1})(1 - q\alpha + c_q\beta^q)^{\frac{1}{q}} < 1, \\ \tau_1\tau_2[(1 - q\lambda_1\sigma_1 + c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}} + \lambda_1\gamma_1\nu]\times \\ \quad \times [(1 - q\lambda_2\sigma_2 + c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}} + \lambda_2\gamma_2\zeta] < \delta_1\delta_2(1 - h), \end{cases} \tag{2.2}$$

where c_q is the same as in Lemma 1.1 and $\lambda_1, \lambda_2 > 0$ are constants. Then problem (1.1) has a solution (x^*, y^*).

Proof. First, we prove the existence of the solution. Define a mapping $Q : X \to CB(X)$ as follows:

$$\begin{aligned} Q(x) = x - p(x) + J_{M_1}^{\lambda_1}(p(x) + J_{M_2}^{\lambda_2}(x - \lambda_2(N_2(A(x), B(x)) - g)) - x \\ - \lambda_1(N_1(S(J_{M_2}^{\lambda_2}(x - \lambda_2(N_2(A(x), B(x)) - g))), \\ T(J_{M_2}^{\lambda_2}(x - \lambda_2(N_2(A(x), B(x)) - g)))) - f)), \quad \forall x \in X. \end{aligned} \tag{2.3}$$

It follows from (2.3) and Lemma 2.1 that (x^*, y^*) is a solution of problem (1.1) if and only if there exists $x^* \in X$ such that $x^* \in Q(x^*)$. Now we prove that Q has a fixed point in X. In fact, for any given $x, y \in X$, $\varepsilon > 0$ and $a \in Q(x)$, there exist $u \in B(x)$ and $v \in T(w)$ such that $a = x - p(x) + J_{M_1}^{\lambda_1}(p(x) + w - x - \lambda_1(N_1(S(w), v) - f))$, where $w = J_{M_2}^{\lambda_2}(x - \lambda_2(N_2(A(x), u) - g))$. Since $B, T : X \to CB(X)$, it follows from Nadler [16] that there exist $u' \in B(y)$, $v' \in T(w')$ such that

$$\begin{aligned} \|u - u'\| \leqslant (1 + \varepsilon)H(B(x), B(y)), \\ \|v - v'\| \leqslant (1 + \varepsilon)H(T(w), T(w')), \end{aligned}$$

where $w' = J_{M_2}^{\lambda_2}(y - \lambda_2(N_2(A(y), u') - g))$. Let $b = y - p(y) + J_{M_1}^{\lambda_1}(p(y) + w' - y - \lambda_1(N_1(S(w'), v') - f))$. Then $b \in Q(y)$. Thus, we obtain

$$
\begin{aligned}
\|a-b\| &= \|x-p(x)+J_{M_1}^{\lambda_1}(p(x)+w-x-\lambda_1(N_1(S(w),v)-f)) \\
&\quad -\{y-p(y)+J_{M_1}^{\lambda_1}(p(y)+w'-y-\lambda_1(N_1(S(w'),v')-f))\}\| \\
&\leqslant \|x-y-(p(x)-p(y))\|+\|J_{M_1}^{\lambda_1}(p(x)+w-x-\lambda_1(N_1(S(w),v)-f)) \\
&\quad -J_{M_1}^{\lambda_1}(p(y)+w'-y-\lambda_1(N_1(S(w'),v')-f))\| \\
&\leqslant \|x-y-(p(x)-p(y))\|+\frac{\tau_1}{\delta_1}\{\|x-y-(p(x)-p(y))\| \\
&\quad +\|w-w'-\lambda_1(N_1(S(w),v)-N_1(S(w'),v))\| \\
&\quad +\lambda_1\|N_1(S(w'),v)-N_1(S(w'),v')\|\} \\
&\leqslant (1+\frac{\tau_1}{\delta_1})\|x-y-(p(x)-p(y))\|+\frac{\tau_1}{\delta_1}\{\lambda_1\|N_1(S(w'),v)-N_1(S(w'),v')\| \\
&\quad +\|w-w'-\lambda_1(N_1(S(w),v)-N_1(S(w'),v))\|\}.
\end{aligned} \tag{2.4}
$$

