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A NOTE FOR BOUNDS OF NORMS OF HADAMARD PRODUCT OF MATRICES

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Abstract. In this paper, we have established upper bounds for the spectral norms of Cauchy-Toeplitz matrix and Cauchy-Hankel matrix, with $g = 1/2$ and $h = 1$. Moreover, we have obtained an upper bound for the spectral norm of Hadamard product of Cauchy-Toeplitz and Cauchy-Hankel matrices. In addition, we have established an upper bound for the norm of Hadamard product of Cauchy-Toeplitz and Cauchy-Hankel matrices.

1. Introduction

It is well known that arbitrary Cauchy-Toeplitz and Cauchy-Hankel matrices of order n are of the form

$$T_n = \left(\frac{1}{g+(i-j)h}\right)_{i,j=1}^{n} \tag{1.1}$$

and

$$H_n = \left(\frac{1}{g-(i+j)h}\right)_{i,j=1}^{n} \tag{1.2}$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.

In [9], E. E. Tyrtyshnikov has obtained lower bounds for the spectral norm of Cauchy-Toeplitz matrix for $h = 1$ and $g = 1/2$.

C. Moler had experimentaly discovered that most of the singular values of Cauchy-Toeplitz matrices are clustered near π. Recently S. Parter explained this phenomenon [2]. In [9], E. E. Tyrtyshnikov has shown that the numbers of the singular values of T_n matrices satisfying $\sigma_{1n} \geqslant \ldots \geqslant \sigma_{nn}$ and $\sigma_{jn} > \pi - \varepsilon$ for $\varepsilon = 10^{-4}$ are 6, 7, 8 for $n = 40, 60, 100$ respectively. In [8], D. Bozkurt has established upper and lower bounds for Euclidean norm of the matrix T_n in (1.1) in general case. In [5] and [6], he has obtained upper and lower bounds for the ℓ_p norms of Cauchy-Toeplitz matrix.

In the section 2, we have established an upper bound for the spectral norm of Cauchy-Toeplitz matrix and Cauchy-Hankel matrix in (1.1) and (1.2) respectively. In the section 3, we have found an upper bound for the spectral norm of Hadamard product of the Cauchy-Toeplitz and Cauchy-Hankel matrices (1.1) and (1.2). Also, we have

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obtained an upper bound for the ℓ_p norm of Hadamard product of the Cauchy-Toeplitz and Cauchy-Hankel matrices in (1.1) and (1.2).

Let A be an $m \times n$ matrix. Then, the Euclidean norm, the ℓ_p norm and the spectral norm of the matrix A are defined by

$$\|A\|_2 = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2},$$

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p}, \quad 1 \leqslant p < \infty$$

and

$$|||A|||_2 = \sqrt{\max_i |\lambda_i(A^*A)|}$$

respectively, where A^* is the conjugate transpose of the matrix A.

A function Ψ is called polygamma function if

$$\Psi(x) = \frac{d}{dx} \{\log [\Gamma(x)]\}$$

where

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

The function $\Psi(m,\ x)$ has the property:

$$\lim_{n \to \infty} \Psi(a,\ n+b) = 0 \tag{1.3}$$

where $a > 0$, b are any numbers and n is a positive integer.

Denote the space of m-by-n complex matrices by $M_{m,n}$ and set $M_n = M_{n,n}$. Then the Hadamard (entry-wise) product of $A = (a_{ij})$ and $B = (b_{ij}) \in M_{m,n}$ is defined by

$$A \circ B = (a_{ij} b_{ij}) \in M_{m,n}.$$

For any $A \in M_{m,n}$, we denote by $c_1(A) \geqslant c_2(A) \geqslant \ldots \geqslant c_n(A) \geqslant 0$ Euclidean lengths of the n columns of A, listed in descending order, and by $r_1(A) \geqslant r_2(A) \geqslant \ldots \geqslant r_n(A) \geqslant 0$ Euclidean lengths of the n rows of A, similarly ordered. The singular values of $A \in M_{m,n}$, which we shall always exhibit in descending order,

$$\sigma_1(A) \geqslant \sigma_2(A) \geqslant \ldots \geqslant \sigma_n(A) \geqslant 0$$

are the non-negative square roots of the eigenvalues of AA^* as well as the non-negative square roots of the n largest eigenvalues of AA^*. Let $\Sigma(A) = (\sigma_{ij}) \in M_{m,n}$ where $\sigma_{ii} = \sigma_i(A)$ for $i = 1, 2, ..., n$ and all other $\sigma_{ij} \equiv 0$ for $i \neq j$. We know that A has a singular value decomposition $A = V\Sigma(A)W^*$, in which $V \in M_m$ and $W \in M_n$ are unitary matrices [1].

