Mathematical
Inequalities
& Applications
Volume 9, Number 2 (2006), 233–246

ONE INEQUALITY FOR CONFORMAL MAPPINGS OF SPHERICAL DOMAINS

LUDMILA BOURCHTEIN AND ANDREI BOURCHTEIN

(*communicated by S. Owa*)

Abstract. We provide an evaluation of variations of the mapping factor for conic mappings from a sphere to a plane. The proved inequality allows to compare the variation coefficients of conic, cylindrical and stereographic projections. Obtained inequality chain for variation coefficients can be used to generate more computationally efficient numerical grids.

1. Introduction

The problem of the uniform grid generation for spherical regions is one of the oldest problems of atmospheric modeling and meteorological analysis [3,5,7]. Different approaches used to solve this problem can be classified according to the type of the transformation of a sphere in the following way: using the original spherical longitude-latitude grids (polar and rotated), conformal mappings from a sphere onto a plane, non-conformal mappings onto a plane (such as gnomonic, icosahedral and geodesic grids) and conformal mappings from a sphere onto a sphere [2,3,7].

Conformal mapping onto a plane is the most widespread approach because it allows to keep a simple form of the primitive equations and guarantees locally isotropic treatment of derivatives and smoothness of physical meshsize variation [6,7]. Commonly used conformal projections are stereographic, conic and cylindrical, which can be considered as specific cases of the conformal separable mappings [1]. The problem of optimization of the numerical grid generation can be formulated as a minimization of the variation coefficient, where the last is defined as the maximum variation of the mapping factor (scale function) over considered domain [1]. Solution of this problem separately for stereographic, conic and cylindrical conformal mappings considered over circular spherical domain Ω with the centerpoint $\overline{P} = \left(\overline{\theta}, \overline{\lambda}\right)$ and radius γ (that is, the domain whose boundary is obtained by the intersection of a plane with a sphere) was done in [1]. In particular, it was shown that the "best" stereographic and cylindrical projections are the rotated ones, which are tangent to the sphere at the centerpoint $\overline{P}$, and the "best" conic mapping is tangent to the sphere at the points $P_0 = (\theta_0, \lambda)$, where

Mathematics subject classification (2000): 26D07, 30C20, 65M50.
Key words and phrases: function inequalities, conformal mappings, grid generation.

Paper MIA-09-24

$\theta_0 \in (\overline{\theta}-\gamma, \overline{\theta})$. The respective minimum values of the variation coefficients were found in the form

$$\alpha_{str} = \frac{2}{1+\cos\gamma}, \quad \alpha_{cyl} = \frac{1}{\cos\gamma}$$

and

$$\alpha_{con} = \frac{\sin\theta_0}{\sin(\overline{\theta}-\gamma)} \left(\frac{\tan((\overline{\theta}-\gamma)/2)}{\tan(\theta_0/2)} \right)^{\cos\theta_0} = \frac{\sin\theta_0}{\sin(\overline{\theta}+\gamma)} \left(\frac{\tan((\overline{\theta}+\gamma)/2)}{\tan(\theta_0/2)} \right)^{\cos\theta_0}$$

for stereographic, cylindrical and conic mappings, respectively [1]. In the last formula parameter θ_0 is defined as

$$\cos\theta_0 = \frac{\ln\sin(\overline{\theta}+\gamma) - \ln\sin(\overline{\theta}-\gamma)}{\ln\tan((\overline{\theta}+\gamma)/2) - \ln\tan((\overline{\theta}-\gamma)/2)}.$$

To choose the best conformal separable mapping from a sphere to a plane we have to compare these results. First, it is evident that

$$\alpha_{str} = \frac{2}{1+\cos\gamma} < \frac{1}{\cos\gamma} = \alpha_{cyl}$$

for any $\gamma \in (0, \pi/2)$. We are also able to prove the second evaluation:

$$\alpha_{cyl} = \frac{1}{\cos\gamma} < \frac{\sin\theta_0}{\sin(\overline{\theta}+\gamma)} \left(\frac{\tan((\overline{\theta}+\gamma)/2)}{\tan(\theta_0/2)} \right)^{\cos\theta_0} = \alpha_{con}$$

for any $\overline{\theta}$ and γ such that $0 < \gamma < \overline{\theta} < \pi/2$. Therefore the following inequality chain is true:

$$\alpha_{str} < \alpha_{cyl} < \alpha_{con}.$$

Hence, for any fixed spherical circular domain Ω, the minimum variation coefficient is reached by choosing the stereographic oblique mapping with the tangent point located at the centercolatitude $\overline{\theta}$ of the considered domain.

This result allows to generate more computationally efficient grids for hydrodynamics limited area models over spherical domains. In particular, it permits to construct more efficient regional and mesoscale numerical weather prediction models. The principal point of this result is the penultimate inequality involving comparison between α_{cyl} and α_{con}, whose demonstration we provide in this paper.

