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SOME NEW INTEGRAL INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper, we introduce and study some new integral inequalities in one variable and two independent variables which provide explicit bounds on unknown functions, and apply these integral inequalities to study the qualitative behavior of the solution for a partial differential equation and an integral equation, respectively.

1. Introduction

It is well known that integral inequalities which provide explicit bounds on unknown functions play an important role in the development of differential and integral equations. For details, we refer to [1]-[9] and references therein.

Recently, Lipovan [3] studied the following integral inequalities

$$u^2(t) \leqslant c^2 + 2\int_0^{\alpha(t)} [f(s)u(s)w(u(s)) + g(s)u(s)]ds, \quad t \in [0, +\infty)$$

and

$$u^2(t) \leqslant c^2 + 2\int_0^{\alpha(t)} f(s)u(s)w(u(s))ds + 2\int_0^{\alpha(t)} g(s)u(s)w(u(s))ds, \quad t \in [0, +\infty).$$

Very recently, Sun [9] studied the following integral inequalities

$$u^m(t) \leqslant c^{\frac{m}{m-n}} + \frac{m}{m-n}\int_0^{\alpha(t)} [f(s)u^n(s)w(u(s)) + g(s)u^n(s)]ds, \quad t \in [0, +\infty)$$

and

$$u^m(t) \leqslant c^{\frac{m}{m-n}} + \frac{m}{m-n}\int_0^{\alpha(t)} f(s)u^n(s)w(u(s))ds + \frac{m}{m-n}\int_0^{t} g(s)u^n(s)w(u(s))ds, \quad t \in [0, +\infty).$$

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On the other hand, Meng and Li [5] studied the following integral inequality involving two independent variables

$$u^p(x,y) \leqslant a(x,y) + b(x,y)\int_0^x\int_0^y [c(s,t)u(s,t)+g(s,t)]dtds, \quad x,y \in [0,+\infty).$$

Inspired and motivated by previous papers, in this paper, we introduce and study some new integral inequalities in one variable and two independent variables which provide explicit bounds on unknown functions. By using these integral inequalities, we study the qualitative behavior of the solution for a partial differential equation and an integral equation, respectively.

2. Integral inequalities

For $T > 0$, let $R^+ = (0,\infty)$ and $I = [0,T)$. We first recall the following Lemma.

LEMMA 1. *[7] Let $a(t), u(t), b(t)$ be nonnegative and continuous functions defined for $t \in R^+$. If*

$$u(t) \leqslant a(t) + \int_0^t b(s)u(s)ds$$

for $t \in R^+$, then

$$u(t) \leqslant a(t)exp(\int_0^t b(s)ds)$$

for $t \in R^+$.

THEOREM 2.1. *Let $k(t), u(t), a_i(t) \in C(I,R^+)$, $\alpha_i(t) \in C^1(I,I)$ be nondecreasing with $\alpha_i(t) < t$ on I, $i = 1,2\cdots,m$. If*

$$u^m(t) \leqslant k(t) + \sum_{i=1}^m \int_0^{\alpha_i(t)} a_i(s)u^i(s)ds \tag{2.1}$$

for $t \in I$, then

$$u(t) \leqslant k^{\frac{1}{m}}(t) + \frac{1}{m}k^{\frac{1}{m}-1}(t)e(t)\exp(\sum_{i=1}^m \frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)ds) \tag{2.2}$$

for $t \in I$, where

$$e(t) = \sum_{i=1}^m \int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}}(s)ds \tag{2.3}$$

for $t \in I$.

Proof. Let

$$z(t) = \sum_{i=1}^m \int_0^{\alpha_i(t)} a_i(s)u^i(s)ds.$$

Then

$$u^m(t) < k(t) + z(t) = k(t)(1+\frac{z(t)}{k(t)}). \tag{2.4}$$

From (2.1) and the generalization of Bernoulli's inequality [4], i.e., $(1+x)^a \leqslant 1+ax$, where $0<a<1$ and $-1<x$, we have