Throughout this paper we will take $h = 1$ and $g = 1/2$, while $[.]$ denotes the greatest integer function and $m = n$.

2. Spectral norm of Cauchy-Toeplitz and Cauchy-Hankel matrices

THEOREM 1. [4]. *Let* $A, B, C \in M_{m,n}$. *If* $A = B \circ C$ *then*

$$|||A|||_2 \leqslant r_1(B)c_1(C) \tag{2.1}$$

where $r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2}$ *and* $c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2}$ *respectively.*

If we substitute $g = 1/2$ and $h = 1$ into (1.1), then we have

$$T_n = \left(\frac{2}{1+2(i-j)}\right)_{i,j=1}^{n} \quad \text{and} \quad H_n = \left(\frac{2}{1-2(i+j)}\right)_{i,j=1}^{n}. \tag{2.2}$$

THEOREM 2. *Let the matrix* T_n *be as in (2.2). Then*

$$\frac{1}{\sqrt{n}}\,|||T_n|||_2 \leqslant \pi$$

is valid where $|||.|||_2$ *is the spectral norm.*

Proof. Let $T_n = A \circ B$, then from (2.1)

$$|||T_n|||_2 \leqslant r_1(A)c_1(B)$$

where $A = (1/(1+2(i-j)))$ and $B = (2)$ respectively.

$$r_1(A) = \max_i \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}$$

$$= \begin{cases} \sqrt{\displaystyle\sum_{k=1}^{(n-1)/2} \frac{1}{(2k-1)^2} + \sum_{k=1}^{[n/2]+1} \frac{1}{(2k-1)^2}}, & \text{if } n \text{ odd} \\ \sqrt{2\displaystyle\sum_{k=1}^{n/2} \frac{1}{(2k-1)^2}}, & \text{if } n \text{ even} \end{cases} \tag{2.3}$$

and

$$c_1(B) = \max_j \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = 2\sqrt{n}$$

If we evaluate sums in (2.3), then

$$r_1(A) = \begin{cases} \sqrt{-\frac{1}{4}\Psi(1,\frac{n}{2}) + \frac{\pi^2}{8} - \frac{1}{4}\Psi\left(1, \left[\frac{n}{2}\right]+1\right) + \frac{\pi^2}{8}}, & \text{if } n \text{ odd} \\ \sqrt{-\frac{1}{2}\Psi\left(1, \frac{n}{2}+\frac{1}{2}\right) + \frac{\pi^2}{4}}, & \text{if } n \text{ even} \end{cases} \tag{2.4}$$

and

$$c_1(B) = 2\sqrt{n}. \tag{2.5}$$

Thus, from (2.4) and (2.5)

$$\frac{1}{\sqrt{n}}\,|||T_n|||_2 \leqslant \begin{cases} \sqrt{-\Psi(1,\frac{n}{2}) - \Psi\left(1,\left[\frac{n}{2}\right]+1\right)+\pi^2}, & \text{if } n \text{ odd} \\ \sqrt{-2\Psi\left(1,\frac{n}{2}+\frac{1}{2}\right)+\pi^2}, & \text{if } n \text{ even.} \end{cases} \tag{2.6}$$

Taking the limit of the right hand side of (2.7) as $n \to \infty$ and from (1.3), we get

$$\frac{1}{\sqrt{n}}\,|||T_n|||_2 \leqslant \pi.$$

This completes the proof. $\square$

THEOREM 3. *Let the matrix H_n be as in (2.2). Then*

$$\frac{1}{\sqrt{n}}\,|||H_n|||_2 \leqslant \sqrt{\frac{\pi^2}{2}-4}$$

is valid, where $|||.|||_2$ is the spectral norm.

