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SCHUR–CONVEXITY FOR A CLASS OF SYMMETRIC FUNCTION AND ITS APPLICATIONS

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Abstract. In this paper, we investigate the symmetric function

$$\prod_n^r(f)=\prod_n^r(f(x))=\left(\prod_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n} f\left(\frac{1}{r}\sum_{j=1}^r x_{i_j}\right)\right)^{\frac{1}{\binom{n}{r}}},$$

where $f(x)$ is a positive function on an interval I. Some analytic inequalities, including "Ky Fan" type inequalities, are established by use of the theory of majorization. An open problem is also solved partly.

1. Introduction

Throughout the paper we assume that $R_+^n=\{x=(x_1,x_2,\ldots,x_n)\mid x_i>0, i=1,2,\ldots,n\}$. The unweighted arithmetic and geometric means of x, denoted by $A_n(x)$, $G_n(x)$, respectively, are defined as follows

$$A_n(x)=\frac{1}{n}\sum_{i=1}^n x_i,\qquad G_n(x)=\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}.$$

Assume that $0\leqslant x_i<1,\ 1\leqslant i\leqslant n$ and define $1-x=(1-x_1,1-x_2,\ldots,1-x_n)$. The symbols $A_n(1-x)$, $G_n(1-x)$ also stand for the unweighted arithmetic, geometric means of $1-x$, respectively.

All kinds of means about numbers and their inequalities have stimulated the interests of many researchers all the time (See, for example, [2, 6, 9, 10] and the references cited therein.). For instances, the r-th order symmetric function (mean)([2, p. 65], [6, p. 78], [10]) is defined as

$$E_n(x,r)=E_n(x_1,x_2,\ldots,x_n;r)=\sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n}\prod_{j=1}^r x_{i_j} \tag{1.1}$$

which is investigated by many authors and some good results are obtained. The Schur-convexity is proved which is very useful in establishing analytic inequalities.

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The complete symmetric function, which is a generalization of (1.1), is

$$c_r = c_r(x) = \sum_{i_1+i_2+...+i_n=r} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}, \tag{1.2}$$

where $i_1, i_2, ..., i_n$ are non-negative integers, $r \in N = \{1, 2, ...\}$, $c_0(x) = 1$. When $1 \leqslant r \leqslant n$, the fact that $c_r(x)$ is Schur-convex in R_+^n has generalized by Baston (see [6, p. 82]). Kaizhong Guan [3] proved that the function $c_r(x)$ and $c_r(x)/c_{r-1}(x)$ is Schur-convex in R_+^n for all $r \in N$. Some analytic inequalities, including "Ky Fan" type inequality, are established by use of the theory of majorization. In [4], the author defined the following symmetric function (mean)

$$\prod_n^r (x) = \left(\prod_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} \frac{1}{r} \sum_{j=1}^r x_{i_j} \right)^{\frac{1}{\binom{n}{r}}}, \tag{1.3}$$

and established the following basic inequality chain:

$$G_n(x) = \prod_n^1 (x) \leqslant \prod_n^2 (x) \leqslant ... \leqslant \prod_n^{n-1} (x) \leqslant \prod_n^n (x) = A_n(x). \tag{1.4}$$

On the other hand, various definitions of convex function have been given and investigated extensively (see [1, 2, 6, 9, 10]). In particular, the weakly logarithmic convex function was pointed out by G. Klambauer [1] and was investigated in [5].

DEFINITION 1. Let $f(x)$ be a positive function defined on an interval I. The function $f(x)$ is a weakly logarithmic convex function if $\forall x_1, x_2 \in I$ implies

$$f\left(\frac{x_1 + x_2}{2}\right) \leqslant \sqrt{f(x_1) f(x_2)}. \tag{1.5}$$

The function $f(x)$ is said to be a weakly logarithmic concave function if (1.5) is reversed.

