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# INEQUALITIES FOR MARKS IN DIGRAPHS

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(*communicated by R. A. Brualdi*)

*Abstract.* A 2-digraph $D$ is an orientation of a multi-graph that is without loops and contains at most two edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and complete 1-digraph is a tournament. Define $p_{v_i}$(or simply $p_i$) $= 2n-2+d^+_{v_i}-d^-_{v_i}$, the mark (2-score) of a vertex $v_i$ in a 2-digraph $D$, where $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of $v_i$ and $n$ is the number of vertices in $D$. In this paper, we obtain some stronger inequalities for marks in 2-digraphs.

## 1. Introduction

A tournament is an orientation of a complete simple graph. The score $s_{v_i}$ (or simply $S_i$) of a vertex $v_i$ in a tournament is the outdegree of $v_i$. The score sequence of a tournament is formed by listing the vertex scores in non-decreasing order. The following result of Landau [3] gives a necessary and sufficient conditions for a sequence of non-negative integers to be the score sequence of some tournament.

THEOREM 1.1. *A sequence $[s_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of some tournament if and only if*

$$\sum_{i=1}^{k} s_i \geqslant \binom{k}{2} \quad \textit{for} \quad 1 \leqslant k \leqslant n$$

*with equality when $k=n$.*

With the marking system, the mark $p_{v_i}$ of a vertex $v_i$ in a tournament is given by $p_{v_i} = 2s_{v_i} + n - 1$ and Landau's conditions become

$$\sum_{i=1}^{k} p_i \geqslant k(n+k-2), \quad \text{for} \quad 1 \leqslant k \leqslant n,$$

with equality when $k=n$.

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Avery [1] defined $a_{v_i}$ (or simply $a_i$) $= n-1+d^+_{v_i}-d^-_{v_i}$, the score of a vertex $v_i$ in an oriented graph, where $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of $v_i$ and $n$ is the number of vertices. The score sequence of an oriented graph is

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formed by listing the vertex scores in non-decreasing order. The following result is due to Avery [1].

THEOREM 1.2. *A sequence $[a_i]_1^n$ of non-negative integers in non-decreasing order is the score sequence of some oriented graph if and only if*

$$\sum_{i=1}^{k} a_i \geqslant k(k-1) \quad for \quad 1 \leqslant k \leqslant n$$

*with equality when* $k = n$.

Once again, with the marking system, the mark $p_{v_i}$ of a vertex $v_i$ in an oriented graph is given by $p_{v_i} = a_{v_i} + n - 1$ and Avery's conditions become

$$\sum_{i=1}^{k} p_i \geqslant k(n+k-2), \quad \text{for} \quad 1 \leqslant k \leqslant n,$$

with equality when k = n.

A digraph $D$ is semicomplete if for any pair of vertices u $\neq v$ in $D$, there is an arc from $u$ to $v$ or an arc from $v$ to $u$ (or both). The following necessary and sufficient conditions for a non-decreasing sequence of integers to be the score sequence for a semicomplete digraph is given by Reid and Zhang [6].

THEOREM 1.3. *A sequence $[s_i]_1^n$ of integers in non-decreasing order is the score sequence of some semicomplete digraph if and only if*

$$\sum_{i=1}^{k} s_i \geqslant \binom{k}{2} \quad and \quad s_k \leqslant n-1, \quad for\ all\ k,\ 1 \leqslant k \leqslant n.$$

A $2$-digraph $D$ is an orientation of a multi-graph that is without loops and contains at most two edges between any pair of distinct vertices. Let $D$ be a $2$-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let $d_{v_i}^{+}$ and $d_{v_i}^{-}$ denote the outdegree and indegree, respectively, of a vertex $v_i$. Define $p_{v_i} = 2n - 2 + d_{v_i}^{+} - d_{v_I}^{-}$, the mark (2-score) of $v_i$, so that $0 \leqslant p_{v_i} \leqslant 4n-4$. The sequence $P = [p_i]_1^n$, where $p_i = p_{v_i}$, in non-decreasing order is the mark sequence of $D$. A $2$-digraph can be interpreted as the result of a competition in which the participants play each other at most twice, with an arc from $u$ to $v$ if and only if $u$ defeats $v$. A player receives two points for each win, and one point for each tie (draw). With this marking system, player $v$ obtains a total of $p_v$ points. A sequence $P$ of non-negative integers in non- decreasing order is said to be realizable if there exists a $2$-digraph with mark sequence $P$. The following existence criteria for realizability is due to Pirzada and Samee [5].