Proof. Let $H_n = A \circ B$, then from (2.1)

$$|||H_n|||_2 \leqslant r_1(A)c_1(B)$$

where $A = (2)$ and $B = (1/(1-2(i+j)))$ respectively.

$$r_1(A) = \max_i \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = 2\sqrt{n} \tag{2.7}$$

$$c_1(B) = \max_j \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{k=1}^{n} \frac{1}{(2k+1)^2}}. \tag{2.8}$$

The values of the sum in (2.8) is to be

$$c_1(B) = \sqrt{\Psi\left(1, n+\frac{1}{2}\right) + \frac{\pi^2}{8} - 1}.$$

If we replace values in (2.8), then

$$\frac{1}{\sqrt{n}}\,|||H_n|||_2 \leqslant \sqrt{-\Psi\left(1, n+\frac{3}{2}\right) - 4 + \frac{\pi^2}{2}}. \tag{2.9}$$

Taking again the limit of the right hand side of inequality (2.9) as $n \to \infty$, we have

$$\frac{1}{\sqrt{n}}\,|||H_n|||_2 \leqslant \sqrt{\frac{\pi^2}{2} - 4}.$$

Thus, the proof is completed. $\square$

3. Spectral norms of the Hadamard product of Cauchy-Toeplitz and Cauchy-Hankel matrices

THEOREM 4. [3]. *Let A, $B \in M_n$. Then*

$$\sum_{i=1}^{k} \sigma_i(A \circ B) \leqslant \begin{cases} \sum_{i=1}^{k} r_i(A)c_i(B), & k = 1,2,...,n \\ \sum_{i=1}^{k} c_i(A)r_i(B), & k = 1,2,...,n. \end{cases} \tag{3.1}$$

Proof. Because Hadamard product is commutative, two inequalities in (3.1) are equivalent; we verify the upper one. We first note the case $k = 1$. Let $\|.\|_2$ denotes Euclidean norm on $\mathbb{C}^n$, let $A = [a_{ij}]$, $B = [b_{ij}]$ and let $x = [x_i] \in \mathbb{C}^n$ be a given unit vector. Then

$$\begin{aligned} \|(A \circ B)x\|_2^2 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} b_{ij} x_j \right|^2 \leqslant \sum_{i=1}^{n} \left[\sum_{l=1}^{n} |a_{il}|^2 \right] \left[\sum_{j=1}^{n} |b_{ij} x_j|^2 \right] \\ &\leqslant r_1(A)^2 \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \, |x_j|^2 = r_1(A)^2 \sum_{j=1}^{n} |x_j|^2 \sum_{j=1}^{n} |b_{ij}|^2 \\ &\leqslant r_1(A)^2 c_1(B)^2 \sum_{j=1}^{n} |x_j|^2 = r_1(A)^2 c_1(B)^2. \end{aligned}$$

Since $\sigma_1(A \circ B) = \max\{\|(A \circ B)x\|_2 : \|x\|_2 = 1\}$, the desired bound has obtained. □

THEOREM 5. *Let T_n and H_n be as in* (2.2). *Then*

$$\||T_n \circ H_n|\|_2 \leqslant \sqrt{\frac{\pi^4}{2} - 4\pi^2}.$$

Proof. From [3],

$$\sum_{i=1}^{k} \sigma_i(T_n \circ H_n) \leqslant \sum_{i=1}^{k} r_i(T_n)c_i(H_n), \quad i = 1,2,...,n$$

where $r_i(T_n) = \max_i \sqrt{\sum_{j=1}^{n} |t_{ij}|^2}$ and $c_i(H_n) = \max_j \sqrt{\sum_{i=1}^{n} |h_{ij}|^2}$, $k = \min\{m,n\}$. This inequality will be proved for special value of $k = 1$ and thus

$$\sigma_1(T_n \circ H_n) \leqslant r_1(T_n)c_1(H_n). \tag{3.2}$$

Hence, we have

$$r_1(T_n) = \begin{cases} \sqrt{\sum_{k=1}^{(n-1)/2} \frac{4}{(2k-1)^2} + \sum_{k=1}^{[n/2]+1} \frac{4}{(2k-1)^2}}, & \text{if } n \text{ odd} \\ \sqrt{\sum_{k=1}^{n/2} \frac{8}{(2k-1)^2}}, & \text{if } n \text{ even} \end{cases} \tag{3.3}$$