$$
\begin{aligned}
z(t) &\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)(k(s)+z(s))^{\frac{i}{m}}ds \\
&= \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}}(s)(1+\frac{z(s)}{k(s)})^{\frac{i}{m}}ds \\
&\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}}(s)(1+\frac{i}{m}\frac{z(s)}{k(s)})ds \qquad (2.5)\\
&= \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}}(s)ds + \sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)z(s)ds \\
&= e(t) + \sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)z(s)ds,
\end{aligned}
$$

where $e(t)$ is defined by (2.3). Clearly, $e(t)$ is nondecreasing in $t\in I$. We assume $e(t)>0$ for $t\in I$. From (2.5), we get

$$\frac{z(t)}{e(t)} \leqslant 1+\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)\frac{z(s)}{e(s)}ds. \qquad (2.6)$$

Let $\frac{z(t)}{e(t)} = p(t)$ for $t\in I$, and define $q(t)$ by the right hand of (2.6). Then $q(t)>0$, $q(0)=1$, $p(t)<q(t)$ and

$$
\begin{aligned}
q'(t) &\leqslant \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))p(t)\alpha_i'(t) \\
&< \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))q(t)\alpha_i'(t).
\end{aligned}
$$

From above inequality, we have

$$\frac{q'(t)}{q(t)} < \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))\alpha_i'(t). \qquad (2.7)$$

Integrating above inequality from 0 to t, $t\in I$, we can get

$$q(t) \leqslant \exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)ds). \qquad (2.8)$$

In light of $\frac{z(t)}{e(t)} = p(t) < q(t)$ and (2.8), we have

$$z(t) \leqslant e(t)\exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)ds). \qquad (2.9)$$

From (2.9) and (2.4), we have

$$u(t) \leqslant k^{\frac{1}{m}}(t) + \frac{1}{m} k^{\frac{1}{m}-1}(t) z(t). \tag{2.10}$$

Now, (2.9) and (2.10) imply (2.2). This completes the proof.

In order to prove following Theorem, we need to denote the class S of nondecreasing function $g \in C(R^+, R^+)$ with $g(x) > 0$ for $x > 0$, $g(tx) > tg(x)$ for $t \geqslant 0$, $g(x) + g(y) > g(x+y)$ and $\int_1^\infty (\frac{dx}{g(x)}) = \infty$.

THEOREM 2.2. *Let $k(t), u(t), a_i(t) \in C(I, R^+)$, $\alpha_i(t) \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) < t$ on I, and $g_i \in S$, $i = 1, 2 \cdots, m$. If*

$$u^m(t) \leqslant k(t) + \sum_{i=1}^{m} \int_0^{\alpha_i(t)} a_i(s) g_i(u^i(s)) ds, \tag{2.11}$$

then for $0 \leqslant t \leqslant t_1$,

$$u(t) \leqslant k^{\frac{1}{m}}(t) + \frac{1}{m} k^{\frac{1}{m}-1}(t) e(t) G^{-1}\left(G(1) + \sum_{i=1}^{m} \frac{i}{m} \int_0^{\alpha_i(t)} a_i(s) k^{\frac{i}{m}-1}(s) ds\right), \tag{2.12}$$

where

$$e(t) = \sum_{i=1}^{m} \int_0^{\alpha_i(t)} a_i(s) g_i(k^{\frac{i}{m}}(s)) ds, \tag{2.13}$$

G^{-1} is the inverse function of

$$G(r) = \int_0^r \frac{ds}{g(s)}, r > 0, \tag{2.14}$$

$g(\cdot) = \max_{1 \leqslant i \leqslant m} g_i(\cdot)$, and $t_1 \in I$ is chosen so that

$$G(1) + \sum_{i=1}^{m} \int_0^{\alpha_i(t)} \frac{i}{m} a_i(s) k^{\frac{i}{m}-1}(s) ds \in Dom(G^{-1})$$

for all $t \in [0, t_1]$.