THEOREM 1.4. *A sequence $[p_i]_1^n$ of non-negative integers in non-decreasing order is the mark sequence of some $2$-digraph if and only if*

$$\sum_{i=1}^{k} p_i \geqslant 2k(k-1), \quad for \quad 1 \leqslant k \leqslant n$$

*with equality when* $k = n$.

Some stronger inequalities for scores in tournaments are given by Brualdi and Shen [2]. Moreover, inequalities for scores in oriented graphs are given by Pirzada and Samee [4].

## 2. Stronger inequalities

A regular 2-digraph on $n$ vertices is one whose all vertices have marks $2(n-1)$. The converse $D'$ of a 2-digraph $D$ is obtained by reversing each arc of $D$.

If $u$ and $v$ are vertices in a 2-digraph, then $u(x-y)v$ denotes that $x$ arcs are directed from $u$ to $v$ and $y$ arcs are directed from $v$ to $u$. Clearly, $0 \leqslant x, y \leqslant 2$ and $0 \leqslant x+y \leqslant 2$. A triple in a 2-digraph is an induced 2-digraph with three vertices and is of the form $u(x_1-x_2)v(y_1-y_2)w(z_1-z_2)u$, where for $1 \leqslant i \leqslant 2$, $0 \leqslant x_i, y_i, z_i \leqslant 2$ and $o \leqslant \sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i, \sum_{i=1}^{2} z_i \leqslant 2$.

In a 2-digraph, a 1- triple is an induced 1-subdigraph with three vertices. A 1-triple is said to be transitive if it is of the following form: $u(1-0)v(1-0)w(0-1)u$ or $u(1-0)v(0-1)w(0-0)u$ or $u(1-0)v(0-0)w(0-1)u$ or $u(1-0)v(0-0)w(0-0)u$ or $u(0-0)v(0-0)w(0-0)u$, otherwise it is intransitive. A 2-digraph is said to be transitive if every of its 1- triple is transitive.

The inequalities given below in Theorems 2.1, 2.2, 2.3, 2.4 are the generalizations of the inequalities on scores in tournaments due to Brualdi and Shen [2]. We use some of the techniques of Brualdi and Shen [2] in obtaining these inequalities.

The following result gives a lower bound for $\sum_{i \in I} p_i$.

THEOREM 2.1. *A sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of a 2-digraph if and only if for every subset $I \subseteq [n] = \{1, 2, \ldots, n\}$,*

$$\sum_{i \in I} p_i \geqslant 2 \sum_{i \in I} (i-1) + |I|\,(|I|-1)\,, \tag{2.1}$$

*with equality when $|I| = n$.*

*Proof. Sufficiency.* Let the sequence $P = [p_i]_1^n$ of non-negative integers in non-decreasing order satisfies the equation (2.1).

Now, for any I $\subseteq$ [n], we have

$$\sum_{i \in I} (i-1) \geqslant \sum_{i=1}^{|I|} (i-1) = \binom{|I|}{2}.$$

Therefore, from equation (2.1), we have

$$\begin{aligned} \sum_{i \in I} p_i &\geqslant 2 \sum_{i \in I} (i-1) + |I|\,(|I|-1) \\ &\geqslant 2\binom{|I|}{2} + |I|\,(|I|-1) \\ &= 2\,|I|\,(|I|-1)\,. \end{aligned}$$

Hence, by Theorem 1.4, $P$ is a mark sequence.

*Necessity.* Assume that $P = [p_i]_1^n$ is a mark sequence of some 2-digraph. For any subset $I \subseteq [n]$, define

$$f(I) = \sum_{i \in I} p_i - 2 \sum_{i \in I} (i-1) - |I|\,(|I|-1)\,.$$

Claim $I = \{i : 1 \leqslant i \leqslant |I|\}$. If not, then there exists $i \notin I$ and $j \in I$ such that $j = i+1$. So, $p_i \leqslant p_j$.