and

$$c_1(H_n) = \sqrt{\sum_{k=1}^{n} \frac{4}{(2k+1)^2}}. \tag{3.4}$$

If we evaluate sums in (3.3) and (3.4), then

$$\sum_{k=1}^{(n-1)/2} \frac{4}{(2k-1)^2} = -\Psi\left(1, \frac{n}{2}\right) + \frac{\pi^2}{2},$$

$$\sum_{k=1}^{[n/2]+1} \frac{4}{(2k-1)^2} = -\Psi\left(1, \left[\frac{n}{2}\right] + 1\right) + \frac{\pi^2}{2},$$

$$\sum_{k=1}^{n/2} \frac{8}{(2k-1)^2} = -2\Psi\left(1, \frac{n}{2} + \frac{1}{2}\right) + \pi^2,$$

and

$$\sum_{k=1}^{n} \frac{4}{(2k+1)^2} = -\Psi\left(1, n + \frac{3}{2}\right) - 4 + \frac{\pi^2}{2}.$$

Thus

$$r_1(T_n) = \begin{cases} \sqrt{-\Psi\left(1, \frac{n}{2}\right) + \frac{\pi^2}{2} - \Psi\left(1, \left[\frac{n}{2}\right] + 1\right) + \frac{\pi^2}{2}}, & \text{if } n \text{ odd} \\ \sqrt{-2\Psi\left(1, \frac{n}{2} + \frac{1}{2}\right) + \pi^2}, & \text{if } n \text{ even} \end{cases} \tag{3.5}$$

and

$$c_1(H_n) = \sqrt{-\Psi\left(1, n + \frac{3}{2}\right) - 4 + \frac{\pi^2}{2}}. \tag{3.6}$$

Taking the limits of the right hand sides of equalities in (3.5) and (3.6) as $n \to \infty$, we have

$$r_1(T_n) = \pi, \quad c_1(H_n) = \sqrt{\frac{\pi^2}{2} - 4}. \tag{3.7}$$

If we replace values in (3.7) at (3.2), then

$$\sigma_1(T_n \circ H_n) \leqslant \sqrt{\frac{\pi^4}{2} - 4}.$$

From [4],

$$|||T_n \circ H_n|||_2 \leqslant \sqrt{\frac{\pi^4}{2} - 4\pi^2}.$$

This completes the proof. $\square$

COROLLARY 6. *Let T_n and H_n be respectively as in (2.2), then*

$$n^{-1/p}\left\|T_n\circ H_n\right\|_p \leqslant 2^{1/p}\left\{\left[\left(2^p-1\right)\zeta(p)\right]\left[1+\left(2^p-1\right)\zeta(p-1)\right.\right.$$
$$\left.\left.-\left(2^{p-1}-1/2\right)\zeta(p)-\ln 2\right]\right\}^{1/p}$$

where $2<p<\infty$ and ζ is Riemann-zeta function.

Proof. From [5] and [6] we have

$$n^{-1/p}\left\|T_n\right\|_p \leqslant 2^{1/p}\left[\left(2^p-1\right)\zeta(p)\right]^{1/p} \tag{3.8}$$

and

$$\left\|H_n\right\|_p \leqslant \left[1+\left(2^p-1\right)\zeta(p-1)-\left(2^{p-1}-1/2\right)\zeta(p)+\ln 2\right]^{1/p}. \tag{3.9}$$

Since

$$\left\|T_n\circ H_n\right\|_p \leqslant \left\|T_n\right\|_p\left\|H_n\right\|_p, \tag{3.10}$$

by multipling both sides of inequality in (3.10) by $n^{-1/p}$ we get:

$$n^{-1/p}\left\|T_n\circ H_n\right\|_p \leqslant n^{-1/p}\left\|T_n\right\|_p\left\|H_n\right\|_p.$$

Now, it is easy to derive from (3.8) and (3.9) that

$$n^{-1/p}\left\|T_n\circ H_n\right\|_p \leqslant 2^{1/p}\left\{\left[\left(2^p-1\right)\zeta(p)\right]\left[1+\left(2^p-1\right)\zeta(p-1)\right.\right.$$
$$\left.\left.-\left(2^{p-1}-1/2\right)\zeta(p)-\ln 2\right]\right\}^{1/p}. \quad \square$$

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