2. Principal inequality

THEOREM 1. *The function*

$$f(x,a) = \frac{\sin y}{\sin(x+a)} \left(\frac{\tan((x+a)/2)}{\tan(y/2)} \right)^{\cos y} = \frac{\sin y}{\sin(x-a)} \left(\frac{\tan((x-a)/2)}{\tan(y/2)} \right)^{\cos y}, \tag{1}$$

$$\cos y = \frac{\ln[\sin(x+a)/\sin(x-a)]}{\ln[\tan((x+a)/2)/\tan((x-a)/2)]}, \tag{2}$$

$$0 < a < x < \pi/2, \;\; 0 < y < \pi \tag{3}$$

satisfies the inequalities

$$\frac{1}{\cos a} < f\,(x,a) < \frac{1}{\cos^2 a} \tag{4}$$

for any x, a *from* (3).

Proof. We divide the demonstration in three steps.

2a. *Preliminary calculations.*

1) Preliminary calculations for evaluation of the function y in (2).

Let us introduce two auxiliary frequently used functions

$$ls\,(x,a) = \ln \frac{\sin\,(x+a)}{\sin\,(x-a)}, \;\; lt\,(x,a) = \ln \frac{\tan\frac{x+a}{2}}{\tan\frac{x-a}{2}} \tag{5}$$

and calculate their partial derivatives

$$ls_x = \frac{-\sin 2a}{\sin\,(x+a)\sin\,(x-a)}, \tag{6}$$

$$ls_a = \frac{\sin 2x}{\sin\,(x+a)\sin\,(x-a)}, \tag{7}$$

$$lt_x = \frac{-2\sin a\cos x}{\sin\,(x+a)\sin\,(x-a)}, \tag{8}$$

$$lt_a = \frac{2\sin x\cos a}{\sin\,(x+a)\sin\,(x-a)}. \tag{9}$$

Now we calculate the limit values (one-sided limits) of the function $\cos y$. The first limit

$$L_1 = \lim_{a\to 0_+} \cos y = \lim_{a\to 0_+} \frac{ls}{lt}$$

is an indeterminate form of type $0/0$, which can be calculated by applying l'Hospital's rule

$$L_1 = \lim_{a\to 0_+} \frac{ls_a}{lt_a} = \lim_{a\to 0_+} \frac{\sin 2x}{2\sin x\cos a} = \cos x. \tag{10}$$

The second limit

$$L_2 = \lim_{a\to x_-} \cos y = \lim_{a\to x_-} \frac{ls}{lt}$$

is an indeterminate form of type ∞/∞ and l'Hospital's rule can be applied to its calculation

$$L_2 = \lim_{a\to x_-} \frac{ls_a}{lt_a} = \lim_{a\to x_-} \frac{\sin 2x}{2\sin x\cos a} = 1. \tag{11}$$

The third limit

$$L_3 = \lim_{x\to a_+} \cos y = \lim_{x\to a_+} \frac{ls}{lt}$$

is an indeterminate form of type ∞/∞ and l'Hospital's rule can be applied once more

$$L_3 = \lim_{x \to a_+} \frac{ls_x}{lt_x} = \lim_{x \to a_+} \frac{\sin 2a}{2 \sin a \cos x} = 1. \tag{12}$$

Finally, the forth limit

$$L_4 = \lim_{x \to \pi/2_-} \cos y = \lim_{x \to \pi/2_-} \frac{ls}{lt}$$

does not represent any kind of indetermination and can be calculated directly

$$L_4 = \frac{\ln \frac{\sin(\pi/2+a)}{\sin(\pi/2-a)}}{\ln \frac{\tan(\pi/4+a/2)}{\tan(\pi/4-a/2)}} = \frac{\ln \frac{\cos a}{\cos a}}{2 \ln \tan(\pi/4 + a/2)} = 0. \tag{13}$$

Summarizing, we have the following results:

$$\lim_{a \to 0_+} y = x, \quad \lim_{a \to x_-} y = 0, \quad \lim_{x \to a_+} y = 0, \quad \lim_{x \to \pi/2_-} y = \pi/2. \tag{14}$$

Let us calculate partial derivatives of $\cos y$. Using results (6)-(9) and simplifying, we obtain

$$(\cos y)_x = \left(\frac{ls}{lt}\right)_x = \frac{1}{lt^2}(ls_x lt - lt_x ls) = \ln^{-1} \frac{\tan \frac{x+a}{2}}{\tan \frac{x-a}{2}} \cdot \frac{2 \sin a (\cos x \cos y - \cos a)}{\sin(x+a) \sin(x-a)}, \tag{15}$$

$$(\cos y)_a = \left(\frac{ls}{lt}\right)_a = \frac{1}{lt^2}(ls_a lt - lt_a ls) = \ln^{-1} \frac{\tan \frac{x+a}{2}}{\tan \frac{x-a}{2}} \cdot \frac{2 \sin x (\cos x - \cos a \cos y)}{\sin(x+a) \sin(x-a)}. \tag{16}$$