For $j \in I$, we have

$$f(I) = \sum_{t\in I} p_t - 2\sum_{t\in I}(t-1) - |I|(|I|-1)$$

$$= \sum_{t\in I, j\notin I} p_t + p_j - 2\left(\sum_{t\in I, j\notin I}(t-1) + (j-1)\right) - |I|(|I|-1).$$

Therefore,

$$\begin{aligned} f(I) - f(I - \{j\}) &= p_j - 2(j-1) - |I|(|I|-1) + (|I|-1)(|I|-2) \\ &= p_j - 2(j-1) - 2|I| + 2 \\ &= p_j - 2(j + |I| - 2). \end{aligned}$$

Since $f(I) - f(I - \{j\}) < 0$, therefore $p_j - 2(j + |I| - 2) < 0$.
Again,

$$f(I \cup \{i\}) = \sum_{t\in I} p_t + p_i - 2\left(\sum_{t\in I}(t-1) + (i-1)\right) - (|I|+1)(|I|).$$

So, $f(I \cup \{i\}) - f(I) = p_i - 2(i-1) - 2|I|$. As $f(I \cup \{i\}) - f(I) \geqslant 0$, therefore $p_i - 2(i-1) - 2|I| \geqslant 0$.
Thus, $p_j < 2(j + |I| - 2)$, and $p_i \geqslant 2(i-1) + 2|I| = 2(i + |I| - 1)$.
Therefore,

$$2(i + |I| - 1) \leqslant p_i \leqslant p_j < 2(j + |I| - 2).$$

Since $j = i+1$, then $2(i+|I|-1) < 2(i+1+|I|-2)$. That is, $2(i+|I|-1) < 2(i+|I|-1)$ which is a contradiction.

Hence,

$$\begin{aligned} f(I) &= \sum_{i=1}^{|I|} p_i - 2\sum_{i=1}^{|I|}(i-1) - |I|(|I|-1) \\ &= \sum_{i=1}^{|I|} p_i - 2\binom{|I|}{2} - |I|(|I|-1) \\ &\geqslant 2|I|(|I|-1) - |I|(|I|-1) - |I|(|I|-1) = 0. \end{aligned}$$

( By Theorem 1.4 )

Thus, $\sum_{i\in I} p_i - 2, \sum_{i\in I}(i-1) - |I|(|I|-1) \geqslant 0$, that is, $\sum_{i\in I} p_i \geqslant 2\sum_{i\in I}(i-1) + |I|(|I|-1)$.
This proves the necessity. □

We note that equality can occur often in equation (2.1). For example, in the transitive 2-digraph of order n with mark sequence $[0, 4, 8, \ldots, 4n-4]$ and in the regular 2-digraph of order $n$ with mark sequence $[2(n-1), 2(n-1), \ldots, 2(n-1)]$. Further we observe that Theorem 2.1 is best possible, since for any real $\varepsilon > 0$, the inequality

$$\sum_{i\in I} p_i \geqslant (1+\varepsilon)2\sum_{i\in I}(i-1) + (1-\varepsilon)|I|(|I|-1)$$

fails for some $I$, and some 2-digraphs. This can been seen, for example, in the transitive 2-digraph of order n with mark sequence $[0,4,8,\ldots,4n-4]$ and in the regular 2-digraph of order n with mark sequence $[2(n-1),2(n-1),\ldots,2(n-1)]$.

The next result gives a set of upper bounds for $\sum_{i\in I} p_i$ and is equivalent to the set of lower bounds for $\sum_{i\in I} p_i$ in Theorem 2.1.

THEOREM 2.2. *A sequence $P=[p_i]_1^n$ of non-negative integers in non-decreasing order is a mark sequence of a 2-digraph if and only if for every subset $I\subseteq[n]=\{1,2,\ldots,n\}$,*

$$\sum_{i\in I} p_i \leqslant 2\sum_{i\in I}(i-1)+|I|\,(2n-|I|-1)\,,$$

*with equality when $|I|=n$.*

*Proof.* We have $[n]=\{1,2,\ldots,n\}$. Let $J=[n]-I$ so that $I+J=[n]$ and $|J|+|I|=n$. Therefore, by Theorem 2.1, $P$ is a mark sequence if and only if

$$\sum_{i\in[n]} p_i = 2n(n-1) \quad \text{and} \quad \sum_{i\in J} p_i \geqslant 2\sum_{i\in J}(i-1)+|J|\,(|J|-1)$$