Proof. Let

$$z(t) = \sum_{i=1}^{m} \int_0^{\alpha_i(t)} a_i(s) g_i(u^i(s)) ds.$$

Then

$$u^m(t) < k(t) + z(t) = k(t)(1 + \frac{z(t)}{k(t)}). \tag{2.15}$$

From (2.15) and the generalization of Bernoulli's inequality, we have

$$\begin{aligned} z(t) &\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)g_i((k(s)+z(s))^{\frac{i}{m}})ds \\ &= \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)g_i(k^{\frac{i}{m}}(s)(1+\frac{z(s)}{k(s)})^{\frac{i}{m}})ds \\ &\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)g_i(k^{\frac{i}{m}}(s)+\frac{i}{m}k^{\frac{i}{m}-1}(s)z(s))ds \\ &\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)g_i(k^{\frac{i}{m}}(s))ds+\sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)g_i(\frac{i}{m}k^{\frac{i}{m}-1}(s)z(s))ds \\ &= e(t)+\sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)\frac{i}{m}k^{\frac{i}{m}-1}(s)\frac{g_i(\frac{i}{m}k^{\frac{i}{m}-1}(s)z(s))}{\frac{i}{m}k^{\frac{i}{m}-1}(s)}ds \\ &\leqslant e(t)+\sum_{i=1}^{m}\int_0^{\alpha_i(t)} a_i(s)\frac{i}{m}k^{\frac{i}{m}-1}(s)g_i(z(s))ds \end{aligned} \tag{2.16}$$

where $e(t)$ is defined by (2.13) which is nondecreasing in $t \in I$. We assume $e(t) > 0$. And the last inequality comes from the property of S that $g(tx) > tg(x)$. From (2.16), we get

$$\begin{aligned} \frac{z(t)}{e(t)} &\leqslant 1+\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)\frac{g_i(z(s))}{e(s)}ds \\ &\leqslant 1+\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s)k^{\frac{i}{m}-1}(s)g_i(\frac{z(s)}{e(s)})ds \end{aligned} \tag{2.17}$$

Let $\frac{z(t)}{e(t)} = p(t)$ for $t \in I$, and define $q(t)$ by the right hand of (2.17). Then $q(t) > 0$, $q(0) = 1$, $p(t) < q(t)$ and

$$\begin{aligned} q'(t) &\leqslant \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))g_i(p(\alpha_i(t))\alpha_i'(t) \\ &< \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))g_i(q(\alpha_i(t)))\alpha_i'(t). \end{aligned} \tag{2.18}$$

Since $g(\cdot) = \max\{g_i(\cdot) : i = 1,2\cdots,m\}$, from (2.18), we know that

$$q'(t) \leqslant g(q(t))\sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))\alpha_i'(t), \tag{2.19}$$

and

$$\frac{dG(q(t))}{dt} = \frac{q'(t)}{g(q(t))} < \sum_{i=1}^{m}\frac{i}{m}a_i(\alpha_i(t))k^{\frac{i}{m}-1}(\alpha_i(t))\alpha_i'(t),$$

where G is defined by (2.14). Integrating above inequality from 0 to t, $t \in I$, we have

$$G(q(t)) \leqslant G(1) + \sum_{i=1}^{m} \frac{i}{m} \int_{0}^{\alpha_i(t)} a_i(s) k^{\frac{i}{m}-1}(s) ds. \tag{2.20}$$

By virtue of the definition of G^{-1} and the fact that G^{-1} is increasing, it follows from (2.20) that

$$q(t) \leqslant G^{-1}(G(1) + \sum_{i=1}^{m} \frac{i}{m} \int_{0}^{\alpha_i(t)} a_i(s) k^{\frac{i}{m}-1}(s) ds). \tag{2.21}$$

In light of $\frac{z(t)}{e(t)} = p(t) < q(t)$ and from (2.21), we have

$$z(t) \leqslant e(t) G^{-1}(G(1) + \sum_{i=1}^{m} \frac{i}{m} \int_{0}^{\alpha_i(t)} a_i(s) k^{\frac{i}{m}-1}(s)) ds). \tag{2.22}$$

From (2.22) and (2.15), we get

$$u(t) \leqslant k^{\frac{1}{m}}(t) + \frac{1}{m} k^{\frac{1}{m}-1}(t) z(t). \tag{2.23}$$

Now, (2.22) and (2.23) imply (2.12). This completes the proof.