Since X is q-uniformly smooth Banach space and $p: X \to X$ is α-strongly monotone and β-Lipschitz continuous, from Lemma 1.1, we know that there exists $c_q > 0$ such that

$$
\|x-y-(p(x)-p(y))\|^q \leqslant (1-q\alpha+c_q\beta^q)\|x-y\|^q. \tag{2.5}
$$

Since N_1 is ϵ_1-Lipschitz continuous and σ_1-strongly accretive with respect to S in the first argument and γ_1-Lipschitz continuous in the second argument, S is μ-Lipschitz continuous and T is ν-H-Lipschitz continuous, we have

$$
\begin{aligned}
\|N_1(S(w'),v)-N_1(S(w'),v')\| &\leqslant \gamma_1\|v-v'\| \\
&\leqslant \gamma_1(1+\varepsilon)H(T(w),T(w')) \leqslant \gamma_1\nu(1+\varepsilon)\|w-w'\|
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
&\|w-w'-\lambda_1(N_1(S(w),v)-N_1(S(w'),v))\|^q \\
&\qquad \leqslant (1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)\|w-w'\|^q.
\end{aligned} \tag{2.7}
$$

Similarly, by ξ-Lipschitz continuity of A, ζ-H-Lipschitz continuity of B, ϵ_2-Lipschitz continuity and σ_2-strongly accretivity with respect to A in the first argument of N_2 and γ_2-Lipschitz continuity in the second argument of N_2, we obtain

$$
\begin{aligned}
\|N_2(A(y),u)-N_2(A(y),u')\| &\leqslant \gamma_2\|u-u'\| \\
&\leqslant \gamma_2(1+\varepsilon)H(B(x),B(y)) \leqslant \gamma_2\zeta(1+\varepsilon)\|x-y\|, \\
\|x-y-\lambda_2(N_2(A(x),u)-N_2(A(y),u))\|^q &\leqslant (1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)\|x-y\|^q,
\end{aligned}
$$

and so

$$\begin{aligned}\|w-w'\| =&\|J_{M_2}^{\lambda_2}(x-\lambda_2(N_2(A(x),u)-g))-J_{M_2}^{\lambda_2}(y-\lambda_2(N_2(A(y),u')-g))\|\\ &\leqslant \frac{\tau_2}{\delta_2}\{\|x-y-\lambda_2(N_2(A(x),u)-N_2(A(y),u))\|\\ &\quad +\lambda_2\|N_2(A(y),u)-N_2(A(y),u')\|\}\\ &\leqslant \frac{\tau_2}{\delta_2}[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta(1+\varepsilon)]\|x-y\|.\end{aligned} \tag{2.8}$$

Combining (2.4)-(2.8), we have

$$\begin{aligned}\|a-b\| &\leqslant (1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}\|x-y\|\\ &\quad +\frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu(1+\varepsilon)]\|w-w'\|\\ &\leqslant \{(1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}+\frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu(1+\varepsilon)]\times\\ &\quad \times\frac{\tau_2}{\delta_2}[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta(1+\varepsilon)]\}\|x-y\|\\ &= [h+\theta_1(\varepsilon)\cdot\theta_2(\varepsilon)]\|x-y\|\\ &= \theta(\varepsilon)\|x-y\|,\end{aligned} \tag{2.9}$$

where

$$\theta(\varepsilon)=h+\theta_1(\varepsilon)\cdot\theta_2(\varepsilon),\quad h=(1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}},$$

$$\theta_1(\varepsilon)=\frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu(1+\varepsilon)],$$

$$\theta_2(\varepsilon)=\frac{\tau_2}{\delta_2}[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta(1+\varepsilon)].$$

From (2.9), we know that

$$\sup_{a\in Q(x)} d(a,Q(y))\leqslant\theta(\varepsilon)\|x-y\|,\quad \forall x,y\in X. \tag{2.10}$$