if and only if

$$\sum_{i\in I} p_i + \sum_{i\in J} p_i = 2n(n-1) \quad \text{and} \quad \sum_{i\in J} p_i \geqslant 2\sum_{i\in J}(i-1)+|J|\,(|J|-1)$$

if and only if

$$\begin{aligned}\sum_{i\in I} p_i &= 2n(n-1)-\sum_{i\in J} p_i\\ &\leqslant 2n(n-1)-\left(2\sum_{i\in J}(i-1)+|J|\,(|J|-1)\right)\\ &= 2n(n-1)-\left(n(n-1)-2\sum_{i\in I}(i-1)+(n-|I|)\,(n-|I|-1)\right)\end{aligned}$$

$$\left(\text{Because}\quad 2\sum_{i\in I}(i-1)+2\sum_{i\in J}(i-1)=n(n-1)\text{ and }|I|+|J|=n\right)$$

$$\begin{aligned}&= 2\sum_{i\in I}(i-1)+2n(n-1)-n(n-1)-(n-|I|)\,(n-|I|-1)\\ &= 2\sum_{i\in I}(i-1)+n^2-n-n^2+n\,|I|+n+n\,|I|-|I|^2-|I|\\ &= 2\sum_{i\in I}(i-1)+|I|\,(2n-|I|-1)\,,\end{aligned}$$

which proves the result. □

Now, we have the following results.

THEOREM 2.3. *If $P=[p_i]_1^n$ is a mark sequence of a 2-digraph, then for each $i$*

$$2(i-1)\leqslant p_i \leqslant 2(n+i-2).$$

*Proof.* Let $I = \{i\}$ in Theorem 2.1 and Theorem 2.2. Then,

$$\sum_{i\in I} p_i \geqslant 2\sum_{i\in I}(i-1) + |I|\,(|I|-1)$$

implies that $p_i \geqslant 2(i-1)$, and $\sum_{i\in I} p_i \leqslant 2\sum_{i\in I}(i-1) + |I|\,(2n-|I|-1)$ implies that $p_i \leqslant 2(i-1) + 1(2n-1-1) = 2(n+i-2)$.

Therefore, $2(i-1) \leqslant p_i \leqslant 2(n+i-2)$. □

*Second proof.* We first show that $2(i-1) \leqslant p_i$. Suppose on contrary that $p_i < 2(i-1)$. Then, for every $k < i$, we have

$$p_k \leqslant p_i < 2(i-1).$$

That is, $p_1 < 2(i-1), p_2 < 2(i-1), \ldots, p_i < 2(i-1)$.

Adding these inequalities, we have

$$\sum_{k=1}^{i} p_k < 2i(i-1),$$

which is a contradiction to Theorem 1.4. Therefore, $2(i-1) \leqslant p_i$.

The second inequality is dual to the first. In the converse 2-digraph with mark sequence $P' = [p_i']_1^n$, we have

$$p'_{n-i+1} \geqslant 2((n-i+1)-1) = 2(n-i).$$

( By the first inequality )

But $p_i = 4(n-1) - p'_{n-i+1}$. So, $p_i \leqslant 4(n-1) - 2(n-i) = 2(n+i-2)$.

Therefore, $p_i \leqslant 2(n+i-2)$. Hence, the result. □

For any integers $r$ and $s$ with $r \leqslant s$, let $[r,s]$ denotes the set of all integers between $r$ and $s$.

THEOREM 2.4. *Let $P = [p_i]_1^n$ be a mark sequence of a 2-digraph. If*

$$\sum_{i\in I} p_i = 2\sum_{i\in I}(i-1) + |I|\,(|I|-1) \tag{2.2}$$

*for some $I \subseteq [n]$, then one of the following holds.*

(*a*) $I = [1,|I|]$ *and* $\sum_{i=1}^{|I|} p_i = 2\,|I|\,(|I|-1)$.