Or, isolating y, we rewrite these results in the form

$$y_x = -\ln^{-1} \frac{\tan((x+a)/2)}{\tan((x-a)/2)} \cdot \frac{2 \sin a (\cos x \cos y - \cos a)}{\sin y \sin(x+a) \sin(x-a)}, \tag{17}$$

$$y_a = -\ln^{-1} \frac{\tan((x+a)/2)}{\tan((x-a)/2)} \cdot \frac{2 \sin x (\cos x - \cos a \cos y)}{\sin y \sin(x+a) \sin(x-a)}. \tag{18}$$

2) Preliminary calculations for evaluation of the function $f(x,a)$ in (1).

We consider the following two functions (in three variables):

$$F(x,a,y) = \frac{\sin y}{\sin(x+a)} \left(\frac{\tan((x+a)/2)}{\tan(y/2)}\right)^{\cos y}, \tag{19}$$

$$G(x,a,y) = \frac{\sin y}{\sin(x-a)} \left(\frac{\tan((x-a)/2)}{\tan(y/2)}\right)^{\cos y}, \tag{20}$$

which coincide with $f(x,a)$ for the values of y from (2). After some algebra, partial derivatives of these functions can be represented as follows

$$F_y = -F \sin y \ln \frac{\tan((x+a)/2)}{\tan(y/2)}, \tag{21}$$

$$F_x = F \frac{\cos y - \cos(x+a)}{\sin(x+a)}, \tag{22}$$

$$G_y = G \sin y \ln \frac{\tan(y/2)}{\tan((x-a)/2)}, \tag{23}$$

$$G_x = G\frac{\cos y - \cos(x-a)}{\sin(x-a)}. \tag{24}$$

Using (17), (21) and (22) for y from (2), we can find the partial derivatives of f in the form

$$f_x = F_x + F_y y_x = f\Phi, \tag{25}$$

where

$$\Phi = \frac{\cos y - \cos(x+a)}{\sin(x+a)} - \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \frac{2\sin a\,(\cos a - \cos x \cos y)}{\sin(x+a)\sin(x-a)}. \tag{26}$$

Similarly, applying (23), (24) and (17), we get

$$f_x = G_x + G_y y_x = f\Phi, \tag{27}$$

where

$$\Phi = \frac{\cos y - \cos(x-a)}{\sin(x-a)} - \frac{\ln\frac{\tan((x-a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \frac{2\sin a\,(\cos a - \cos x \cos y)}{\sin(x+a)\sin(x-a)}. \tag{28}$$

Of course, expressions (26) and (28) coincide for y from (2). Calculation of the half-sum of (25) and (27) yields the third representation for the same partial derivative

$$f_x = f\Phi, \tag{29}$$

where

$$\Phi = -\frac{\sin x\,(\cos x - \cos a \cos y)}{\sin(x+a)\sin(x-a)} - \frac{\ln\frac{\tan((x+a)/2)\tan((x-a)/2)}{\tan^2(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \frac{\sin a\,(\cos a - \cos x \cos y)}{\sin(x+a)\sin(x-a)} \tag{30}$$

is the third representation for the same function Φ.

Using direct substitution of the limit value and the last limit in (14), we can calculate the following limit

$$\lim_{x\to\pi/2_-} \Phi = \frac{\ln 1}{2\ln\tan(\pi/4 + a/2)}\frac{\sin a \cos a}{\cos^2 a} - \frac{1\cdot 0}{\cos^2 a} = 0. \tag{31}$$

Using direct substitution and the last limit in (14), we can find

$$\lim_{x\to\pi/2_-} y_x = \frac{1}{\ln\tan(\pi/4 + a/2)}\frac{\sin a}{\cos a}. \tag{32}$$

The following limit

$$L_5 = \lim_{x\to\pi/2_-} \frac{\ln\frac{\tan((x+a)/2)\tan((x-a)/2)}{\tan^2(y/2)}}{\cos x - \cos a \cos y}$$

represents an indeterminate form of type $0/0$ and can be transformed using l'Hospital's rule to

$$L_5 = \lim_{x\to\pi/2_-} \frac{\frac{1}{\sin(x+a)} + \frac{1}{\sin(x-a)} - \frac{2}{\sin y}y_x}{-\sin x + \cos a \sin y \cdot y_x}.$$

The above limit is calculated by direct substitution, using (32) and the last limit in (14):

$$L_5 = \frac{\frac{2}{\cos a} - 2\frac{\sin a}{\cos a}\frac{1}{\ln\tan(\pi/4+a/2)}}{-1+\cos a\frac{\sin a}{\cos a}\frac{1}{\ln\tan(\pi/4+a/2)}} = -\frac{2}{\cos a}. \tag{33}$$

