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ZEROS OF CERTAIN TRINOMIAL EQUATIONS

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Abstract. We begin by studying the zero distribution of the trinomial equation $bz^n - az^m + a - b = 0$, where $b > a > 0$ are real and $n > m > 0$ are integers. And we study the location of zeros of more general class of trinomial equations $(t-2)z^n + (t-1)z - s = 0$, where $t > 2$ and $s > 0$.

1. Introduction

Throughout this paper, n is an integer $\geqslant 2$ and we denote the unit circle by U. There have been a number of literatures about zero distributions of trinomial equations. For examples, Fell [3] studied the transition of the zeros of

$$\alpha z^{r+s} + (1-\alpha)z^r - 1 = 0 \quad (0 < \alpha < 1)$$

that is a weighted sum of the two binomial equations, $z^{r+s} - 1 = 0$ and $z^r - 1 = 0$. Cella and Lettl [1] showed how power series could be used to obtain solutions of

$$a_1 z^{n_1} + a_2 z^{n_2} + a_3 z^{n_3} = 0,$$

where $a_j, n_j \in \mathbb{C}$ and the n_j's are pairwise different. Glasser [4] expressed the zeros of trinomial equations

$$x^n - x + t = 0$$

as a finite sum of generalized hypergeometric functions. Studying this equation is essentially same as doing

$$z^n - az^{n-1} + a = 0$$

by letting $z = 1/x$, $a = 1/t$.

Many of classical inequalities of analysis have been obtained from trinomial equations. For x real and n positive even integer, an inequality,

$$x^n - nx + n - 1 \geqslant 0$$

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with equality if and only if $x = 1$, has been used in a number of cases as a starting point in the process of finding other inequalities [6, pp. 126-129]. In this direction, Dilcher, Nulton, Stolarsky [2] studied zero distribution of

$$bz^n - az^m + a - b, \tag{1.1}$$

where $a > b > 0$ are real and $n > m$ are positive integers. For $a = n$ and $b = m$ in (1.1), we have an inequality

$$mx^n - nx^m + n - m > 0$$

for $0 < x < 1$. Now we obtain, upon replacing x by $(x/y)^{1/n}$ and m/n by λ, the weighted arithmetic-geometric mean inequality

$$\lambda x + (1-\lambda)y \geqslant x^\lambda y^{1-\lambda}. \tag{1.2}$$

In Section 2, we will begin by studying the zero distribution of the polynomial (1.1) in case $b > a > 0$. In fact, we will prove in Theorem 2.2 that this polynomial has exactly $d = \gcd(m,n)$ zeros (dth roots of unity) on U, and all others strictly inside U. A specific case for this when $t > 2$, $b = 2t-3$, $a = t-1$, $m = n-1$ has a real zero 1 and all others strictly inside U. Thus the reciprocal polynomial of this multiplied by -1,

$$h_{2t-3}(z) := (t-2)z^n + (t-1)z - (2t-3) \quad (t > 2),$$

has a real zero 1 and all others strictly outside U. Our goal in this paper is to study the zero distribution of

$$h_s(z) = (t-2)z^n + (t-1)z - s \quad (s > 0)$$

which is a generalization of $h_{2t-3}(z)$. We will show in Theorem 2.4 that if $z = re^{i\theta}$ is a nonreal zero of $h_s(z)$ and $x = \cos\theta$, then, for $t \geqslant 3$ and $s > 0$, we have

$$U_{n-2}(x) = \frac{s}{(t-2)r^n}$$

and

$$\left(\frac{s}{(t-2)(n-1)}\right)^{\frac{1}{n}} \leqslant |z| \leqslant 1 + \frac{s+1}{n(t-2)-(t-1)}. \tag{1.3}$$

The regions (1.3) where the nonreal zeros of $h_s(z)$ occur are sharp. For example, for $(s,t,n) = (5,6,7)$, the lower bound in (1.3) is greater than 0.7992 while the minimum modulus is in fact 0.8126. There is a bit more to be said in general about the polynomial $h_s(z)$. Kim [5] showed that the locus of

$$|(z-2)^n| = \left|(z-1)(z-(n+1))^{n-1}\right|$$

has exactly two connected components in the complex plane; one oval and one $\infty-$component. Each component has zeros of $(z-2)^n + (z-1)(z-(n+1))^{n-1} = 0$. Kim [5] also investigated the locus of zeros of the more general polynomial

$$g(z,t) := (z-2)^n + (z-1)(z-t)^{n-1}, \quad t \geqslant 4,$$

with starting the observation: The zeros of $g(z,t)$ are $\frac{2+a_{n,t}}{1+a_{n,t}}$, where each $a_{n,t}^{-\frac{1}{n-1}}$ is a zero of the trinomial equation $(t-2)z^n+(t-1)z-1=0$. The $h_s(z)=0$ is a generalization of this trinomial equation.