(*b*) $I = [t, t+|I|-1]$ *for some* $t, 2 \leqslant t \leqslant n-|I|+1$,

$$\sum_{i=1}^{t+|I|-1} p_i = 2\,(t+|I|-1)\,(t+|I|-2) \text{ and } p_i = 2(t+|I|-2) \text{ for all } i \leqslant t+|I|-1.$$

(*c*) $I = [1,r] \cup [r+t, t+|I|-1]$ *for some $r$ and $t$ such that* $1 \leqslant r \leqslant |I|-1$ *and* $2 \leqslant t \leqslant n-|I|+1$, $\sum_{i=1}^{r} p_i = 2r(r-1)$, $\sum_{i=1}^{t+|I|-1} p_i = 2\,(t+|I|-1)\,(t+|I|-2)$ *and* $p_i = 2(r+t+|I|-2)$ *for all* $i, r+1 \leqslant i \leqslant t+|I|-1$.

*Proof.* For any subset $J \subseteq [n]$, define

$$f(J) = \sum_{i\in J} p_i - 2\sum_{i\in J}(i-1) - |J|(|J|-1),$$

so that $f(I) = \sum_{i\in I} p_i - 2\sum_{i\in I}(i-1) - |I|(|I|-1)$ for $I \subseteq [n]$.

Therefore, by equation (2.2), we have $f(I) = 0$ and $f(J) \geqslant 0$ for all $J \subseteq [n]$.

We prove that I is one of the three types as given in the Theorem.

Assume to the contrary that this is not true. Then, there exist indices $i, j, k$ and $l$ with $j = i+1$ and $l = k+1$ such that $\{i,k\} \cap I = \varphi$ and $\{j,l\} \subseteq I$. Thus, $p_i \leqslant p_j$ and $p_k \leqslant p_l$.

Since $f(I) = 0$ and $f(J) \geqslant 0$ for all $J \subseteq [n]$, therefore $f(I) - f(I - \{j,l\}) \leqslant 0$ and $f(I \cup \{i,k\}) - f(I) \geqslant 0$.

Consider

$$f(I-\{j,l\}) = \sum_{i\in I} p_i - p_j - p_l - 2\left(\sum_{i\in I}(i-1) - (j-1) - (l-1)\right) - (|I|-2)(|I|-3)$$

$$= \sum_{i\in I} p_i - 2\sum_{i\in I}(i-1) - p_j - p_l + 2j + 2l - 4 - (|I|-2)(|I|-3)$$

and

$$f(I \cup \{i,k\}) = \sum_{i\in I} p_i + p_i + p_k - 2\left(\sum_{i\in I}(i-1) + (i-1) + (k-1)\right) - (|I|+2)(|I|+1)$$

$$= \sum_{i\in I} p_i - 2\sum_{i\in I}(i-1) + p_i + p_k - 2i - 2k + 4 - (|I|+2)(|I|+1).$$

Therefore,

$$\begin{aligned} f(I) - f(I - \{j,l\}) &= p_j + p_l - 2j - 2l + 4 - 4|I| + 6 \\ &= p_j + p_l - 2(j + l + 2|I| - 5) \end{aligned}$$

and

$$\begin{aligned} f(I \cup \{i,k\}) - f(I) &= p_i + p_k - 2i - 2k + 4 - 4|I| - 2 \\ &= p_i + p_k - 2(i + k + 2|I| - 1). \end{aligned}$$

So, $p_j + p_l - 2(j + l + 2|I| - 5) \leqslant 0$ and $p_i + p_k - 2(i + k + 2|I| - 1) \geqslant 0$.
This gives, $p_j + p_l \leqslant 2(j + l + 2|I| - 5)$ and $p_i + p_k \geqslant 2(i + k + 2|I| - 1)$.
Therefore, $2(i + k + 2|I| - 1) \leqslant p_i + p_k \leqslant p_j + p_l \leqslant 2(j + l + 2|I| - 5)$.
That is, $i + k + 2|I| - 1 \leqslant i + 1 + k + 1 + 2|I| - 5$, or $-1 \leqslant -3$, a contradiction.
This shows that $I$ satisfies one of the conditions $(a)$, $(b)$ or $(c)$.

*Case* $(a)$ $I = [1, |I|]$.

Then, $f(I) = \sum_{i=1}^{|I|} p_i - 2|I|(|I|-1)$ and so $\sum_{i=1}^{|I|} p_i = 2|I|(|I|-1)$.

*Claim.* If there exist indices $i$ and $j$ with $j = i+1$ such that $i \notin I$ and $j \in I$, then $f(I \cup \{i\}) = f(I - \{j\}) = 0$.