For fixed $n \geqslant 2$, let

$$x = (x_1, x_2, ..., x_n), \quad y = (y_1, y_2, ..., y_n)$$

be two n-tuples of real numbers. And let

$$x_{[1]} \geqslant x_{[2]} \geqslant ... \geqslant x_{[n]}, \quad y_{[1]} \geqslant y_{[2]} \geqslant ... \geqslant y_{[n]},$$

be their ordered components.

DEFINITION 2. ([6, p. 55; 14, p. 1]) The $n-$tuple x is said to be majorized by y (in symbols $x \prec y$), if

$$\sum_{i=1}^m x_{[i]} \leqslant \sum_{i=1}^m y_{[i]}, \quad m = 1, 2, ..., n-1; \tag{1.6}$$

and

$$\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}. \tag{1.7}$$

The Schur-convex function was introduced by I. Schur in 1923 [6]. It has many important applications in analytic inequalities. Hardy, Littlewood, and Pólya were also interested in some inequalities that are related to Schur-convex functions [8]. Its definition is following

DEFINITION 3. ([6, p. 54]) A real-valued function ϕ defined on a set $\Omega \subset R^n$ is said to be Schur-convex function on Ω if

$$x \prec y \; on \; \Omega \implies \phi(x) \leqslant \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on Ω. ϕ is Schur-concave function on Ω if and only if $-\phi$ is Schur-convex function; ϕ is a strictly Schur-concave function on Ω if and only if $-\phi$ is strictly Schur-convex function on Ω.

Recently, K. Z. Guan [7] defined the following symmetric function which generalized (1.3)

$$\prod_n^r (f(x)) = \left(\prod_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n} f\left(\frac{1}{r}\sum_{j=1}^r x_{i_j}\right) \right)^{\frac{1}{\binom{n}{r}}}, \quad r = 1, 2, ..., n, \tag{1.8}$$

where $f(x)$ is a non-negative function on an interval I, $x_1, x_2, ..., x_n \in I, and \binom{n}{r} = \frac{n!}{r!(n-r)!}$. We also established the basic inequality which is very interesting, that is,

THEOREM A. *If $f(x)$ is a weakly logarithmic convex function, then*

$$\prod_n^{r+1}(f) \leqslant \prod_n^r (f), \quad r = 1, 2, ..., n-1. \tag{1.9}$$

The inequality (1.9) is reversed if $f(x)$ is a weakly logarithmic concave function.

The main purpose of this paper is to consider the symmetric function $\prod_n^r(f(x))$, simply denote it as $\prod_n^r(f)$. In section 3, the Schur-convexity is discussed when $f(x)$ is a weakly logarithmic convex (concave) function, some analytic inequalities are established by use of theory of majorization which is a useful method of establishing inequalities, and an open problem is solved partly. Some well-known inequalities are refined and generalized by use of Theorem A in the section 4.

2. Lemmas

In order to verify our main results, the following lemmas are necessary.

LEMMA 2.1. ([9, *p.* 259; 6, *p.* 57]) *Assume that $f(x) = f(x_1, x_2, ..., x_n)$ is symmetric, and has continuous partial derivatives on I^n, where I is an open interval. Then $f : I^n \to R$ is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \tag{2.1}$$

on I^n. It is strictly Schur-convex if (2.1) is a strict inequality for $x_i \neq x_j$, $1 \leqslant i, j \leqslant n$.

Since $f(x)$ is symmetric, Schur's condition, i.e. (2.1), can be reduced as [6, p. 57]

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geqslant 0, \tag{2.2}$$

and f is strictly Schur-convex if (2.2) is a strict inequality for $x_1 \neq x_2$. The Schur's condition that guarantees a symmetric function being Schur-concave is the same as (2.1) or (2.2) except for the direction of the inequality. In Schur's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n. Lemma 2.1 remains true if we replace I^n by a set $A \subseteq R^n$ with the following properties ([6, p. 57]):

(i) A is convex and has a nonempty interior;

(ii) A is symmetric in the sense that $x \in A$ implies $Px \in A$ for any $n \times n$ permutation matrix P.

For convenience, we quote the following lemma which can be found in the convex literatures (See, for example [2], [9]and [10].).

