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MONOTONICITY OF SEQUENCES INVOLVING CONVEX FUNCTION AND SEQUENCE

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Abstract. Let f be an increasing convex (concave, respectively) function defined on $[0,1]$ and $\{a_i\}_{i\in\mathbb{N}}$ be an increasing positive sequence such that $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases $\left(\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}\text{ increases, respectively}\right)$, then the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{a_i}{a_n}\right)\right\}_{n\in\mathbb{N}}$ is decreasing.

Let f be an increasing convex (concave, respectively) positive function defined on $[0,1]$ and φ be an increasing convex positive function defined on $[0,\infty)$ such that $\varphi(0)=0$ and the sequence $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)}-1\right]\right\}_{i\in\mathbb{N}}$ decreases, then the sequence $\left\{\frac{1}{\varphi(n)}\sum_{i=1}^{n}f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}_{n\in\mathbb{N}}$ is decreasing.

As applications, taking special sequence $\{a_i\}_{i\in\mathbb{N}}$ and special functions f and φ, many new inequalities between ratios of means are obtained, and the Alzer's inequality, the Minc-Sathre's inequality, and the like, are recovered.

1. Introduction

Let I be an interval in $\mathbb{R}$. Then $f: I\to\mathbb{R}$ is said to be convex if for all $x,y\in I$ and $\lambda\in[0,1]$,

$$f(\lambda x+(1-\lambda)y)\leqslant \lambda f(x)+(1-\lambda)f(y). \tag{1}$$

If (1) is strict for all $x\neq y$ and $\lambda\in(0,1)$, then f is said to be strictly convex.

If the inequality in (1) is reversed, then f is said to be concave. If inequality (1) is reversed and strict for all $x\neq y$ and $\lambda\in(0,1)$, then f is said to be strictly concave.

The finite difference of a sequence $\{a_i\}_{i\in\mathbb{N}}$ can be defined by

$$\Delta^0 a_i=a_i,\quad \Delta a_i=a_{i+1}-a_i,\quad \Delta^m a_i=\Delta(\Delta^{m-1}a_i). \tag{2}$$

We shall say that a sequence $\{a_i\}_{i\in\mathbb{N}}$ is convex of order m (m-convex) if $\Delta^m a_i\geqslant 0$ for $m\geqslant 0$, $i\in\mathbb{N}$. If $\{a_i\}_{i\in\mathbb{N}}$ is 2-convex, we have $a_{i+1}+a_{i-1}\geqslant 2a_i$ for $i\geqslant 2$, the sequence $\{a_i\}_{i\in\mathbb{N}}$ is called convex; if $a_{i+1}+a_{i-1}\leqslant 2a_i$ for $i\geqslant 2$,, we call $\{a_i\}_{i\in\mathbb{N}}$ being concave.

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Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geqslant a_i^2$ for $i \geqslant 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leqslant a_i^2$ for $i \geqslant 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence.

Let f be a strictly increasing convex (or concave) function in $(0,1]$, J.-Ch. Kuang in [8] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)dx. \tag{3}$$

In [15], the first author generalized the results in [8] and obtained the following main result and some corollaries: Let f be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right)$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$, that is,

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\left(\frac{i}{n+k+1}\right) > \int_0^1 f(t)dt, \tag{4}$$

where k is a nonnegative integer, n a natural number.

With the help of these conclusions, we can deduce the Alzer's inequality, the Minc-Sathre's inequality, and more other inequalities involving the sum of powers of positive numbers or the ratios of the arithmetic means of n numbers. These inequalities have been investigated by many mathematicians. For more information, please refer to the references in this paper.

In this article, by similar procedure as in [8, 15], considering the convexity of a given function or sequence, using the Hermite-Hadamard inequality in [7, 11], we obtain

THEOREM 1. *Let f be an increasing convex (concave) function defined on $[0,1]$ and $\{a_i\}_{i\in\mathbb{N}}$ be an increasing positive sequence such that the sequence $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases $\left(\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}\text{ increases}\right)$, then the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right)\right\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \geqslant \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \geqslant \int_0^1 f(t)dt. \tag{5}$$

THEOREM 2. *Let f be an increasing convex (concave) positive function defined on $[0,1]$ and φ be an increasing convex positive function defined on $[0,\infty)$ such that $\varphi(0)=0$ and the sequence $\left\{\varphi(i)\left[\frac{\varphi(i)}{\varphi(i+1)}-1\right]\right\}_{i\in\mathbb{N}}$ decreases, then the sequence $\left\{\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}_{n\in\mathbb{N}}$ is decreasing. That is*

$$\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) \geqslant \frac{1}{\varphi(n+1)}\sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \tag{6}$$

As applications, taking special sequence $\{a_i\}_{i\in\mathbb{N}}$ and special functions f and φ satisfying Theorem 1 and Theorem 2, many new inequalities between ratios of mean values are obtained, and the Alzer's inequality, the Minc-Sathre's inequality, and the like, are recovered.