Similarly, we have

$$\sup_{b\in Q(y)} d(b,Q(x))\leqslant\theta(\varepsilon)\|x-y\|,\quad \forall x,y\in X. \tag{2.11}$$

It follows from (2.10), (2.11) and the definition of Hausdorff metric that

$$H(Q(x),Q(y))\leqslant\theta(\varepsilon)\|x-y\|,\quad \forall x,y\in X.$$

Letting $\varepsilon\to 0$, we get

$$H(Q(x),Q(y))\leqslant\theta\|x-y\|,\quad \forall x,y\in X, \tag{2.12}$$

where $\theta = h+\theta_1\theta_2$, $h=(1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}$, $\theta_1=\frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu]$, $\theta_2=\frac{\tau_2}{\delta_2}[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta]$. It follows from

(2.2) that $0 < \theta < 1$ and so by (2.12) and Lemma 2.2, we know that Q has a fixed point in X, i.e., there exists a point $x^* \in X$ such that $x^* \in Q(x^*)$. This completes the proof.

If $p \equiv I$, then Theorem 2.1 becomes the following existence theorem of the solution for problem (1.2).

THEOREM 2.2. *Assume that* $X, S, A, T, B, N_i, \eta_i$ *and* M_i *for* $i = 1,2$ *are the same as in Theorem* 2.1. *If there exist constants* $\lambda_1, \lambda_2 > 0$ *such that*

$$\begin{aligned} \tau_1\tau_2[(1 - q\lambda_1\sigma_1 + c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}} + \lambda_1\gamma_1\nu]\times \\ \times [(1 - q\lambda_2\sigma_2 + c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}} + \lambda_2\gamma_2\zeta] < \delta_1\delta_2, \end{aligned} \tag{2.13}$$

where c_q *is the same as in Lemma* 1.1, *then problem* (1.2) *has a solution* (x^*, y^*).

If $X = X^* = H$ is a Hilbert space and $M_i = \triangle\phi_i$, where $\triangle\phi_i$ denote the η-subdifferential of proper convex lower semi-continuous functions ϕ_i on H, $i = 1,2$, then from Theorem 2.1 and Ding [2], we can obtain the following theorem.

THEOREM 2.3. *Suppose that* S, A, T, B, N_1 *and* N_2 *are the same as in Theorem 2.1. Let* $X = X^* = H$ *be a Hilbert space, for all* $i = 1,2$, $\eta_i : X \to X$ *be* τ_i*-Lipschitz continuous and* δ_i*-strongly monotone such that* $\eta_i(x,y) = -\eta_i(y,x)$ *for all* $x, y \in X$ *and for each given* $x \in X$, *the function* $h(y,u) = \langle x - u, \eta_i(y,u)\rangle$ *be* 0*-diagonally quasi-convex* (*in short,* $0-DQCV$) *in* y (*see* [2], [20]) *and* $\phi_i : X \to R \cup \{+\infty\}$ *be proper convex lower semi-continuous and* η*-subdifferentiable functionals. If there exist constants* $\lambda_1, \lambda_2 > 0$ *such that condition* (2.2) *in Theorem* 2.1 *holds, then problem* (1.3) *has a solution* (x^*, y^*).

Proof. Let $M_i = \triangle\phi_i$ and $J_{M_i}^{\lambda_i}(x) = J_{\lambda_i}^{\triangle\phi_i}(x) = (I + \lambda_i\triangle\phi_i)^{-1}(x)$ for all $x \in H$, where $\triangle\phi_i$ denote the η-subdifferential of ϕ_i on H and $i = 1,2$. Then it follows from Theorem 2.2 of Ding [2] that the η-proximal mappings $J_{\lambda_i}^{\triangle\phi_i}$ of ϕ_i is τ_i/δ_i-Lipschitz continuous for $i = 1,2$. Thus the conclusion follows from Theorem 2.1.