The following limit

$$L_6 = \lim_{x\to a_+}\frac{\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} = \lim_{x\to a_+}\frac{\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}{lt}$$

represents an indeterminate form of type ∞/∞ and can be calculated by l'Hospital's rule:

$$L_6 = \lim_{x\to a_+}\frac{-\frac{\sin a}{2\sin((x+a)/2)\sin((x-a)/2)}}{-\frac{2\sin a\cos x}{\sin(x+a)\sin(x-a)}} = 1. \tag{34}$$

The limit

$$L_7 = \lim_{x\to a_+}\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} = \lim_{x\to a_+}\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{lt}$$

represents an indeterminate form of type ∞/∞ and by applying l'Hospital's rule it can be transformed to

$$L_7 = \lim_{x\to a_+}\frac{\frac{1}{\sin(x+a)} - \frac{1}{\sin y}y_x}{-\frac{2\sin a\cos x}{\sin(x+a)\sin(x-a)}}.$$

Substitution of expression (17) for y_x and simplification yields

$$L_7 = \lim_{x\to a_+}\left[-\frac{\sin(x-a)}{2\sin a\cos x} + \frac{1}{\cos x}\frac{\cos a - \cos x\cos y}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}\cdot\sin^2 y}\right].$$

Since the limit of the first summand equals 0, the considered limit can be rewritten in the form

$$L_7 = \frac{1}{\cos a}\lim_{x\to a_+}\frac{\cos a - \cos x\cos y}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}\cdot\sin^2 y}.$$

Substituting expressions for $\cos y$ and $\sin^2 y$ and applying some algebra, we obtain

$$L_7 = \frac{1}{\cos a}\lim_{x\to a_+}\left[\frac{\cos a - \cos x}{4\ln\frac{\cos((x-a)/2)}{\cos((x+a)/2)}} + \frac{\cos a + \cos x}{4\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}\right] = 0. \tag{35}$$

The following limit is the simple consequence of (34), (35):

$$\lim_{x\to a_+}\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}\cdot\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}$$

$$= \lim_{x\to a_+}\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\cdot\lim_{x\to a_+}\frac{\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} = 0. \tag{36}$$

Using the well known result

$$\lim_{t \to 0_+} t^k \ln^m t = 0\,, \quad \forall k, m > 0, \tag{37}$$

and making the following substitution

$$t = \frac{\tan\left((x-a)/2\right)}{\tan\left((x+a)/2\right)}, \quad \lim_{x \to a_+} t = 0_+,$$

we can calculate one more limit

$$\lim_{x \to a_+} \frac{\ln \frac{\tan((x+a)/2)}{\tan(y/2)}}{\tan \frac{x-a}{2} \ln^2 \frac{\tan((x+a)/2)}{\tan((x-a)/2)}} = \frac{\lim_{x \to a_+} \ln \frac{\tan((x+a)/2)}{\tan(y/2)}}{\lim_{x \to a_+} \tan \frac{x+a}{2} \cdot \lim_{t \to 0_+} t \ln^2 t} = +\infty. \tag{38}$$

2*b*. *Auxiliary inequality chain*

Now we prove the following inequality chain

$$\cos x < \cos y < \frac{\cos x}{\cos a} < \cos\left(x-a\right) \tag{39}$$

for any a and x from (3).

Let us fix any x in the interval $(0, \pi/2)$ and consider the function $g(a) = \cos y$ with respect to one variable a. According to (10),(11) and (16) this function has one-sided limits

$$\lim_{a \to 0_+} g(a) = \cos x, \quad \lim_{a \to x_-} g(a) = 1 \tag{40}$$

and its derivative has the following expression

$$g'(a) = \ln^{-1} \frac{\tan\left((x+a)/2\right)}{\tan\left((x-a)/2\right)} \cdot \frac{2 \sin x}{\sin\left(x+a\right) \sin\left(x-a\right)} \left[\cos x - \cos a \cos y\right]. \tag{41}$$

Since the first two quotients in (41) are positive, the sign of the first derivative depends on the expression in the square brackets. Let us suppose that at least one critical point $a_0 \in (0, x)$ exists, that is, the equation

$$\cos x - \cos a \cdot g(a) = 0 \tag{42}$$

has at least one solution in the indicated interval. In this case, calculating the second derivative, simplifying obtained expression and considering it at any critical point a_0, we have

$$g''(a_0) = \ln^{-1} \frac{\tan\left((x+a)/2\right)}{\tan\left((x-a)/2\right)} \cdot \frac{2 \sin x}{\sin\left(x+a_0\right) \sin\left(x-a_0\right)} \cdot \sin a_0 \cdot g(a_0) > 0. \tag{43}$$

Therefore a_0 is the local minimum point of the function $g(a)$. Because $g(a)$ and its first derivative $g'(a)$ are continuos functions on interval $(0, x)$ and inequality (43) is true for any critical point a_0, then it can exist at most one critical point and this point is the local minimum point. According to (43) the first derivative changes sign passing through the point a_0 and therefore the function $g(a)$ is decreasing to the left of the point a_0 and increasing to the right of this point. This guarantees that a_0 is the absolute minimum point on the interval $(0,x)$. Hence, $g(a)$ attains an absolute minimum value at this point:

$$g(a_0) = \frac{\cos x}{\cos a_0}$$

However, this value is bigger than the values of the function $g(a)$ in some right-hand neighborhood of point 0, because (40) holds. Therefore we obtain a contradiction and this means that our supposition about existence of critical points is false.