THEOREM 2.3. *Let $k(t), u(t), a_i(t)$ be the same as in Theorem 2.1., and F : $R^+ \times R^+ \to R^+$ be a continuous function satisfying the condition*

$$0 \leqslant F(t, u_i) - F(t, v_i) \leqslant H_i(t)(u_i - v_i), \tag{2.24}$$

for $u_i > v_i > 0, i = 1, 2, \cdots, m$, where $H_i(t)$ is a nonnegative continuous function on I, $i = 1, 2, \cdots, m$. If

$$u^m(t) \leqslant k(t) + \int_0^t b(t)(u^m(s)) ds + \sum_{i=1}^{m} \int_0^t F(s, u^i(s)) ds \tag{2.25}$$

for $t \in I$, then

$$u(t) \leqslant B^{\frac{1}{m}}(t) \left[k^{\frac{1}{m}}(t) + \frac{1}{m} k^{\frac{1}{m}-1}(t) G(t) exp \left(\sum_{i=1}^{m} \frac{i}{m} \int_0^t H_i(s) B^{\frac{i}{m}}(s) k^{\frac{i}{m}-1}(s) dt ds \right) \right] \tag{2.26}$$

for $t \in I$, where

$$G(t) = \sum_{i=1}^{m} \int_0^t F\left(s, B^{\frac{i}{m}}(s) k^{\frac{i}{m}}(s)\right) ds \quad and \quad B(t) = exp(\int_0^t b(t) ds) \tag{2.27}$$

for $t \in I$.

Proof. Let

$$z(t) = k(t) + \sum_{i=1}^{m} \int_0^t F(s, u^i(s)) ds. \tag{2.28}$$

Then (2.25) can be restated as

$$u^m(t) \leqslant z(t) + \int_0^t b(s) u^m(s) ds. \tag{2.29}$$

It is obvious that $z(t)$ is a nonnegative continuous and nondecreasing function in t for $t \in R^+$. Using Lemma 1, we get

$$u^m(t) \leqslant B(t)z(t), \tag{2.30}$$

where $B(t)$ is defined by (2.27). From (2.28) and (2.30), we have

$$u^m(t) \leqslant B(t)[k(t) + v(t)], \tag{2.31}$$

where $v(t) = \sum_{i=1}^{m} \int_0^t F(s, u^i(s))ds$. Using the generalization of Bernoulli's inequality to (2.31), we get

$$\begin{aligned} u^i(t) &\leqslant B^{\frac{i}{m}}(t)[k(t) + v(t)]^{\frac{i}{m}} \\ &\leqslant B^{\frac{i}{m}}(t)[k^{\frac{i}{m}}(t) + \frac{i}{m}k^{\frac{i}{m}-1}(t)v(t)]. \end{aligned} \tag{2.32}$$

From (2.32) and the hypotheses on F, it follows that

$$\begin{aligned} v(t) &\leqslant \sum_{i=1}^{m} \int_0^t (F(s, B^{\frac{i}{m}}(s)[k^{\frac{i}{m}}(s) + \frac{i}{m}k^{\frac{i}{m}-1}(s)v(s)]ds \\ &= \sum_{i=1}^{m} \int_0^t (F(s, B^{\frac{i}{m}}(s)(k^{\frac{i}{m}}(s) + \frac{i}{m}k^{\frac{i}{m}-1}(s)v(s))) - F(s, B^{\frac{i}{m}}(s)k^{\frac{i}{m}}(s)))ds \\ &+ \sum_{i=1}^{m} \int_0^t F(s, B^{\frac{i}{m}}(s)k^{\frac{i}{m}}(s))ds \\ &\leqslant G(t) + \sum_{i=1}^{m} \int_0^t H_i(s)B^{\frac{i}{m}}(s)\frac{i}{m}k^{\frac{i}{m}-1}(s)v(s)ds, \end{aligned} \tag{2.33}$$

where $G(t)$ is defined by (2.27) which is nondecreasing in $t \in I$. We assume $G(t) > 0$ for $t \in I$. Then from (2.33), we have

$$\frac{v(t)}{G(t)} \leqslant 1 + \sum_{i=1}^{m} \int_0^t H_i(s)B^{\frac{i}{m}}(s)\frac{i}{m}k^{\frac{i}{m}-1}(st)\frac{v(s)}{G(s)}ds. \tag{2.34}$$