Finally we will find a lower bound for the positive real zero of $h_s(z)$ for $s \leqslant (t-1)(t-2)^{-1/n}+1$ in Proposition 2.6.

2. Proofs of the results

Our first aim is to get a result on the zeros of the polynomial

$$p(z) = bz^n - az^m + a - b, \tag{2.1}$$

where $b>a>0$ are real and $n>m>0$ are integers. For this, we use the same method as in [2] where the zero distribution of $bz^n-az^m+a-b=0$ in case $a>b>0$ was examined. We first need following lemma (see [2] for the proof).

LEMMA 2.1. *If $b>a>0$ and p, q are all real, then*

$$bz^q - az^p = b-a, \quad z=e^{i\theta},$$

implies $p\theta \equiv q\theta \equiv 0$ (mod 2π).

By using above lemma, we have

THEOREM 2.2. *Let $b>a>0$ be real and $n>m>0$ be integers, and $d=$* gcd(m,n). *Then the polynomial*

$$p(z) = bz^n - az^m + a - b \tag{2.2}$$

has exactly d zeros (dth roots of unity) on U, and $n-d$ zeros strictly inside U.

Proof. For small $\epsilon>0$, consider

$$p_\epsilon(z) = bz^n - az^m + a - b + \epsilon.$$

By Rouché's theorem, $p_\epsilon(z)$ has exactly n zeros, say $\alpha_1,\cdots,\alpha_n$ inside U. As $\epsilon\to 0$, some of these may tend to U. If

$$\alpha_j \to e^{i\theta_j} = e^{i\theta}, \quad \theta\in\mathbb{R}.$$

By above lemma, $n\theta\equiv m\theta\equiv 0 \pmod{2\pi}$. Hence $e^{i\theta}$ is a dth root of unity, where $d=\gcd(m,n)$. Conversely, if $w^d=1$, then $p(w)=0$. Now suppose that $\alpha_j\to w$, where $w^d=1$. Since $p'(z)=nbz^{n-1}-amz^{m-1}$ and $nb\neq am$, the root w is simple. Now set

$$\beta = rw, \quad 0<r<\infty,$$

and

$$u(r) := p_\epsilon(\beta) = br^n - ar^m + a - b + \epsilon.$$

It is easy to see that, for sufficiently small $\epsilon>0$, both $u(r)$ and $u'(r)$ have only one positive zero, say r_0 and r', between 0 and 1, respectively. For $0<r<r_0$ the

function $u(r)$ is negative and $0 < r' < r_0$, where of course r' is independent of the value of ϵ. Thus r_0 is an inside zero that goes to U as $\epsilon \to 0$. This completes the proof. □

REMARK 2.3. For $a = m$ and $b = n$ in (2.2), we have an inequality

$$nx^n - mx^m + m - n < 0$$

for $0 < x < 1$. Using same method to get the weighted arithmetic-geometric mean inequality (1.2), we get the lower bound for $x^\lambda y^{1-\lambda}$ as following:

$$x^\lambda y^{1-\lambda} > \frac{1}{\lambda}x - (1-\lambda)y,$$

where $0 < x < y$ and $0 < \lambda < 1$. If y is replaced by 1 in above inequality, we obtain an inequality

$$\lambda(1-\lambda) - x + \lambda x^\lambda > 0,$$

where $0 < x < 1$ and $0 < \lambda < 1$.

The polynomial (2.2) of the case $b = 2t-3$, $a = t-1$, $m = n-1$ is

$$h^*_{2t-3}(z) = (2t-3)z^n - (t-1)z^{n-1} - (t-2).$$

By Theorem 2.2, the reciprocal polynomial of this multiplied by -1, i.e.,

$$h_{2t-3}(z) = (t-2)z^n + (t-1)z - (2t-3)$$

has a real zero 1 and $n-1$ zeros strictly outside U. We generalize this as

$$h_s(z) = (t-2)z^n + (t-1)z - s \quad (s > 0) \tag{2.3}$$

and want to study locations of its zeros. First, we note that $h_s(z)$ has a multiple zero only when

$$s = \frac{(n-1)(t-1)}{n}\left(\frac{1-t}{n(t-2)}\right)^{\frac{1}{n-1}}$$

by the calculation of the discriminant of $h_s(z)$, and hence all zeros of $h_s(z)$ are simple. For the formula for the discriminant of trinomials, see p. 184 of [9]. Now we establish our main result of this paper.