LEMMA 2.2. *If $f(x)$ is a positive and twice continuously differentiable function on interval I, then*

(i) $f(x)$ is a weakly logarithmic convex function on I if and only if $(f'(x))^2 \leqslant f(x)f''(x)$ or $\frac{f'(x)}{f(x)}$ is increasing on I;

(ii) $f(x)$ is a weakly logarithmic concave function on I if and only if $(f'(x))^2 \geqslant f(x)f''(x)$ or $\frac{f'(x)}{f(x)}$ is decreasing on I.

LEMMA 2.3. ([14, p. 5]) *Assume that $x = (x_1, x_2, ..., x_n) \in R^n$. Then*

$$(\overline{x}, \overline{x}, ..., \overline{x}) \prec (x_1, x_2, ..., x_n), \tag{2.3}$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

3. Schur-convexty of $\prod_n^r(f(x))$

In the section, we investigate the Schur-convexity of $\prod_n^r(f(x))$. Some analytic inequalities are established by use of the theory of majorization. An open problem is also solved partly.

THEOREM 3.1. *If $f(x)$ is a weakly logarithmic convex, and has twice continuous derivatives function on an interval I, then $\prod_n^r(f)$ is Schur-convex function on I^n; the fact that $f(x)$ is a weakly logarithmic concave function with two order continuous derivatives on I implies that $\prod_n^r(f)$ is a Schur-concave function on I^n.*

Proof. We only consider the case that $f(x)$ is a weakly logarithmic convex function on I. The case where $f(x)$ is a weakly logarithmic concave function on I is similar and is omitted.

Obviously, $\prod_n^r(f)$ is symmetric and continuously differential on I^n. By Lemma 2.1, we only need to prove

$$(x_1 - x_2)\left(\frac{\partial \prod_n^r(f)}{\partial x_1} - \frac{\partial \prod_n^r(f)}{\partial x_2}\right) \geqslant 0. \tag{3.1}$$

To this end, we consider the following three possible cases for r.

Case 1. When $r=1$. By taking logarithm on $\prod_n^1(f)$, we have

$$\ln\prod_n^1(f)=\frac{1}{n}\sum_{i=1}^n \ln f(x_i). \tag{3.2}$$

Differentiating the both sides of Eq. (3.2) with respect to x_i, we obtain

$$\frac{\partial\prod_n^1(f)}{\partial x_i}=\frac{1}{n}\prod_n^1(f)\frac{f'(x_i)}{f(x_i)},\ i=1,2,...,n.$$

And so

$$(x_1-x_2)\left(\frac{\partial\prod_n^1(f)}{\partial x_1}-\frac{\partial\prod_n^1(f)}{\partial x_2}\right)=\frac{1}{n}\prod_n^1(f)(x_1-x_2)\left(\frac{f'(x_1)}{f(x_1)}-\frac{f'(x_2)}{f(x_2)}\right).$$

Since $f(x)$ is a weakly logarithmic convex function, from Lemma 2.2, it follows that inequality (3.1).

Case 2.

When $r=2$. If $n=2$, we have

$$\prod_2^2(f)=f\left(\frac{x_1+x_2}{2}\right).$$

And so (3.1) is obvious. If $n\geqslant 3$, we can easily derive

$$\begin{aligned}\ln\prod_n^2(f)&=\frac{1}{\binom{n}{2}}\sum_{1\leqslant i<j\leqslant n}\ln f\left(\frac{x_i+x_j}{2}\right)\\&=\frac{1}{\binom{n}{2}}\left[\sum_{j=2}^n \ln f\left(\frac{x_1+x_j}{2}\right)+\sum_{2\leqslant i<j\leqslant n}\ln f\left(\frac{x_i+x_j}{2}\right)\right].\end{aligned}$$