Let $\frac{v(t)}{G(t)} = p(t)$ for $t \in I$, and define $q(t)$ by the right hand of (2.34). Then $q(t) > 0$, $q(0) = 1$, $p(t) < q(t)$ and

$$\begin{aligned} q'(t) &\leqslant \sum_{i=1}^{m} H_i(t)B^{\frac{i}{m}}(t)\frac{i}{m}k^{\frac{i}{m}-1}(t)p(t) \\ &\leqslant \sum_{i=1}^{m} H_i(t)B^{\frac{i}{m}}(t)\frac{i}{m}k^{\frac{i}{m}-1}(t)q(t) \\ &\leqslant (t)\sum_{i=1}^{m} H_i(t)B^{\frac{i}{m}}(t)\frac{i}{m}k^{\frac{i}{m}-1}(t). \end{aligned}$$

From above inequality, we have

$$\frac{q'(t)}{q(t)} \leqslant \sum_{i=1}^{m} H_i(t)B^{\frac{i}{m}}(t)\frac{i}{m}k^{\frac{i}{m}-1}(t).$$

Integrating above inequality from 0 to t, $t \in I$, we can get

$$q(t) \leqslant \exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^t H_i(s)B^{\frac{i}{m}}(s)k^{\frac{i}{m}-1}(s)ds). \tag{2.35}$$

In light of $\frac{v(t)}{G(t)} = p(t) < q(t)$ and (2.35), we have

$$v(t) \leqslant G(t)\exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^t H_i(s)B^{\frac{i}{m}}(s)k^{\frac{i}{m}-1}(s)ds). \tag{2.36}$$

From (2.35) and (2.36), we have

$$\begin{aligned} u(t) &\leqslant B^{\frac{1}{m}}(t)[k(t)+v(t)]^{\frac{1}{m}} \\ &\leqslant B^{\frac{1}{m}}(t)[k^{\frac{1}{m}}(t)+\frac{1}{m}k^{\frac{1}{m}-1}(t)v(t)]. \end{aligned} \tag{2.37}$$

Now, (2.36) and (2.37) imply (2.26). This completes the proof.

3. The integral inequalities of two variables

In this section, we consider the two-independent-variable version of Section 3. Set $I_1 = [0,X)$, $I_2 = [0,Y)$, where $X, Y \in R^+$, and denote $\Omega = I_1 \times I_2$.

THEOREM 3.1. *Let* $k(x,y), u(x,y), a_i(x,y) \in C(\Omega, R^+)$, $\alpha_i(x) \in C^1(I_1, I_1), \beta_i(y) \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha_i(x) < x$ *on* I_1, $\beta_i(y) < y$ *on* I_2, $i = 1, 2\cdots, m$. *If*

$$u^m(x,y) \leqslant k(x,y) + \sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)} a_i(s,t)u^i(s,t)dtds \tag{3.1}$$

for $(x,y) \in \Omega$, *then*

$$\begin{aligned} u(x,y) \leqslant\ & k^{\frac{1}{m}}(x,y) \\ & + \frac{1}{m}k^{\frac{1}{m}-1}(x,y)e(x,y)exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)} a_i(s,t)k^{\frac{i}{m}-1}(s,t)dtds) \end{aligned} \tag{3.2}$$

for $(x,y) \in \Omega$, *where*

$$e(x,y) = \sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)} a_i(s,t)k^{\frac{i}{m}}(s,t)dtds \tag{3.3}$$

for $(x,y) \in \Omega$.

Proof. Let

$$z(x,y)=\sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)u^i(s,t)dtds. \tag{3.4}$$