3. Algorithms and convergence

In this section, we construct some new perturbed iterative algorithms with mixed errors for solving problems (1.1)-(1.3), respectively. We also prove the convergence of the iterative sequences generated by the algorithms in Banach spaces.

Now, based on Lemma 2.1, we give a new iterative algorithm for solving problem (1.1).

Algorithm 3.1. For any given $x_0 \in X$, the generalized Mann iterative sequence with mixed errors $\{x_n\}$ and $\{y_n\}$ in X is defined as follows:

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n\,[x_n - p(x_n) + J_{M_1}^{\lambda_1}(p(x_n) + y_n - x_n \\ \qquad - \lambda_1(N_1(S(y_n), T(y_n)) - f))\,] + \alpha_n e_n + s_n, \\ y_n \in J_{M_2}^{\lambda_2}(x_n - \lambda_2(N_2(A(x_n), B(x_n)) - g)) + f_n, \qquad n = 0,1,2,\ldots, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{e_n\},\{f_n\},\{s_n\}$ are three sequences in X satisfying the following conditions:

(i) $e_n = e_n' + e_n''$;

(ii) $\lim_{n\to\infty} \|e_n'\| = \lim_{n\to\infty} \|f_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|e_n''\| < \infty, \quad \sum_{n=0}^{\infty} \|s_n\| < \infty.$

If $p = I$, then Algorithm 3.1. reduces to the following algorithm for solving problem (1.2).

Algorithm 3.2. For any given $x_0 \in X$, define the Mann iterative sequence with mixed errors $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n[J_{M_1}^{\lambda_1}(y_n - \lambda_1(N_1(S(y_n),T(y_n)) - f))] + \alpha_n e_n + s_n, \\ y_n \in J_{M_2}^{\lambda_2}(x_n - \lambda_2(N_2(A(x_n),B(x_n)) - g)) + f_n, \qquad n = 0,1,2,\ldots, \end{cases}$$

where $\{\alpha_n\}$, $\{e_n\}$, $\{f_n\}$ and $\{s_n\}$ are the same as in Algorithm 3.1.

If $X = X^* = H$ is a Hilbert space and $M_i = \triangle\phi_i$, the η-subdifferential of ϕ_i for $i = 1,2$, then we have the following algorithm for solving problem (1.3).

Algorithm 3.3. For any given $x_0 \in X$, we obtain the following iterative sequence $\{x_n\}$ and $\{y_n\}$ in X:

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n\,[x_n - p(x_n) + J_{\lambda_1}^{\triangle\phi_1}(p(x_n) + y_n - x_n \\ \qquad\qquad - \lambda_1(N_1(S(y_n),T(y_n)) - f))\,] + \alpha_n e_n + s_n, \\ y_n \in J_{\lambda_2}^{\triangle\phi_2}(x_n - \lambda_2(N_2(A(x_n),B(x_n)) - g)) + f_n, \qquad n = 0,1,2,\ldots, \end{cases}$$

where $J_{\lambda_i}^{\triangle\phi_i}(x)$ $(i = 1,2)$ are the same as in the proof of Theorem 2.3 and $\{\alpha_n\}$, $\{e_n\}$, $\{f_n\}$ and $\{s_n\}$ are the same as in Algorithm 3.1.

LEMMA 3.1. *Let $\{a_n\},\{b_n\},\{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \leqslant (1-t_n)a_n + b_n t_n + c_n, \quad \forall n \geqslant n_0,$$

where $t_n \in [0,1]$ with $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

Proof. The proof directly follows from the proof of Lemma 2 in Liu [14].