Differentiating the above equation with respect to x_1, we have

$$\begin{aligned}\frac{\partial\prod_n^2(f)}{\partial x_1}&=\frac{1}{2\binom{n}{2}}\prod_n^2(f)\sum_{j=2}^n\frac{f'\left(\frac{x_1+x_j}{2}\right)}{f\left(\frac{x_1+x_j}{2}\right)}\\&=\frac{1}{2\binom{n}{2}}\prod_n^2(f)\left[\frac{f'\left(\frac{x_1+x_2}{2}\right)}{f\left(\frac{x_1+x_2}{2}\right)}+\sum_{j=3}^n\frac{f'\left(\frac{x_1+x_j}{2}\right)}{f\left(\frac{x_1+x_j}{2}\right)}\right].\end{aligned}$$

Similarly, we can also obtain

$$\frac{\partial\prod_n^2(f)}{\partial x_2}=\frac{1}{2\binom{n}{2}}\prod_n^2(f)\left[\frac{f'\left(\frac{x_1+x_2}{2}\right)}{f\left(\frac{x_1+x_2}{2}\right)}+\sum_{j=3}^n\frac{f'\left(\frac{x_2+x_j}{2}\right)}{f\left(\frac{x_2+x_j}{2}\right)}\right].$$

Set

$$u=\frac{x_1+x_j}{2},\quad v=\frac{x_2+x_j}{2},\quad\text{and}\quad \varphi(x)=\frac{f'(x)}{f(x)}.$$

And so

$$(x_1 - x_2)\left(\frac{\partial \prod_n^2(f)}{\partial x_1} - \frac{\partial \prod_n^2(f)}{\partial x_2}\right) = \frac{(x_1 - x_2)}{2\binom{n}{2}} \prod_n^2(f)(\varphi(u) - \varphi(v))$$

$$= \frac{1}{\binom{n}{2}} \prod_n^2(f) \sum_{j=3}^{n} (\varphi(u) - \varphi(v))(u - v).$$

Since $f(x)$ is a weakly logarithmic convex function, from Lemma 2.2 and $u, v \in I$, we get inequality (3.1).

Case 3.

When $3 \leqslant r \leqslant n$. Taking logarithm on $\prod_n^r(f)$ and calculating simply yields

$$\ln \prod_n^r(f) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n} \ln f\left(\frac{x_{i_1} + x_{i_2} + \ldots + x_{i_r}}{r}\right)$$

$$= \frac{1}{\binom{n}{r}} \left[\sum_{2 \leqslant i_1 < \ldots < i_r \leqslant n} \ln f\left(\frac{x_{i_1} + x_{i_2} + \ldots + x_{i_r}}{r}\right) + \sum_{2 \leqslant i_1 < \ldots < i_{r-1} \leqslant n} \ln f\left(\frac{x_1 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right) \right].$$

Differentiating the above equation with respect to x_1, we obtain

$$\frac{\partial \prod_n^r(f)}{\partial x_1} = \frac{1}{r\binom{n}{r}} \prod_n^r(f) \cdot \left[\sum_{2 \leqslant i_1 < \ldots < i_{r-1} \leqslant n} \frac{f'\left(\frac{x_1 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)}{f\left(\frac{x_1 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)} \right]$$

$$= \frac{1}{r\binom{n}{r}} \prod_n^r(f) \cdot \left[\sum_{3 \leqslant i_1 < \ldots < i_{r-1} \leqslant n} \frac{f'\left(\frac{x_1 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)}{f\left(\frac{x_1 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)} + \sum_{3 \leqslant i_1 < \ldots < i_{r-2} \leqslant n} \frac{f'\left(\frac{x_1 + x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-2}}}{r}\right)}{f\left(\frac{x_1 + x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-2}}}{r}\right)} \right].$$

Similarly, we can also get

$$\frac{\partial \prod_n^r(f)}{\partial x_2} = \frac{1}{r\binom{n}{r}} \prod_n^r(f) \cdot \left[\sum_{3 \leqslant i_1 < \ldots < i_{r-1} \leqslant n} \frac{f'\left(\frac{x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)}{f\left(\frac{x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-1}}}{r}\right)} + \sum_{3 \leqslant i_1 < \ldots < i_{r-2} \leqslant n} \frac{f'\left(\frac{x_1 + x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-2}}}{r}\right)}{f\left(\frac{x_1 + x_2 + x_{i_1} + x_{i_2} + \ldots + x_{i_{r-2}}}{r}\right)} \right].$$