Since $f(I - \{j\}) = \sum_{i\in I} p_i - p_j - 2\left(\sum_{i\in I}(i-1) - (j-1)\right) - (|I|-1)(|I|-2)$ and

$$f(I \cup \{i\}) = \sum_{i\in I} p_i + p_i - 2\left(\sum_{i\in I}(i-1) + (i-1)\right) - (|I|+1)|I|.$$

Therefore, $f(I) - f(I - \{j\}) = p_j - 2(j-1) - 2|I| + 2 = p_j - 2(j - 2 + |I|)$ and $f(I \cup \{i\}) - f(I) = p_i - 2(i-1) - 2|I| = p_i - 2(i - 1 + |I|)$.
But $f(I) - f(I - \{j\}) \leqslant 0$ and $f(I \cup \{i\}) - f(I) \geqslant 0$.
So, $p_j - 2(j - 2 + |I|) \leqslant 0$ and $p_i - 2(i - 1 + |I|) \geqslant 0$.
This gives, $p_j \leqslant 2(j - 2 + |I|)$ and $p_i \geqslant 2(i - 1 + |I|)$. Therefore,

$$2(i - 1 + |I|) \leqslant p_i \leqslant p_j \leqslant 2(j - 2 + |I|) = 2(i + 1 - 2 + |I|) = 2(i - 1 + |I|).$$

This implies that equalities hold throughout all the above inequalities. Thus,

$$f(I \cup \{i\}) = f(I) = f(I - \{j\}) = 0.$$

*Case* $(b)$ $I = [t, t + |I| - 1]$ for some $t, 2 \leqslant t \leqslant n - |I| + 1$.

By applying above claim recursively, we have $f(\{t + |I| - 1\}) = 0$, that is, $p_{t+|I|-1} = 2(t + |I| - 2)$.

Since $\sum\limits_{i=1}^{t+|I|-1} p_i \geqslant 2(t + |I| - 1)(t + |I| - 2)$ ( By Theorem 1.4 ) and $p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{t+|I|-1}$, equalities hold throughout all the above inequalities.

*Case* $(c)$ $I = [1, r] \cup [r + t, t + |I| - 1]$ for some $r$ and $t$ such that $1 \leqslant r \leqslant |I| - 1$ and $2 \leqslant t \leqslant n - |I| + 1$.

Again, by applying above claim recursively, we have

$$f([1, r]) = f([1, t + |I| - 1]) = f([1, r] \cup \{t + |I| - 1\}) = 0.$$

Therefore, $\sum\limits_{i=1}^{r} p_i - 2r(r-1) = f([1, r]) = 0,$

$$\sum_{i=1}^{t+|I|-1} p_i - 2(t + |I| - 1)(t + |I| - 2) = f([1, t + |I| - 1]) = 0,$$

and $f([1, r] \cup \{t + |I| - 1\}) - f([1, r]) = p_{t+|I|-1} - 2(|I| + r + t - 2)$.

But $f([1, r] \cup \{t + |I| - 1\}) - f([1, r]) = 0$. So, $p_{t+|I|-1} = 2(|I| + r + t - 2)$. Since

$$\sum_{i=r+1}^{t+|I|-1} p_i = \sum_{i=1}^{t+|I|-1} p_i - \sum_{i=1}^{r} p_i = 2(t + |I| - 1)(t + |I| - 2) - 2r(r-1)$$
$$= 2(|I| + r + t - 2)(|I| - r + t - 1)$$

and $p_{r+1} \leqslant p_{r+2} \leqslant \ldots \leqslant p_{t+|I|-1}$, we have

$$p_i = 2(r + t + |I| - 2) \quad \text{for all} \quad i, r + 1 \leqslant i \leqslant t + |I| - 1. \qquad \square$$

THEOREM 2.5. *If* $P = [p_i]_1^n$ *is a mark sequence of a* 2*-digraph, then*

$(a)$ $\sum\limits_{i=1}^{k} p_i^2 \geqslant \sum\limits_{i=1}^{k} (4k - 4 - p_i)^2$, *for* $1 \leqslant k \leqslant n$ *with equality when* $k = n$.

$(b)$ *for* $1 < g < \infty$,

$$\sum_{i=1}^{k} p_i^g \geqslant k(2k-2)^g,$$

*where* $1 \leqslant k \leqslant n$ *with equality when* $k = n$ *and* $p_1 = p_2 = \ldots = p_k$.