Then

$$u^m(x,y)\leqslant k(x,y)+z(x,y)=k(x,y)(1+\frac{z(x,y)}{k(x,y)}). \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned} z(x,y) &\leqslant \sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)k^{\frac{i}{m}}(s,t)(1+\frac{i}{m}\frac{z(s,t)}{k(s,t)})dtds \\ &= \sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)k^{\frac{i}{m}}(s,t)dtds \\ &\quad +\sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)\frac{i}{m}k^{\frac{i}{m}}(s,t)\frac{z(s,t)}{k(s,t)}dtds \\ &= e(x,y)+\sum_{i=1}^{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)\frac{i}{m}k^{\frac{i}{m}-1}(s,t)z(s,t)dtds, \end{aligned} \tag{3.6}$$

where $e(x,y)$ is defined by (3.3) which is nondecreasing in each of variables $(x,y)\in\Omega$. We assume that $e(x,y)>0$ for $(x,y)\in\Omega$. From (3.6), we get

$$\frac{z(x,y)}{e(x,y)}\leqslant 1+\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)k^{\frac{i}{m}-1}(s,t)\frac{z(s,t)}{e(s,t)}ds. \tag{3.7}$$

Let $\frac{z(x,y)}{e(x,y)}=p(x,y)$ for $(x,y)\in\Omega$, and define $q(x,y)$ by the right hand of (3.7). Then $q(x,y)>0$, $q(0,y)=q(x,0)=1$, $p(x,y)<q(x,y)$ and

$$\begin{aligned} \frac{\partial q(x,y)}{\partial x} &\leqslant \sum_{i=1}^{m}\int_0^{\beta_i(y)}a_i(\alpha(x),t)k^{\frac{i}{m}-1}(\alpha(x),t)p(\alpha(x),t)\alpha'(t)dt \\ &\leqslant \sum_{i=1}^{m}\int_0^{\beta_i(y)}a_i(\alpha(x),t)k^{\frac{i}{m}-1}(\alpha(x),t)q(\alpha(x),t)\alpha'(t)dt \\ &\leqslant q(x,y)\sum_{i=1}^{m}\int_0^{\beta_i(y)}a_i(\alpha(x),t)k^{\frac{i}{m}-1}(\alpha(x),t)\alpha'(t)dt. \end{aligned}$$

From above inequality, we have

$$\frac{\partial q(x,y)}{\partial x}\frac{1}{q(x,y)}\leqslant\sum_{i=1}^{m}\int_0^{\beta_i(y)}a_i(\alpha(x),t)k^{\frac{i}{m}-1}(\alpha(x),t)\alpha'(t)dt.$$

Integrating above inequality from 0 to x, $x\in I_1$, we can get

$$q(x,y)\leqslant\exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)k^{\frac{i}{m}-1}(s,t)dtds). \tag{3.8}$$

In light of $\frac{z(x,y)}{e(x,y)} = p(x,y) < q(x,y)$ and (3.8), we have

$$z(x,y) \leqslant e(x,y)\exp(\sum_{i=1}^{m} \frac{i}{m} \int_0^{\alpha_i(x)} \int_0^{\beta_i(y)} a_i(s,t)k^{\frac{i}{m}-1}(s,t)dtds). \tag{3.9}$$

From (3.9) and (3.5), we have

$$u(x,y) \leqslant k^{\frac{1}{m}}(x,y) + \frac{1}{m}k^{\frac{1}{m}-1}(x,y)z(x,y). \tag{3.10}$$

Now, (3.9) and (3.10) imply (3.2). This completes the proof.

THEOREM 3.2. *Let* $k(x,y), u(x,y), a_i(x,y)$ *be the same as in Theorem* 3.1*, and* $g_i \in S,\ i = 1,2\cdots,m$. *If*

$$u^m(x,y) \leqslant k(x,y) + \sum_{i=1}^{m} \int_0^{\alpha_i(t)} a_i(s,t)g_i(u^i(s,t))dtds, \tag{3.11}$$

then for $0 \leqslant x \leqslant x_1$ *and* $0 \leqslant y \leqslant y_1$,

$$u(x,y) \leqslant k^{\frac{1}{m}}(x,y)+\frac{1}{m}k^{\frac{1}{m}-1}(x,y)e(x,y)G^{-1}(G(1)+\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(t)} a_i(s,t)k^{\frac{i}{m}-1}(s,t))dtds), \tag{3.12}$$

where

$$e(t) = \sum_{i=1}^{m} \int_0^{\alpha_i(t)} \int_0^{\beta_i(y)} a_i(s,t)g_i(k^{\frac{i}{m}}(s,t))ds, \tag{3.13}$$

G^{-1} *is the inverse function of*

$$G(r) = \int_0^r \frac{ds}{g(s)}, r > 0, \tag{3.14}$$

$g(\cdot) = \max_{1\leqslant i\leqslant m} g_i(\cdot)$, *and* $x_1 \in I_1, y_1 \in I_2$ *are chosen so that*

$$G(1) + \sum_{i=1}^{m} \int_0^{\alpha_i(x)} \int_0^{\beta_i(y)} \frac{i}{m}a_i(s,t)k^{\frac{i}{m}-1}(s,t)dtds \in Dom(G^{-1})$$

for all $x \in [0,x_1], y \in [0,y_1]$.