Hence, no critical point exists and $g'(a)$ keeps the same sign for all values of $a \in (0,x)$. According to the limit values (40) the function $g(a)$ is strictly increasing, that is, $g'(a)$ is positive on $(0,x)$. Therefore, expression in the square brackets in (41) is positive and

$$g(a) < \frac{\cos x}{\cos a}$$

for any $a \in (0,x)$ and any fixed $x < \pi/2$. Consequently,

$$\cos y < \frac{\cos x}{\cos a} \tag{44}$$

for any a and x from (3).

From the limit values (40) and positiveness of the derivative it follows also that $\cos x < \cos y$ for the same values of a and x.

Finally, using the inequality

$$\cos(x-a) > \frac{\cos x}{\cos a},$$

which comes from simple evaluation

$$\cos(x-a)\cos a > \cos(x-a)\cos a - \sin(x-a)\sin a = \cos x,$$

we can conclude that the inequality chain (39) is true.

$2c$. *Study of the function* f_x

Let us recall that by (25)

$$f_x = f\Phi,$$

where function Φ has different representations (26), (28) or (30). Therefore the properties of the derivative f_x depend on the properties of the function Φ, which we will study first.

1) Details of the left-sided limit of the function Φ at $x = \pi/2$ for any fixed a from (3).

First, we study the behavior of the function Φ in the left-sided neighborhood of the point $x = \pi/2$ for any fixed a from (3). From (31) we know that function Φ approaches 0 as x approaches $\pi/2$ from the left. Let us investigate whether function Φ keeps determinate sign under this approaching. For that, we use representation (30) of the Φ, which can be written in the form

$$\Phi = \frac{\sin x(\cos x - \cos a\cos y)}{\sin(x+a)\sin(x-a)}\Psi \tag{45}$$

where

$$\Psi = -1 - \frac{\ln \frac{\tan((x+a)/2)\tan((x-a)/2)}{\tan^2(y/2)}}{\ln \frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \frac{\sin a\,(\cos a - \cos x \cos y)}{\sin x\,(\cos x - \cos a \cos y)}. \tag{46}$$

By (39),

$$\frac{\sin x\,(\cos x - \cos a \cos y)}{\sin (x+a)\sin (x-a)} > 0, \tag{47}$$

for any x, a such that $0 < a < x < \pi/2$ and

$$\lim_{x\to\pi/2_-} \frac{\sin x\,(\cos x - \cos a \cos y)}{\sin (x+a)\sin (x-a)} = 0 \tag{48}$$

according to (14).

The limit of Ψ can be calculated by using (33)

$$\lim_{x\to\pi/2_-} \Psi = \lim_{x\to\pi/2_-}\left[-1 - \frac{\ln \frac{\tan((x+a)/2)\tan((x-a)/2)}{\tan^2(y/2)}}{\cos x - \cos a \cos y} \cdot \frac{\sin a\,(\cos a - \cos x \cos y)}{\sin x \ln \frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\right]$$

$$= -1 + \frac{2}{\cos a}\cdot\frac{\sin a \cos a}{2\ln\tan(\pi/4 + a/2)} = \frac{1}{\ln\tan(\pi/4+a/2)}\left(\sin a - \ln\tan(\pi/4 + a/2)\right).$$

Using Taylor series expansion [4]

$$\ln t = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{t-1}{t+1}\right)^{2k-1}, \; t > 0$$

for $t = \tan(\pi/4 + a/2)$, we can evaluate this limit in the form

$$\lim_{x\to\pi/2_-} \Psi = -\frac{1}{\ln\tan(\pi/4+a/2)} \sum_{k=2}^{\infty} \frac{\sin^{2k-1} a}{2k-1} < 0. \tag{49}$$

Therefore, function Ψ is negative in some left-sided neighborhood of the point $\pi/2$ for any fixed a. Based on (45), (47)-(49) we can conclude that function Φ approaches 0 (as x approaches $\pi/2$ from the left) keeping negative values in the left-sided neighborhood of the point $\pi/2$.

2) Details of the right-sided limit of the function Φ at $x = a$ for any fixed a from (3).