*Proof.* $(a)$ By Theorem 1.4, we have

$$2k(k-1) \leqslant \sum_{i=1}^{k} p_i, \quad \text{for} \quad 1 \leqslant k \leqslant n$$

with equality when $k = n$, or

$$\sum_{i=1}^{k} p_i^2 + 2(4k-4)2k(k-1) \leqslant \sum_{i=1}^{k} p_i^2 + 2(4k-4)\sum_{i=1}^{k} p_i, \ \text{for } 1 \leqslant k \leqslant n$$

with equality when $k = n$, or

$$\sum_{i=1}^{k} p_i^2 + k(4k-4)^2 - 2(4k-4)\sum_{i=1}^{k} p_i \leqslant \sum_{i=1}^{k} p_i^2, \ \text{for } 1 \leqslant k \leqslant n$$

with equality when $k = n$, or

$$p_1^2 + \ldots + p_k^2 + \underbrace{(4k-4)^2 + \ldots + (4k-4)^2}_{k-times} - 2(4k-4)p_1 - \ldots - 2(4k-4)p_k \leqslant \sum_{i=1}^{k} p_i^2,$$

for $1 \leqslant k \leqslant n$ with equality when $k = n$, or

$$(4k-4-p_1)^2 + \ldots + (4k-4-p_k)^2 \leqslant \sum_{i=1}^{k} p_i^2,$$

for $1 \leqslant k \leqslant n$ with equality when $k = n$, or $\sum\limits_{i=1}^{k} (4k-4-p_i)^2 \leqslant \sum\limits_{i=1}^{k} p_i^2$, for $1 \leqslant k \leqslant n$ with equality when $k = n$.

$(b)$ Again, by Theorem 1.4, we have

$$\begin{aligned}
2k(k-1) &\leqslant \sum_{i=1}^{k} p_i, \quad \text{for} \quad 1 \leqslant k \leqslant n \quad \text{with equality when} \quad k = n \\
&= \sum_{i=1}^{k} p_i \cdot 1, \quad \text{for} \quad 1 \leqslant k \leqslant n \quad \text{with equality when} \quad k = n \\
&\leqslant \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}} \left(\sum_{i=1}^{k} 1^h\right)^{\frac{1}{h}}, \quad \text{for} \quad 1 \leqslant k \leqslant n \quad \text{with equality when} \quad k = n
\end{aligned}$$

and $p_1 = p_2 = \ldots = p_k$, where $\frac{1}{g} + \frac{1}{h} = 1$ (By Holder's Inequality)

$$= \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}} k^{\frac{1}{h}}, \quad \text{for} \quad 1 \leqslant k \leqslant n \quad \text{with equality when} \quad k = n$$

and $p_1 = p_2 = \ldots = p_k$.

That is, $2k^{1-\frac{1}{h}}(k-1) \leqslant \left(\sum_{i=1}^{k} p_i^g\right)^{\frac{1}{g}}$, for $1 \leqslant k \leqslant n$ with equality when $k = n$ and $p_1 = p_2 = \ldots = p_k$.

Hence, $\sum_{i=1}^{k} p_i^g \geqslant k(2k-2)^g$, for $1 \leqslant k \leqslant n$ with equality when $k = n$ and $p_1 = p_2 = \ldots = p_k$ $\left(\text{Since } \frac{1}{g} + \frac{1}{h} = 1\right)$. □

THEOREM 2.6. *Let $D$ be a 2-digraph on $r$ vertices with mark sequence $[p_i]_1^r$. Then for each $k \geqslant 1$, there exists a 2-digraph on $kr$ vertices with mark sequence $[p_i + 2(k-1)r]_1^{kr}$.*

*Proof.* For each $i, 1 \leqslant i \leqslant k$, let $D^i$ be a copy of $D$ with $r$ vertices. Define a 2-digraph $D_1$ as

$$D_1 = D^1 \cup D^2 \cup \ldots \cup D^k,$$

such that vertices and arcs of $D_1$ are that of $D^1, D^2, \ldots, D^k$. Then, $D_1$ is a 2-digraph on $kr$ vertices with mark sequence $[p_i + 2(k-1)r]_1^{kr}$. □

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