Proof. With the same way in Theorems 2.2 and 3.1, we can prove that (3.12) holds.

THEOREM 3.3. *Let* $k(x,y), u(x,y), a_i(x,y)$ *be the same as in Theorem* 3.1 *and* $F : R^+ \times R^+ \times R^+ \to R^+$ *be a continuous function satisfying the condition*

$$0 \leqslant F(x,y,u_i) - F(x,y,v_i) \leqslant H_i(x,y)(u_i - v_i), \tag{3.15}$$

for $u_i > v_i > 0, i = 1,2,\cdots,m$, *where* $H_i(x,y)$ *is a nonnegative continuous function defined for* $x,y \in R^+$, $i = 1,2,\cdots,m$. *If*

$$u^m(x,y) \leqslant k(x,y)+ \int_0^x\int_0^y b(s,t)(u^m(s,t))dtds+ \sum_{i=1}^{m} \int_0^x\int_0^y F(s,t,u^i(s,t))dtds \tag{3.16}$$

for $x,y \in R^+$, *then*

$$\begin{aligned} u(x,y) &\leqslant B^{\frac{1}{m}}(x,y)[k^{\frac{1}{m}}(x,y)+\frac{1}{m}k^{\frac{1}{m}-1}(x,y)\times \\ &\times G(x,y)\exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^x\int_0^y H_i(s,t)B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}-1}(s,t)dtds)] \end{aligned} \tag{3.17}$$

for $x,y\in R^+$, *where*

$$\begin{aligned} G(x,y) &= \sum_{i=1}^{m}\int_0^x\int_0^y F\left(s,t,B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}}(s,t)\right)dtds, \\ B(x,y) &= \exp(\int_0^x b(s,y)ds) \end{aligned} \tag{3.18}$$

for $x,y\in R^+$.

Proof. Let

$$z(x,y)=k(x,y)+\sum_{i=1}^{m}\int_0^x\int_0^y F(s,t,u^i(s,t))dtds. \tag{3.19}$$

Then (3.16) can be restated as

$$u^m(x,y)\leqslant z(x,y)+\int_0^x\int_0^y b(s,t)u^m(s,t)dtds. \tag{3.20}$$

It is obvious that $z(x,y)$ is a nonnegative continuous and nondecreasing function in $x\in R^+$. Treating y, $y\in R^+$ fixed in (3.20), and using Lemma 1, we get

$$u^m(x,y)\leqslant B(x,y)z(x,y), \tag{3.21}$$

where $B(x,y)$ is defined by (3.18). From (3.19) and (3.21), we have

$$u^m(x,y)\leqslant B(x,y)[k(x,y)+v(x,y)], \tag{3.22}$$

where $v(x,y)=\sum_{i=1}^{m}\int_0^x\int_0^y F(s,t,u^i(s,t))dtds$. Using the generalization of Bernoulli's inequality to (3.22), we get

$$\begin{aligned} u^i(x,y) &\leqslant B^{\frac{i}{m}}(x,y)[k(x,y)+v(x,y)]^{\frac{i}{m}} \\ &\leqslant B^{\frac{i}{m}}(x,y)[k^{\frac{i}{m}}(x,y)+\frac{i}{m}k^{\frac{i}{m}-1}(x,y)v(x,y)]. \end{aligned} \tag{3.23}$$