2. Proofs of Theorems

Proof. [Proof of Theorem 1] The left inequality in (5) is equivalent to

$$\begin{aligned} n\sum_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) &\leqslant (n+1)\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right), \\ n\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) + nf(1) &\leqslant (n+1)\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right), \\ n\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) &\leqslant \sum_{i=1}^{n}\left[(i-1)f\left(\frac{a_{i-1}}{a_n}\right) + (n-i+1)f\left(\frac{a_i}{a_n}\right)\right], \\ \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) &\leqslant \sum_{i=1}^{n}\left[\frac{i-1}{n}f\left(\frac{a_{i-1}}{a_n}\right) + \left(1-\frac{i-1}{n}\right)f\left(\frac{a_i}{a_n}\right)\right], \end{aligned} \tag{7}$$

where we let $a_0 = 0$.

Since the sequence $\left\{i\left(\frac{a_i}{a_{i+1}} - 1\right)\right\}_{i\in\mathbb{N}}$ decreases and $\left\{i\left(\frac{a_{i+1}}{a_i} - 1\right)\right\}_{i\in\mathbb{N}}$ increases, then we have

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leqslant (i-1)\left(\frac{a_{i-1}}{a_i} - 1\right), \tag{8}$$

$$n\left(\frac{a_{n+1}}{a_n} - 1\right) \geqslant i\left(\frac{a_{i+1}}{a_i} - 1\right). \tag{9}$$

Inequality (8) can be rewritten as

$$\frac{(i-1)a_{i-1} + (n-i+1)a_i}{na_n} \geqslant \frac{a_i}{a_{n+1}}, \tag{10}$$

and inequality (9) yields

$$\begin{aligned} (n+1)\left(\frac{a_{n+1}}{a_n} - 1\right) &\geqslant i\left(\frac{a_{i+1}}{a_i} - 1\right), \\ \frac{ia_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}} &\leqslant \frac{a_i}{a_n}. \end{aligned} \tag{11}$$

Since f is increasing, from (10) and (11), we have

$$f\left(\frac{(i-1)a_{i-1} + (n-i+1)a_i}{na_n}\right) \geqslant f\left(\frac{a_i}{a_{n+1}}\right), \tag{12}$$

$$f\left(\frac{ia_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}}\right) \leqslant f\left(\frac{a_i}{a_n}\right). \tag{13}$$

If f is convex, then

$$\frac{i-1}{n}f\left(\frac{a_{i-1}}{a_n}\right) + \left(1-\frac{i-1}{n}\right)f\left(\frac{a_i}{a_n}\right) \geqslant f\left(\frac{(i-1)a_{i-1} + (n-i+1)a_i}{na_n}\right). \tag{14}$$

Combination of (14) with (12) leads to

$$f\left(\frac{a_i}{a_{n+1}}\right) \leqslant \left[\frac{i-1}{n} f\left(\frac{a_{i-1}}{a_n}\right) + \left(1 - \frac{i-1}{n}\right) f\left(\frac{a_i}{a_n}\right)\right], \tag{15}$$

inequality (7) follows.

If f is concave, then

$$\begin{aligned} &\frac{i}{n+1} f\left(\frac{a_{i+1}}{a_{n+1}}\right) + \left(1 - \frac{i}{n+1}\right) f\left(\frac{a_i}{a_{n+1}}\right) \\ &\qquad \leqslant f\left(\frac{i}{n+1} \cdot \frac{a_{i+1}}{a_{n+1}} + \frac{n-i+1}{n+1} \cdot \frac{a_i}{a_{n+1}}\right) \\ &\qquad = f\left(\frac{i a_{i+1} + (n-i+1) a_i}{(n+1) a_{n+1}}\right). \end{aligned} \tag{16}$$

From (13) and (16), we obtain

$$\sum_{i=1}^{n} \left[\frac{i}{n+1} f\left(\frac{a_{i+1}}{a_{n+1}}\right) + \left(1 - \frac{i}{n+1}\right) f\left(\frac{a_i}{a_{n+1}}\right)\right] \leqslant \sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right),$$

that is

$$\begin{aligned} &\sum_{i=1}^{n} \frac{i}{n+1} f\left(\frac{a_{i+1}}{a_{n+1}}\right) + \sum_{i=1}^{n} \left(\frac{n}{n+1} - \frac{i-1}{n+1}\right) f\left(\frac{a_i}{a_{n+1}}\right) \\ &\qquad = \frac{n}{n+1} f(1) + \frac{n}{n+1} \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) \\ &\qquad = \frac{n}{n+1} \sum_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \leqslant \sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right). \end{aligned} \tag{17}$$

The final line in (17) implies the left inequality in (5).