Let

$$u_* = \frac{x_1 + x_{i_1} + \ldots + x_{i_{r-1}}}{r}, v_* = \frac{x_2 + x_{i_1} + \ldots + x_{i_{r-1}}}{r}, \quad \text{and} \quad \varphi(x) = \frac{f'(x)}{f(x)}.$$

Thus, we obtain

$$(x_1-x_2)\left(\frac{\partial \prod_n^r(f)}{\partial x_1}-\frac{\partial \prod_n^r(f)}{\partial x_2}\right) = \frac{(x_1-x_2)}{r\binom{n}{r}}\prod_n^r(f)\cdot \sum_{3\leqslant i_1<i_2<\ldots<i_{r-1}\leqslant n}(\varphi(u_*)-\varphi(v_*))$$
$$= \frac{1}{\binom{n}{r}}\prod_n^r(f)\cdot \sum_{3\leqslant i_1<i_2<\ldots<i_{r-1}\leqslant n}(\varphi(u_*)-\varphi(v_*))(u_*-v_*).$$

From Lemma 2.2 and $u_*, v_* \in I$, it follows that

$$(\varphi(u_*)-\varphi(v_*))(u_*-v_*)\geqslant 0.$$

And so

$$(x_1-x_2)\left(\frac{\partial \prod_n^r(f)}{\partial x_1}-\frac{\partial \prod_n^r(f)}{\partial x_2}\right)\geqslant 0.$$

Combing the cases 1-3, we have completed the proof of Theorem 3.1.

REMARK 1. Let $f(x)$ be a weakly logarithmic convex, and has twice continuous derivatives function on an interval I. A. W. Marshall and I. Olkin [6, p. 73] proved that the function

$$\phi(x)=\prod_{i=1}^n f(x_i), \quad x\in I^n,$$

is Schur-convex on I^n. By Theorem 3.1 (let $r=1$), we prove that the following so-called geometric mean of $f(x_i)$:

$$\prod_n^1(f)=\left(\prod_{i=1}^n f(x_i)\right)^{\frac{1}{n}}$$

is also Schur-convex on I^n.

By Theorem 3.1, we can get the following

COROLLARY 3.2. *Assume that $f(x)$ is a weakly logarithmic convex function with twice continuous derivatives in R, $x=(x_1,x_2,\ldots,x_n)$, $y=(y_1,y_2,\ldots,y_n)$, and $x\prec y$. Then*

$$\prod_n^r(f(x))\leqslant \prod_n^r(f(y)), \quad r=1,2,\ldots,n. \tag{3.3}$$

The inequality (3.3) is reversed if $f(x)$ is a weakly logarithmic concave function with twice continuous derivatives in R.

The fact that the function $\psi(x)=\prod_{i_1<i_2<\ldots<i_k}\sum_{j=1}^k x_{i_j}$ is Schur-concave in R_+^n was given by Marshall and Olkin in [6, p. 86]. Now we establish the following conclusions.

THEOREM 3.3. *Assume that $x_i\in(0,1), i=1,2,\ldots,n$, and that*

$$E_k^*(x)=E_k^*(x_1,x_2,\ldots,x_n)=\prod_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n}\sum_{j=1}^k x_{i_j}, \quad k=1,2,\ldots,n.$$

Then

(*i*) $x_i \in (0, \frac{1}{2}]$ *implies that* $\frac{E_k^*(1-x)}{E_k^*(x)}$ *is Schur-convex function in* $(0, \frac{1}{2}]^n$;

(*ii*) $x_i \in [\frac{1}{2}, 1)$ *implies that* $\frac{E_k^*(1-x)}{E_k^*(x)}$ *is Schur-concave function in* $[\frac{1}{2}, 1)^n$.