Now we study the behavior of the function Φ in right-sided neighborhood of the point $x = a$ for any fixed a from (3). First, let us calculate the limit

$$\lim_{x\to a_+} X(x,a),$$

where

$$X(x,a) = \frac{\ln \frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln \frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \frac{\cos a - \cos x \cos y}{\sin (x-a)}.$$

Substituting the expression for $\cos y$ and rearranging the terms yields

$$\lim_{x\to a_+} X(x,a)$$
$$= \lim_{x\to a_+} \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \frac{(\cos a - \cos x)\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)} + (\cos a + \cos x)\ln\frac{\cos((x-a)/2)}{\cos((x+a)/2)}}{2\sin\frac{x-a}{2}\cos\frac{x-a}{2}}$$
$$= \lim_{x\to a_+} \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \cdot \left[\frac{\sin\frac{x+a}{2}}{\cos\frac{x-a}{2}}\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)} + \frac{\cos\frac{x+a}{2}}{\sin\frac{x-a}{2}}\ln\frac{\cos((x-a)/2)}{\cos((x+a)/2)}\right]$$
$$= \sin a \lim_{x\to a_+} \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}\ln\frac{\sin((x+a)/2)}{\sin((x-a)/2)}}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}$$
$$+ \cos a \ln\frac{1}{\cos a} \lim_{x\to a_+} \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\tan\frac{x-a}{2}\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}.$$

The first limit is (36) and it equals 0, while the second limit is (38) and it equals $+\infty$. Therefore,

$$\lim_{x\to a_+} X(x,a) = +\infty.$$

Another limit we need is quite simple

$$\lim_{x\to a_+} \frac{\cos y - \cos(x+a)}{\sin(x+a)} = \frac{1-\cos 2a}{\sin 2a} = \frac{\sin a}{\cos a}$$

Now, using the representation (26) for function Φ, we find

$$\lim_{x\to a_+} \Phi = \lim_{x\to a_+} \frac{\cos y - \cos(x+a)}{\sin(x+a)} - \lim_{x\to a_+} \frac{2\sin a}{\sin(x+a)} X(x,a) = -\infty.$$

Thus, the values of the function Φ are negative in both left-sided neighborhood $\delta_{\pi/2_-}$ of the point $x=\pi/2$ and right-sided neighborhood δ_{a_+} of the point $x=a$.

3) The sign of the function Φ on the entire interval $x\in(a,\pi/2)$ for any fixed a. Now we show that function Φ is negative on the entire interval $x\in(a,\pi/2)$, that is, $\Phi(x,a)<0$ for any x,a from (3).

Let us suppose that the function Φ assumes positive values at some points. First, we make the following observation: since the function Φ is continuous and negative in neighborhoods δ_{a_+} and $\delta_{\pi/2_-}$, the assumption implies that there exist at least two zero points of function Φ on the interval $x\in(a,\pi/2)$.

Then we investigate some properties of the function at these zero points. Rewriting (26) in the form

$$\Phi(x,a) = \frac{2\sin a(\cos a - \cos x\cos y)}{\sin(x+a)\sin(x-a)}\Omega(x,a), \tag{50}$$

$$\Omega(x,a) = \frac{\sin(x-a)(\cos y - \cos(x+a))}{2\sin a(\cos a - \cos x\cos y)} - \frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}} \tag{51}$$

and observing that

$$\frac{2\sin a\,(\cos a-\cos x\cos y)}{\sin(x+a)\sin(x-a)}>0$$

for any x, a from (3) because

$$\cos a>\cos x>\cos x\cos y, \tag{52}$$

we conclude that the sign of Ω coincides with the sign of Φ and $\Omega(x,a)=0$ if, and only if, $\Phi(x,a)=0$. Therefore, according to our assumption, there are at least two points in which $\Omega(x,a)=0$, that is, at these points the following equality holds

$$\frac{\sin(x-a)\,(\cos y-\cos(x+a))}{2\sin a\,(\cos a-\cos x\cos y)}=\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}. \tag{53}$$

Now we find the partial derivative Ω_x at these points. General expression for this partial derivative can be given in the form:

$$\begin{aligned}\Omega_x=&\frac{[\cos(x-a)\,(\cos y-\cos(x+a))+\sin(x-a)\,(-\sin y\cdot y_x+\sin(x+a))]}{2\sin a\,(\cos a-\cos x\cos y)^2}\times\\&\times(\cos a-\cos x\cos y)-\frac{\sin(x-a)\,(\cos y-\cos(x+a))\,(\sin x\cos y+\cos x\sin y\cdot y_x)}{2\sin a\,(\cos a-\cos x\cos y)^2}\\&-\frac{1}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\left[\left(\frac{1}{\sin(x+a)}-\frac{y_x}{\sin y}\right)\ln\frac{\tan\frac{x+a}{2}}{\tan\frac{x-a}{2}}\right.\\&\left.-\left(\frac{1}{\sin(x+a)}-\frac{1}{\sin(x-a)}\right)\ln\frac{\tan\frac{x+a}{2}}{\tan\frac{y}{2}}\right]\end{aligned}$$