Finally, by definition of definite integral, the right inequality in (5) follows.

Proof. [Proof of Theorem 2] Since

$$\varphi(i) \left(\frac{\varphi(i)}{\varphi(i+1)} - 1\right) \leqslant \varphi(i-1) \left(\frac{\varphi(i-1)}{\varphi(i)} - 1\right), \tag{18}$$

therefore we obtain

$$\varphi(n) \left(\frac{\varphi(n)}{\varphi(n+1)} - 1\right) \leqslant \varphi(i-1) \left(\frac{\varphi(i-1)}{\varphi(i)} - 1\right), \tag{19}$$

that is

$$\frac{\varphi(i)}{\varphi(n+1)} \leqslant \frac{\varphi^2(i-1) + [\varphi(n) - \varphi(i-1)]\varphi(i)}{\varphi^2(n)}. \tag{20}$$

From monotonicity of f and letting $\varphi(0)=0$, we have

$$\begin{aligned} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) &\leqslant f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right), \\ \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) &\leqslant \sum_{i=1}^{n} f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right). \end{aligned} \tag{22}$$

Since φ is convex and f is positive, if f is convex, then

$$\begin{aligned} &\sum_{i=1}^{n} f\left(\frac{\varphi^2(i-1)+[\varphi(n)-\varphi(i-1)]\varphi(i)}{\varphi^2(n)}\right) \\ &\qquad \leqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)} f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n)-\varphi(i-1)}{\varphi(n)} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\} \\ &\qquad \leqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)} f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n+1)-\varphi(i)}{\varphi(n)} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}. \end{aligned} \tag{23}$$

From (22) and (23), we get

$$\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \leqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i-1)}{\varphi(n)} f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+\frac{\varphi(n+1)-\varphi(i)}{\varphi(n)} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\},$$

that is

$$\begin{aligned} &\varphi(n) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \\ &\qquad \leqslant \sum_{i=1}^{n}\left\{\varphi(i-1) f\left(\frac{\varphi(i-1)}{\varphi(n)}\right)+[\varphi(n+1)-\varphi(i)] f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\} \\ &\qquad =\varphi(n+1) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right)-\varphi(n) f(1). \end{aligned} \tag{24}$$

Inequality (24) is equivalent to

$$\begin{aligned} \varphi(n+1) \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) &\geqslant \varphi(n) \sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right), \\ \frac{1}{\varphi(n)} \sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) &\geqslant \frac{1}{\varphi(n+1)} \sum_{i=1}^{n+1} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right). \end{aligned} \tag{25}$$

Now assume f is concave. Then

$$\begin{aligned} &\frac{\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i+1)}{\varphi(n+1)}\right)+\left(1-\frac{\varphi(i)}{\varphi(n+1)}\right) f\left(\frac{\varphi(i)}{\varphi(n+1)}\right) \\ &\qquad \leqslant f\left(\frac{\varphi(i)\varphi(i+1)+\varphi(i)\varphi(n+1)-\varphi^2(i)}{\varphi^2(n+1)}\right). \end{aligned} \tag{26}$$

Since φ is increasing and convex, then easy computation gives us

$$\frac{\varphi(i)\varphi(i+1)+\varphi(i)\varphi(n+1)-\varphi^2(i)}{\varphi^2(n+1)} \leqslant \frac{\varphi(i)}{\varphi(n)}. \tag{26}$$

Therefore, from convexity of φ, we have

$$\begin{aligned}
\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right) &\geqslant \sum_{i=1}^{n}\left\{\frac{\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i+1)}{\varphi(n+1)}\right)+\left(1-\frac{\varphi(i)}{\varphi(n+1)}\right) f\left(\frac{\varphi(i)}{\varphi(n+1)}\right)\right\}\\
&=\sum_{i=1}^{n+1}\frac{\varphi(n+1)+\varphi(i-1)-\varphi(i)}{\varphi(n+1)} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right)\\
&\geqslant \frac{\varphi(n)}{\varphi(n+1)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n+1)}\right).
\end{aligned}$$

The proof is complete.