Proof. Let $f(t) = \frac{1-t}{t}, t \in (0,1)$, a simple calculation reveals that

$$(f'(t))^2 - f(t)f''(t) = \frac{2t-1}{t^4}.$$

(*i*) When $t \in (0, \frac{1}{2}]$. By Lemma 2.2, we see that $f(t)$ is a weak logarithmical convex function in $t \in (0, \frac{1}{2}]$. For a fixed number k, calculating directly and using Theorem 3.1, we see that $\prod_n^k(f) = \left(\frac{E_k^*(1-x)}{E_k^*(x)}\right)^{\frac{1}{\binom{n}{k}}}$ is Schur-convex function in $(0, \frac{1}{2}]^n$. Thus, for $x_i, y_i \in (0, \frac{1}{2}]$ and $x \prec y$, we have

$$\left(\frac{E_k^*(1-x)}{E_k^*(x)}\right)^{\frac{1}{\binom{n}{k}}} \leqslant \left(\frac{E_k^*(1-y)}{E_k^*(y)}\right)^{\frac{1}{\binom{n}{k}}},$$

or

$$\left(\frac{E_k^*(1-x)}{E_k^*(x)}\right) \leqslant \left(\frac{E_k^*(1-y)}{E_k^*(y)}\right).$$

By Definition 3, the function $\frac{E_k^*(1-x)}{E_k^*(x)}$ is Schur-convex function in $(0, \frac{1}{2}]^n$.

(*ii*) When $t \in [\frac{1}{2}, 1)$. By Lemma 2.2, $f(t)$ is a weak logarithmical concave function in $t \in [\frac{1}{2}, 1)$. Similar to the proof of (i), we may prove that the function $\frac{E_k^*(1-x)}{E_k^*(x)}$ is Schur-concave function in $[\frac{1}{2}, 1)^n$. Thus, the proof is complete.

REMARK 2. Theorem 3.3 partly answer the problem pointed out by Prof. Shi and published in the following website: "http://zgbdsyjxz.nease.net".

COROLLARY 3.4. *Suppose that* $0 < x_i \leqslant \frac{1}{2}, i = 1, 2, ..., n,$ *and* $\sum_{i=1}^n x_i \leqslant 1$. *Then*

$$\frac{E_k^*(1-x)}{E_k^*(x)} \geqslant \left(\frac{n-1}{n}\right)^k. \tag{3.4}$$

Proof. Using Theorem 3.3 and Lemma 2.3, we have

$$\frac{E_k^*(1-x)}{E_k^*(x)} \geqslant \frac{E_k^*(1-\overline{x})}{E_k^*(\overline{x})}. \tag{3.5}$$

Or,

$$\frac{E_k^*(1-x)}{E_k^*(x)} \geqslant \left(\frac{1-\overline{x}}{\overline{x}}\right)^k. \tag{3.6}$$

Since $f(x) = (\frac{1-x}{x})^k$ is decreasing in $(0, \frac{1}{n}]$, it follows from (3.6) that

$$\frac{E_k^*(1-x)}{E_k^*(x)} \geqslant \left(\frac{n-1}{n}\right)^k.$$

REMARK 3. Inequality (3.4) give the blower bound of $\frac{E_k^*(1-x)}{E_k^*(x)}$.

4. Some applications

In this section, we establish several analytic inequalities by use of Theorem A. Our proofs are briefer than those of the literatures. Which illustrates the advantage of our results.

THEOREM 4.1. ([4]) *Assume that* $x_i > 0, i = 1, 2, ..., n$. *Then*

$$G_n(x) = \prod_n^1(x) \leqslant \prod_n^2(x) \leqslant ... \leqslant \prod_n^n(x) = A_n(x). \tag{4.1}$$

Proof. Let $f(x) = x, x \in (0, +\infty)$, we can easily verify that $f(x) = x$ is a weak logarithmical concave function in $(0, +\infty)$. By Theorem A, we have (4.1).

REMARK 4. The inequality (4.1) refines A-G inequality, and my proof is simpler than that of [4].