THEOREM 2.4. *If $z = re^{i\theta}$ is a nonreal zero of $h_s(z)$ and $x = \cos\theta$, then, for $t \geqslant 3$ and $s > 0$,*

$$U_{n-2}(x) = \frac{s}{(t-2)r^n},$$

and

$$\left(\frac{s}{(t-2)(n-1)}\right)^{\frac{1}{n}} \leqslant |z| \leqslant 1 + \frac{s+1}{n(t-2)-(t-1)}. \tag{2.4}$$

Proof. For the right inequality of (2.4), we use Rouché's theorem. Let $z = (1+\delta)e^{it}$ with $\delta > 0$ and $t \in \mathbb{R}$. Then since

$$(t-2)(1+\delta)^n > (t-2)(n\delta+1)$$

and

$$(t-1)(1+\delta)+s > |(t-1)(1+\delta)e^{it}-s|,$$

we have

$$|(t-2)z^n| > |(t-1)z-s| \tag{2.5}$$

when

$$(t-2)(n\delta+1) \geqslant (t-1)(1+\delta)+s$$

or equivalently

$$\delta \geqslant \delta_0 := \frac{s+1}{n(t-2)-(t-1)}.$$

Hence by Rouché with (2.5), $h_s(z)$ has the same number of zeros inside the circle $|z| = 1+\delta_0$ as $(t-2)z^n$, namely all n zeros. This proves the right inequality of (2.4).

For the left inequality of (2.4), we let, for $t \geqslant 3$ and $s > 0$,

$$h_s^*(z) = -z^n h_s\left(\frac{1}{z}\right) = sz^n - (t-1)z^{n-1} - (t-2).$$

Let $0 < b = (t-2)^{-1/n} \leqslant 1$ and $z = (t-2)^{1/n}y$. Then $h_s^*(z) = 0$ becomes

$$s(t-2)y^n - (t-1)(t-2)^{\frac{n-1}{n}}y^{n-1} - (t-2) = 0,$$

and

$$sy^n - (t-1)by^{n-1} - 1 = 0,$$

so

$$y^{n-1}(sy-(t-1)b) = 1. \tag{2.6}$$

Let $y = r(\cos\theta + i\sin\theta)$ be a nonreal zero of (2.6). Then simple calculations yield that

$$r^{n-1}(rs\cos n\theta - b(t-1)\cos(n-1)\theta) = 1$$

and

$$r^{n-1}(rs\sin n\theta - b(t-1)\sin(n-1)\theta) = 0.$$

Hence, with $x = \cos\theta$,

$$r^{n-1}(rsT_n(x) - b(t-1)T_{n-1}(x)) = 1 \tag{2.7}$$

and

$$rsU_{n-1}(x) - b(t-1)U_{n-2}(x) = 0, \tag{2.8}$$

where T_n and U_n are the Chebyshev polynomials of the first kind of degree n and of the second kind of degree n, respectively. If $U_{n-2}(x) = 0$, then it follows from (2.8) that $U_{n-1}(x) = 0$ and so $T_n(x) = 1$, i.e., $n = 0$ which is a contradiction. From (2.8), for x with $U_{n-2}(x) \neq 0$, we have

$$b(t-1) = \frac{rsU_{n-1}(x)}{U_{n-2}(x)}$$

and so, from (2.7),

$$r^{n-1}\left(rsT_n(x) - \frac{rsU_{n-1}(x)}{U_{n-2}(x)}T_{n-1}(x)\right) = 1. \tag{2.9}$$

Multiplying $U_{n-2}(x)$ of each side in (2.9) gives

$$T_n(x)U_{n-2}(x) - U_{n-1}(x)T_{n-1}(x) = \frac{1}{r^n s}U_{n-2}(x).$$

But, by using well known equalities on Chebyshev polynomials (see p. 71 of [7] and p. 9 of [8]), we have

$$\begin{aligned} &T_n(x)U_{n-2}(x) - U_{n-1}(x)T_{n-1}(x) \\ =&(xT_{n-1}(x) - (1-x^2)U_{n-2}(x))U_{n-2}(x) - (xU_{n-2}(x) + T_{n-1}(x))T_{n-1}(x) \\ =&-(1-x^2)U_{n-2}^2(x) - T_{n-1}^2(x) \\ =&-1. \end{aligned}$$