3. Corollaries

In this section, as applications, taking special sequence $\{a_i\}_{i\in\mathbb{N}}$ and special functions f and φ satisfying Theorem 1 and Theorem 2, many new inequalities between ratios of means will be obtained , and the Alzer's inequality, the Minc-Sathre's inequality, and the like, are recovered.

COROLLARY 1. *Let f be an increasing convex (or concave, respectively) function defined on $[0,1]$, $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex (or a logarithmically concave, respectively) increasing positive sequence, then the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right)\right\}_{n\in\mathbb{N}}$ is decreasing.*

COROLLARY 2. *Let f be an increasing convex (or concave, respectively) positive function defined on $[0,1]$, and let φ be an increasing, convex, logarithmically convex, positive function defined on $(0,\infty)$, then the sequence $\left\{\frac{1}{\varphi(n)}\sum_{i=1}^{n} f\left(\frac{\varphi(i)}{\varphi(n)}\right)\right\}_{n\in\mathbb{N}}$ is decreasing.*

It is clear that the function $f(x)=x^r$ is strictly increasing in $[0,1]$ for $r>0$, convex for $r\geqslant 1$, and concave for $0<r<1$. Taking $a_i=i$ in Theorem 1, we recover the Alzer's inequality as follows:

COROLLARY 3. ([1])*Let $n\in\mathbb{N}$, then for any $r>0$, we have*

$$\frac{n}{n+1} \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}. \tag{27}$$

The lower bound is best possible.

The first easy proof of Alzer's inequality is due to J. Sándor who used Cauchy mean value theorem and mathematical induction in his proof, see [22]. Recently, some new proofs were given in [3].

If let $f(x)=x^r$, $r>0$, $x\in[0,1]$, and $a_i=i+k$, where k is a given natural number, in Theorem 1, then we obtain

COROLLARY 4. ([14]) *Let n and m be natural numbers, k a nonnegative integer. Then*

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{28}$$

where r is any given positive real number. The lower bound is best possible.

Let $f(x) = x^r$, $r > 0$, $x \in [0,1]$, $\varphi = x + k$, where k is a given nonnegative integer, in Theorem 2, then we have

COROLLARY 5. *Let n and m be natural numbers, k a nonnegative integer, then*

$$\left(\frac{1}{n+k}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+k+m}\sum_{i=k+1}^{n+k+m} i^r\right)^{1/r} > \frac{n+k}{n+k+m}, \tag{29}$$

where r is any given positive real number.

Since $\ln(1+x)$ and $\ln\frac{x}{1+x}$ are strictly increasing concave function in $(0,1]$, let $f(x) = \ln(1+x)$ or $f(x) = \ln\frac{x}{1+x}$ in (5) respectively, by direct calculation, we have

COROLLARY 6. *If $\{a_i\}_{i\in\mathbb{N}}$ is an increasing positive sequence such that $\big\{i\big(\frac{a_{i+1}}{a_i} - 1\big)\big\}_{i\in\mathbb{N}}$ increases, then we have*

$$\frac{a_n}{a_{n+1}} \leqslant \frac{\sqrt[n]{\prod_{i=1}^{n}(a_i + a_n)}}{\sqrt[n+1]{\prod_{i=1}^{n+1}(a_i + a_{n+1})}} \leqslant \frac{\sqrt[n]{\prod_{i=1}^{n} a_i}}{\sqrt[n+1]{\prod_{i=1}^{n+1} a_i}}. \tag{30}$$

Let $f(x) = \ln(1+x)$ in (6), by direct computation, we obtain

COROLLARY 7. *If the function φ is an increasing convex positive function defined on $(0,\infty)$ such that $\big\{\varphi(i)\big[\frac{\varphi(i)}{\varphi(i+1)} - 1\big]\big\}_{i\in\mathbb{N}}$ decreases, then*

$$\frac{[\varphi(n)]^{n/\varphi(n)}}{[\varphi(n+1)]^{(n+1)/\varphi(n+1)}} \leqslant \frac{\sqrt[\varphi(n)]{\prod_{i=1}^{n}[\varphi(i)+\varphi(n)]}}{\sqrt[\varphi(n+1)]{\prod_{i=1}^{n+1}[\varphi(i)+\varphi(n+1)]}}. \tag{31}$$

REMARK 1. The inequalities (30) and (31) generalize those obtained in [8], [15], and [21]. If taking more special functions f, φ, and $\{a_i\}_{i\in\mathbb{N}}$ in Theorem 1 and 2, we can obtain more new concrete inequalities between the ratios of mean values involving sums or products of positive sequences.

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