For $0 < x_i \leqslant \frac{1}{2}$, $i = 1, 2, ..., n$, the following inequality

$$\frac{G_n(x)}{G_n(1-x)} \leqslant \frac{A_n(x)}{A_n(1-x)}, \tag{4.2}$$

commonly referred to as the Ky Fan inequality ([12, p.5]) has stimulated an interest of many researchers. New proofs, improvements and generalizations of the inequality (4.2). See, for example, [2, 3, 11, 15] and the references cited therein. Now we shall investigate it further.

THEOREM 4.2. *Assume that* $0 < x_i \leqslant \frac{1}{2}, i = 1, 2, ..., n$, *and let*

$$S_r(x) = S_r(x_1, x_2, ..., x_n) = \left(\prod_{1 \leqslant i_1 < i_2 < ... < i_r \leqslant n} \frac{\sum_{j=1}^n x_{i_j}}{\sum_{j=1}^n (1 - x_{i_j})} \right)^{\frac{1}{\binom{n}{r}}}, \quad r = 1, 2, ..., n.$$

Then

$$\frac{G_n(x)}{G_n(1-x)} = S_1(x) \leqslant S_2(x) \leqslant ... \leqslant S_n(x) = \frac{A_n(x)}{A_n(1-x)}. \tag{4.3}$$

Proof. Let $f(x) = \frac{x}{1-x}, x \in (0, \frac{1}{2}]$. A simple calculation reveals that

$$(f'(x))^2 - f(x)f''(x) = \frac{1-2x}{(1-x)^4} \geqslant 0, x \in (0, \frac{1}{2}].$$

It is clear that $\prod_n^r(f) = S_r(x)$. Using Lemma 2.2 and Theorem A, we immediately obtain (4.3).

REMARK 5. The inequality (4.3) refines Ky Fan inequality.

THEOREM 4.3. ([11]) *Assume that* $0 < x_i < 1, i = 1, 2, ..., n$, *then*

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1 - x_i)} \leqslant \prod_{i=1}^n \left(\frac{x_i}{1 - x_i}\right)^{\beta_i}, \tag{4.4}$$

where $S_n = \sum_{i=1}^n x_i$ *and* $\beta_i = \frac{x_i}{S_n}$.

Proof. Let $f(x) = (\frac{x}{1-x})^x, \quad x \in (0,1)$, by taking logarithm on the function $f(x)$ and differentiating it with respect to x, we have

$$f'(x) = f(x)\left(\frac{1}{1-x} + \ln\frac{x}{1-x}\right), \tag{4.5}$$

and

$$f''(x) = f(x)\left[\left(\frac{1}{1-x} + \ln\frac{x}{1-x}\right)^2 + \frac{1}{x} + \frac{1}{1-x} + \left(\frac{1}{1-x}\right)^2\right]. \tag{4.6}$$

By Lemma 2.2, and from (4.5) and (4.6), we can see that $f(x)$ is a weakly logarithmic convex function. From Theorem A, we obtain

$$\left(\frac{A_n(x)}{1-A_n(x)}\right)^{A_n(x)} = \prod_n^n(f) \leqslant \prod_n^{n-1}(f) \leqslant \dots \leqslant \prod_n^2(f) \leqslant \prod_n^1(f) = \left(\prod_{i=1}^n(\frac{x_i}{1-x_i})^{x_i}\right)^{\frac{1}{n}}.$$

In particular, we have

$$\left(\frac{A_n(x)}{1-A_n(x)}\right)^{A_n(x)} \leqslant \left(\prod_{i=1}^n\left(\frac{x_i}{1-x_i}\right)^{x_i}\right)^{\frac{1}{n}}. \tag{4.7}$$

Simplifying (4.7), we can get the inequality (4.4).

REMARK 6. Our proof is briefer than that of [11] and our result also refines inequality (4.4).