THEOREM 3.1. *Suppose that X,S,A,T,B,p,N_i,η_i and M_i are the same as in Theorem 2.1 for $i = 1,2$. If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and condition (2.2) holds, then the generalized Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 3.1 converge strongly to the solution (x^*,y^*) of problem (1.1).*

Proof. Let (x^*, y^*) be the solution of problem (1.1). It follows from Lemma 2.1 that

$$\begin{cases} p(x^*) = J_{M_1}^{\lambda_1}(p(x^*) + y^* - x^* - \lambda_1(N_1(S(y^*), v^*) - f)), \quad \forall v^* \in T(y^*), \\ y^* = J_{M_2}^{\lambda_2}(x^* - \lambda_2(N_2(A(x^*), u^*) - g)), \quad \forall u^* \in B(x^*). \end{cases} \tag{3.2}$$

Since $B(x^*), B(x_n), T(y^*), T(y_n) \in CB(X)$ for all $n \geqslant 0$, for any given $n \geqslant 0$ and $\varepsilon > 0$, it follows from Nadler [16] that there exist $u_n \in B(x_n)$, $v_n \in T(y_n)$ such that

$$\|u_n - u^*\| \leqslant (1+\varepsilon)H(B(x_n), B(x^*)),$$
$$\|v_n - v^*\| \leqslant (1+\varepsilon)H(T(y_n), T(y^*)).$$

From (3.1), (3.2) and the proof of (2.9), for all $v_n \in T(y_n)$ and $v^* \in T(y^*)$, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n[x_n - p(x_n) + J_{M_1}^{\lambda_1}(p(x_n) + y_n - x_n \\ &\quad - \lambda_1(N_1(S(y_n), v_n) - f))] + \alpha_n e_n + s_n - x^*\| \\ &\leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (p(x_n) - p(x^*))\| \\ &\quad + \alpha_n\|J_{M_1}^{\lambda_1}(p(x_n) + y_n - x_n - \lambda_1(N_1(S(y_n), v_n) - f)) \\ &\quad - J_{M_1}^{\lambda_1}(p(x^*) + y^* - x^* - \lambda_1(N_1(S(y^*), v^*) - f))\| + \alpha_n\|e_n\| + \|s_n\| \\ &\leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n\{(1+\frac{\tau_1}{\delta_1})\|x_n - x^* - (p(x_n) - p(x^*))\| \\ &\quad + \frac{\tau_1}{\delta_1}[\|y_n - y^* - \lambda_1(N_1(S(y_n), v_n) - N_1(S(y^*), v_n))\| \\ &\quad + \lambda_1\|N_1(S(y^*), v_n) - N_1(S(y^*), v^*)\|]\} + \alpha_n\|e_n'\| + \alpha_n\|e_n''\| + \|s_n\| \\ &\leqslant (1-\alpha_n)\|x_n - x^*\| + \alpha_n h\|x_n - x^*\| + \alpha_n\theta_1(\varepsilon)\|y_n \\ &\quad - y^*\| + \alpha_n\|e_n'\| + (\|e_n''\| + \|s_n\|), \end{aligned} \tag{3.3}$$

where $h = (1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}$, $\theta_1(\varepsilon)=\frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu(1+\varepsilon)]$.

Similarly, we get

$$\begin{aligned} \|y_n - y^*\| &= \|J_{M_2}^{\lambda_2}(x_n - \lambda_2(N_2(A(x_n), u_n) - g)) + f_n - J_{M_2}^{\lambda_2}(x^* - \lambda_2(N_2(A(x^*), u^*) - g))\| \\ &\leqslant \theta_2(\varepsilon)\|x_n - x^*\| + \|f_n\|, \end{aligned}$$

where $\theta_2(\varepsilon) = \frac{\tau_2}{\delta_2}[(1 - q\lambda_2\sigma_2 + c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}} + \lambda_2\gamma_2\zeta(1+\varepsilon)]$.