Substituting y_x from (17), rearranging the terms and simplifying, we have

$$\begin{aligned}\Omega_x=&\frac{\cos a\,(\cos y-\cos(x+a))\,(\cos(x-a)-\cos y)}{2\sin a\,(\cos a-\cos x\cos y)^2}\\&+\frac{\sin(x-a)\sin(x+a)\,(\cos a-\cos x\cos y)}{2\sin a\,(\cos a-\cos x\cos y)^2}\\&-\frac{\sin(x-a)\sin y\,(\cos a-\cos x\cos(x+a))}{2\sin a\,(\cos a-\cos x\cos y)^2}\frac{1}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\times\\&\times\frac{2\sin a\,(\cos a-\cos x\cos y)}{\sin y\sin(x+a)\sin(x-a)}\\&-\frac{1}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\frac{1}{\sin(x+a)}+\frac{1}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\frac{2\sin a\,(\cos a-\cos x\cos y)}{\sin^2 y\sin(x+a)\sin(x-a)}\\&-\frac{\ln\frac{\tan((x+a)/2)}{\tan(y/2)}}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\frac{2\sin a\cos x}{\sin(x-a)\sin(x+a)}.\end{aligned}$$

Considering this derivative at the points where $\Omega = 0$, substituting (53) in the last term and simplifying, we obtain

$$\Omega_x|_{\Omega=0} = \frac{\cos a\,(\cos y - \cos(x+a))\,(\cos(x-a) - \cos y)}{2\sin a\,(\cos a - \cos x\cos y)^2} + \frac{\sin(x-a)\sin(x+a)\,(\cos a - \cos x\cos y)}{2\sin a\,(\cos a - \cos x\cos y)^2}$$
$$= -\frac{1}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\,\frac{2\cos a - 2\cos x\cos(x+a)}{\sin(x+a)\,(\cos a - \cos x\cos y)} + \frac{1}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\,\frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin^2 y\sin(x+a)\sin(x-a)}$$

Finally, applying some algebra to the first and second terms, we are able to write result as follows:

$$\Omega_x|_{\Omega=0} = -\frac{1}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\,\frac{2\sin x}{(\cos a - \cos x\cos y)}$$
$$+\frac{\cos a\,(\cos y - \cos(x+a))\,(\cos(x-a) - \cos y) + \left(\cos^2 a - \cos^2 x\right)(\cos a - \cos x\cos y)}{2\sin a\,(\cos a - \cos x\cos y)^2}$$
$$+\frac{1}{\ln^2\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\,\frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin^2 y\sin(x+a)\sin(x-a)}$$
$$= \left[\frac{1}{\ln\frac{\tan((x+a)/2)}{\tan((x-a)/2)}}\,\frac{1}{\sin y}\sqrt{\frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin(x+a)\sin(x-a)}} - \frac{\sin x\sin y}{\cos a - \cos x\cos y}\cdot\sqrt{\frac{\sin(x+a)\sin(x-a)}{2\sin a\,(\cos a - \cos x\cos y)}}\right]^2$$
$$+\frac{\cos a\,(\cos y - \cos(x+a))\,(\cos(x-a) - \cos y) + \left(\cos^2 a - \cos^2 x\right)(\cos a - \cos x\cos y)}{2\sin a\,(\cos a - \cos x\cos y)^2}$$
$$-\frac{\left(\cos^2 a - \cos^2 x\right)\sin^2 x\sin^2 y}{2\sin a\,(\cos a - \cos x\cos y)^3}.$$

The terms outside the square brackets can be represented in the form

$$Q = \frac{q}{p}$$

where

$$p = 2\sin a\,(\cos a - \cos x\cos y)^3$$

is positive due to (52) and numerator can be simplified to the form

$$q = [\cos a\,(\cos y - \cos(x+a))\,(\cos(x-a) - \cos y) + \left(\cos^2 a - \cos^2 x\right)\times$$
$$\times(\cos a - \cos x\cos y)]\,(\cos a - \cos x\cos y) - \sin^2 x\sin^2 y\left(\cos^2 a - \cos^2 x\right)$$
$$= \cos x\,(\cos y - \cos(x+a))\,(\cos(x-a) - \cos y)\,(\cos x - \cos a\cos y).$$

Each factor is positive due to (39) and, therefore, q is positive too. Hence, Q is positive and $\Omega_x|_{\Omega=0}$ is positive too. Therefore, it can exist exactly one point $x \in (a, \pi/2)$ such that $\Omega = 0$. By (50), the same is true for function Φ. Therefore we obtain a contradiction with the initial observation that function Φ must have at least two zero points on the interval $x \in (a, \pi/2)$. Hence, our supposition about existence of positive values of the function Φ is false.