Hence, for x with $U_{n-2}(x) \neq 0$, we get

$$U_{n-2}(x) = -r^n s = -|y|^n s = -(t-2)|z|^n s,$$

where z is a nonreal zero of $h_s^*(z)$. It follows from the fact $|U_{n-2}(x)| \leqslant n-1$ that

$$|z|^n \leqslant \frac{(t-2)(n-1)}{s}$$

and

$$|z| \leqslant \left(\frac{(t-2)(n-1)}{s}\right)^{\frac{1}{n}}. \tag{2.10}$$

Since the zeros of $h_s(z)$ are the inverse of the zeros of $h_s^*(z)$, it follows from (2.10) that all nonreal zeros of $h_s(z)$ lies in

$$|z| \geqslant \left(\frac{s}{(t-2)(n-1)}\right)^{\frac{1}{n}}.$$

This completes the proof of the left inequality of (2.4). □

From above theorem, the following is immediately obtained.

COROLLARY 2.5. *If* $s > (t-2)(n-1)$, *then all nonreal zeros of* $h_s(z)$ *lie outside* U.

Finally, we investigate the location of the positive real zero of $h_s(z)$.

PROPOSITION 2.6. *For* $t \geqslant 3$, *if* $0 < s \leqslant (t-1)(t-2)^{-1/n} + 1$, *then the positive real zero of* $h_s(z)$ *is at least*

$$\frac{1}{(t-2)^{1/n}(d_1+1)},$$

where $c = (t-2)^{-1/n} - 1$ *and*

$$d_1 = \frac{-ns+(c+1)(n-1)(t-1)+\sqrt{(ns-(c+1)(n-1)(t-1))^2-4(n-1)s(c+s-(c+1)t)}}{2(n-1)s}.$$

Proof. For $t \geqslant 3$ and $s > 0$, let

$$h_s^*(z) = -z^n h_s\left(\frac{1}{z}\right) = sz^n - (t-1)z^{n-1} - (t-2),$$
$$0 < b = (t-2)^{-1/n} \leqslant 1, \qquad z = (t-2)^{1/n}y.$$

Now $h_s^*(z) = 0$ becomes

$$s(t-2)y^n - (t-1)(t-2)^{\frac{n-1}{n}}y^{n-1} - (t-2) = 0,$$

and

$$sy^n - (t-1)by^{n-1} - 1 = 0. \tag{2.11}$$

Lets assume that z is real and positive, $b = c+1$ where $-1 < c \leqslant 0$, and $y = x+1$. Then $x > -1$ and (2.11) becomes

$$s(x+1)^n - (t-1)(c+1)(x+1)^{n-1} - 1 = 0$$

and

$$(x+1)^{n-1}(sx+s-(t-1)(c+1)) = 1.$$

Suppose that $s \leqslant (t-1)(c+1)+1$ and

$$(x+1)^{n-1}(sx+s-(t-1)(c+1)) > 1. \tag{2.12}$$

Then $x > 0$. On the other hand, for $-1 < x \leqslant 0$

$$sx+s-(t-1)(c+1) > 1,$$

which leads to a contradiction. The inequality (2.12) is fulfilled when

$$((n-1)x+1)(sx+s-(t-1)(c+1)) > 1,$$

and it holds when

$$x > d_1 \geqslant 0,$$

where

$$d_1 = \frac{-ns+(c+1)(n-1)(t-1)+\sqrt{(ns-(c+1)(n-1)(t-1))^2-4(n-1)s(c+s-(c+1)t)}}{2(n-1)s}.$$

Here the fact that d_1 is nonnegative follows from $s \leqslant (t-1)(c+1)+1$ and

$$(ns-(c+1)(n-1)(t-1))^2-4(n-1)s(c+s-(c+1)t)-(-ns+(c+1)(n-1)(t-1))^2$$
$$= -4(n-1)s(c+s-t(c+1)) \geqslant 0$$

Thus the positive zero z of $h_s(z)$ (the reciprocal of $h_s^*(z)$) satisfies

$$z \geqslant \frac{1}{(t-2)^{1/n}(d_1+1)}. \quad \square$$

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