Combining (3.3) and (3.4), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leqslant [1 - \alpha_n + \alpha_n h + \alpha_n\theta_1(\varepsilon)\cdot\theta_2(\varepsilon)]\|x_n - x^*\| \\ &\quad + \alpha_n[\|e_n'\| + \theta_1(\varepsilon)\|f_n\|] + (\|e_n''\| + \|s_n\|) \\ &= [1 - \alpha_n(1 - \theta(\varepsilon))]\|x_n - x^*\| \\ &\quad + \alpha_n[\|e_n'\| + \theta_1(\varepsilon)\|f_n\|] + (\|e_n''\| + \|s_n\|), \end{aligned} \tag{3.4}$$

where $\theta(\varepsilon) = h + \theta_1(\varepsilon)\cdot\theta_2(\varepsilon)$. Let $\varepsilon \to 0$. Then we have $\theta_i(\varepsilon) \to \theta_i$ for $i = 1,2$, $\theta(\varepsilon) \to \theta$, where $\theta = h + \theta_1\theta_2$, $h = (1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}$, $\theta_1 = \frac{\tau_1}{\delta_1}[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu]$, $\theta_2 = \frac{\tau_2}{\delta_2}[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta]$.

Since $0<\theta<1$, $1-\theta>0$, it follows from (3.5) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leqslant [1-\alpha_n(1-\theta)]\|x_n-x^*\| \\ &\quad + \alpha_n(1-\theta)\cdot\frac{1}{1-\theta}(\|e'_n\|+\theta_1\|f_n\|)+(\|e''_n\|+\|s_n\|).\end{aligned} \tag{3.5}$$

Since $\sum_{n=0}^{\infty}\alpha_n=\infty$, it follows from Lemma 3.1 and (3.6) that $\|x_n-x^*\|\to 0\ (n\to\infty)$, i.e., $x_n\to x^*$. This completes the proof.

REMARK 3.1. If X is a 2-uniformly smooth Banach space and there exist constants $\lambda_1,\lambda_2>0$ such that

$$\begin{cases} h=(1+\frac{\tau}{\delta})\sqrt{1-2\alpha+c_2\beta^2}<1, \\ \tau_1\tau_2(\sqrt{1-2\lambda_1\sigma_1+c_2\lambda_1^2\epsilon_1^2\mu^2}+\lambda_1\gamma_1\nu)\times \\ \qquad \times(\sqrt{1-2\lambda_2\sigma_2+c_2\lambda_2^2\epsilon_2^2\xi^2}+\lambda_2\gamma_2\zeta)<\delta_1\delta_2(1-h), \end{cases}$$

then (2.2) holds. We note that Hilbert spaces and L_p (or l_p) spaces, $2\leqslant p<\infty$, are 2-uniformly smooth.

From Theorem 3.1, we can get the following theorems.

THEOREM 3.2. *Assume that X,S,A,T,B,N_i,η_i and M_i for $i=1,2$ are the same as in Theorem 2.2. If $\sum_{n=0}^{\infty}\alpha_n=\infty$ and condition (2.13) holds, then the Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 3.2 converge strongly to the solution (x^*,y^*) of problem (1.2).*

THEOREM 3.3. *Let X,S,A,T,B,N_i,η_i and ϕ_i for $i=1,2$ be the same as in Theorem 2.3. If $\sum_{n=0}^{\infty}\alpha_n=\infty$ and the following conditions hold:*

$$\begin{cases} h=(1+\frac{\tau_1}{\delta_1})(1-q\alpha+c_q\beta^q)^{\frac{1}{q}}<1, \\ \tau_1\tau_2[(1-q\lambda_1\sigma_1+c_q\lambda_1^q\epsilon_1^q\mu^q)^{\frac{1}{q}}+\lambda_1\gamma_1\nu]\times \\ \quad\times[(1-q\lambda_2\sigma_2+c_q\lambda_2^q\epsilon_2^q\xi^q)^{\frac{1}{q}}+\lambda_2\gamma_2\zeta]<\delta_1\delta_2(1-h), \end{cases}$$

where c_q is the same as in Lemma 1.1 and $\lambda_1,\lambda_2>0$ are constants, then the iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 3.3 converge strongly to the solution (x^*,y^*) of problem (1.3).

REMARK 3.2. For an appropriate choice of the mappings S,A,T,B,N_i,η_i and M_i for $i=1,2$ and the space X, Theorems 3.1 ∼ 3.3 include many known results of generalized variational inclusions as special cases, see [3], [4], [8], [13], [15], [19] and the references therein.

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