It remains to show that function Φ cannot assume zero values, which is the simple consequence of the previous calculations. In fact, because the zero points of the functions Φ and Ω coincide (if they would exist), the partial derivative of Φ at zero points is positive:

$$\begin{aligned}\Phi_x|_{\Phi=0} &= \left(\frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin(x+a)\sin(x-a)}\right)_x \Omega\bigg|_{\Omega=0} + \frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin(x+a)\sin(x-a)}\Omega_x\bigg|_{\Omega=0} \\ &= \frac{2\sin a\,(\cos a - \cos x\cos y)}{\sin(x+a)\sin(x-a)}\Omega_x\bigg|_{\Omega=0} > 0.\end{aligned}$$

Therefore, if zero points would exist, then the function Φ have to be positive in their right-hand neighborhoods, but this is false according to the above conclusions. Thus Φ can assume only negative values on whole interval $x \in (a, \pi/2)$ for any fixed a.

4) Concluding derivations

We have shown that $\Phi(x,a) < 0$ for any $x \in (a, \pi/2)$ and any fixed a from (3). By (25), it means that $f_x(x,a) < 0$, that is, function $f(x,a)$ is strictly decreasing in x on the interval $(a, \pi/2)$. Let us find the limit values of $f(x,a)$ at the end points of this interval. The left-hand limit at the point $\pi/2$ can be calculated by direct substitution and using the last result in (14):

$$\begin{aligned}\lim_{x\to\pi/2_-} f(x,a) &= \lim_{x\to\pi/2_-} \frac{\sin y}{\sin(x+a)}\left(\frac{\tan((x+a)/2)}{\tan(y/2)}\right)^{\cos y} \\ &= \frac{1}{\cos a}\left(\tan\left(\frac{\pi}{4}+\frac{a}{2}\right)\right)^0 = \frac{1}{\cos a}.\end{aligned} \tag{54}$$

By applying the third limit in (14), the right-hand limit at the point a can be rewritten in the following form:

$$\begin{aligned}\lim_{x\to a_+} f(x,a) &= \lim_{x\to a_+} \frac{\sin y}{\sin(x+a)}\left(\frac{\tan((x+a)/2)}{\tan(y/2)}\right)^{\cos y} \\ &= \frac{\tan a}{\sin 2a}\lim_{y\to 0_+}\frac{\sin y}{(\tan(y/2))^{\cos y}}.\end{aligned}$$

This limit represents indeterminate form of type $0/0$ and can be transformed to the form:

$$\begin{aligned}\lim_{x\to a_+} f(x,a) &= \frac{1}{2\cos^2 a}\lim_{y\to 0_+}\frac{2\sin(y/2)\cos(y/2)(\cos(y/2))^{\cos y}}{(\sin(y/2))^{\cos y}} \\ &= \frac{1}{\cos^2 a}\lim_{y\to 0_+}\left(\sin\frac{y}{2}\right)^{2\sin^2\frac{y}{2}}.\end{aligned}$$

The last limit can be calculated by substitution

$$t = \sin \frac{y}{2}, \quad \lim_{y \to 0_+} t = 0_+$$

and by using the limit (37):

$$\lim_{x \to a_+} f(x,a) = \frac{1}{\cos^2 a} \lim_{t \to 0_+} t^{2t^2} = \frac{1}{\cos^2 a} \lim_{t \to 0_+} e^{2t^2 \ln t} = \frac{1}{\cos^2 a}. \tag{55}$$

Finally, since function $f(x,a)$ has the limit values (54), (55) and it is strictly decreasing in x on the entire interval $(a, \pi/2)$, we can conclude that

$$\frac{1}{\cos a} < f(x,a) < \frac{1}{\cos^2 a}$$

for any x, a such that

$$0 < a < x < \pi/2.$$

This completes the proof of the theorem.

Acknowledgements. This research was supported by Brazilian science foundation CNPq.

REFERENCES

[1] A. BOURCHTEIN, L. BOURCHTEIN, *Comparative analysis of conformal mappings used in limited area models of numerical weather prediction*, Mon. Wea. Rew., **131**, (2003), 1759–1768.
[2] P. COURTIER, J. F. GELEYN, *A global numerical weather prediction model with variable resolution: application to the shallow water equations*, Quart. J. Royal Met. Soc, **114**, (1988), 1321–1346.
[3] H. R. GLAHN, *Characteristics of map projections and implications for AWIPS-* 90 , TDL Office Note 88-5, (National Weather Service, NOAA, 1988).
[4] I. S. GRADSHTEYN, I. M. RYZHIK, *Tables of integrals, series and products* (Academic Press, 1994).
[5] W. J. SAUCIER, *Principles of meteorological analysis*, (The University of Chicago Press, 1962).
[6] J. F. THOMPSON, Z. U. A. WARSI AND C. W. MASTIN, *Numerical grid generation*, (North-Holland, 1985).
[7] D. L. WILLIAMSON, *Difference approximations for numerical weather prediction over a sphere*, (GARP Publ. Ser., **17**, (II), 1979).

(Received September 18, 2004)

Department of Mathematics
Pelotas State University
Rua Anchieta 4715 bloco K, ap.304
Pelotas 96020-250
Brazil
e-mail: burstein@terra.com.br