THEOREM 4.4. *Assume that $f(x)$ is an increasing and weakly logarithmic convex function with two order derivatives in $(0,+\infty)$, and that $g(x) = f(\frac{x}{1-x}), x \in (0,1)$. Then*

$$\prod_n^{r+1}(g) \leqslant \prod_n^r(g), r = 1,2,\dots,n-1. \tag{4.8}$$

(4.8) *reverses if $f(x)$ is a decreasing and weakly logarithmic concave function with twice derivatives.*

Proof. We consider the case where $f(x)$ is an increasing and weakly logarithmic convex function. Calculating and using the assumption of the theorem yields

$$(g'(x))^2 - g(x)g''(x) = \frac{1}{(1-x)^4}(f'(u))^2 - f(u)f''(u)) - \frac{1}{(1-x)^3}f(u)f'(u) \leqslant 0,$$

where $u = \frac{x}{1-x}$. By Lemma 2.2, $g(x)$ is a weakly logarithmic convex function. From Theorem A, we obtain (4.8).

The case that $f(x)$ is a decreasing and weakly logarithmic concave function with two order derivatives is similar to the above, and so be omitted.

REMARK 7. Inequality (4.8) can be rewritten as the following inequality chain:

$$f\left(\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n}(1-x_i)}\right) = \prod_{n}^{1}(g) \leqslant \prod_{n}^{2}(g) \leqslant \ldots \leqslant \prod_{n}^{n}(g) = \left(\prod_{i=1}^{n} f\left(\frac{x_i}{1-x_i}\right)\right)^{\frac{1}{n}}. \quad (4.9)$$

Thus we establish a class of more general "Ky Fan" type inequalities.

The well-known Jensen's inequality for positive convex function is

$$f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \leqslant \frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

In [13], J. Pečarić and D. Svrtan give new refinement of it, that is,

THEOREM 4.5. *Assume that* $f : I \to R$ *is convex function,* $x_k \in I, k = 1,2,\ldots,n$. *Let* $f_{k,n} = \frac{1}{\binom{n}{k}} \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right)$. *Then*

$$f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = f_{n,n} \leqslant \ldots \leqslant f_{k+1,n} \leqslant f_{k,n} \leqslant \ldots \leqslant f_{1,n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k). \quad (4.10)$$

Proof. Let $\phi(x) = \exp(f(x))$. By Definition 1, it is easy to verify that $\phi(x)$ is a weak logarithmical convex function in I. Using Theorem A, we have

$$\prod_{n}^{n}(\phi) \leqslant \prod_{n}^{n-1}(\phi) \leqslant \ldots \leqslant \prod_{n}^{2}(\phi) \leqslant \prod_{n}^{1}(\phi). \quad (4.11)$$

By taking logarithm on (4.11), we establish (4.10).

REMARK 8. The proof of the theorem is briefer than that of the paper [13].

THEOREM 4.6. *Assume that* $0 < x_i < 1, i = 1,2,\ldots,n$, *and let*

$$\psi_r(x) = \psi_r(x_1, x_2, \ldots, x_n) = \left[\prod_{1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n} \frac{\sum_{j=1}^{r}(1+x_{i_j})}{\sum_{j=1}^{r}(1-x_{i_j})}\right]^{\frac{1}{\binom{n}{r}}}.$$

Then

$$\psi_{r+1}(x) \leqslant \psi_r(x), r = 1,2,\ldots,n-1. \quad (4.12)$$

Proof. Let $f(x) = \frac{1+x}{1-x}, x \in (0,1)$. Calculating simply and using Lemma 2.2, we see that $f(x)$ is a weakly logarithmic convex function in $(0,1)$. By Theorem A, we prove the inequality (4.12).

REMARK 9. The inequality (4.12) can be written as the following inequality chain:

$$\left(\prod_{i=1}^{n} \frac{1+x_i}{1-x_i}\right)^{\frac{1}{n}} = \psi_1(x) \geqslant \psi_2(x) \geqslant \ldots \geqslant \psi_n(x) = \frac{n+\sum_{i=1}^{n} x_i}{n-\sum_{i=1}^{n} x_i}. \quad (4.13)$$

Thus, when $\sum_{i=1}^{n} x_i = 1$, (4.13) refined Klamkin inequality [6, p. 78]:

$$\prod_{i=1}^{n} \frac{1+x_i}{1-x_i} \geqslant \left(\frac{n+1}{n-1}\right)^n.$$

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