From (3.23) and the hypotheses on F, it follows that

$$\begin{aligned} v(x,y) &\leqslant \sum_{i=1}^{m}\int_0^x\int_0^y F\left(s,t,B^{\frac{i}{m}}(s,t)[k^{\frac{i}{m}}(s,t)+\frac{i}{m}k^{\frac{i}{m}-1}(s,t)v(s,t)]\right)dtds \\ &= \sum_{i=1}^{m}\int_0^x\int_0^y F\left(s,t,B^{\frac{i}{m}}(s,t)(k^{\frac{i}{m}}(s,t)+\frac{i}{m}k^{\frac{i}{m}-1}(s,t)v(s,t))\right) \\ &- F(s,t,B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}}(s,t)))dtds+\sum_{i=1}^{m}\int_0^x\int_0^y F(s,t,B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}}(s,t))dtds \end{aligned}$$

$$\leqslant \; G(x,y)+\sum_{i=1}^{m}\int_0^x\int_0^y H_i(s,t)B^{\frac{i}{m}}(s,t)\frac{i}{m}k^{\frac{i}{m}-1}(s,t)v(s,t)dtds. \qquad (3.24)$$

where $G(x,y)$ is defined by (3.18) which is nondecreasing in each of variables $x,y\in R^+$. We assume that $G(x,y)>0$ for $x,y\in R^+$. Then from (3.24), we have

$$\frac{v(x,y)}{G(x,y)}\leqslant 1+\sum_{i=1}^{m}\int_0^x\int_0^y H_i(s,t)B^{\frac{i}{m}}(s,t)\frac{i}{m}k^{\frac{i}{m}-1}(s,t)\frac{v(s,t)}{G(s,t)}dtds. \qquad (3.25)$$

Let $\frac{v(x,y)}{G(x,y)}=p(x,y)$ for $x,y\in R^+$, and define $q(x,y)$ by the right hand of (3.25). Then $q(x,y)>0$, $q(0,y)=q(x,0)=1$, $p(x,y)<q(x,y)$ and

$$\begin{aligned}\frac{\partial q(x,y)}{\partial x} &\leqslant \sum_{i=1}^{m}\int_0^y H_i(x,t)B^{\frac{i}{m}}(x,t)\frac{i}{m}k^{\frac{i}{m}-1}(x,t)p(x,t)dt\\ &\leqslant \sum_{i=1}^{m}\int_0^y H_i(x,t)B^{\frac{i}{m}}(x,t)\frac{i}{m}k^{\frac{i}{m}-1}(x,t)q(x,t)dt\\ &\leqslant q(x,y)\sum_{i=1}^{m}\int_0^y H_i(x,t)B^{\frac{i}{m}}(x,t)\frac{i}{m}k^{\frac{i}{m}-1}(x,t)dt.\end{aligned}$$

From above inequality, we have

$$\frac{\partial q(x,y)}{\partial x}\frac{1}{q(x,y)}\leqslant \sum_{i=1}^{m}\int_0^y H_i(x,t)B^{\frac{i}{m}}(x,t)\frac{i}{m}k^{\frac{i}{m}-1}(x,t)dt.$$

Integrating above inequality from 0 to x, $x\in R^+$, we can get

$$q(x,y)\leqslant \exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^x\int_0^y H_i(s,t)B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}-1}(s,t)dtds). \qquad (3.26)$$

In light of $\frac{v(x,y)}{G(x,y)}=p(x,y)<q(x,y)$ and (3.26), we have

$$v(x,y)\leqslant G(x,y)\exp(\sum_{i=1}^{m}\frac{i}{m}\int_0^x\int_0^y H_i(s,t)B^{\frac{i}{m}}(s,t)k^{\frac{i}{m}-1}(s,t)dtds). \qquad (3.27)$$

From (3.22) and (3.23), we have

$$\begin{aligned}u(x,y) \;&\leqslant\; B^{\frac{1}{m}}(x,y)[k(x,y)+v(x,y)]^{\frac{1}{m}}\\ &\leqslant\; B^{\frac{1}{m}}(x,y)[k^{\frac{1}{m}}(x,y)+\frac{1}{m}k^{\frac{1}{m}-1}(x,y)v(x,y)]. \qquad (3.28)\end{aligned}$$

Now (3.27) and (3.28) imply (3.17). This complete the proof.

4. Some applications

In this section, we utilize the integral inequalities presented in Sections 2 and 3 to analyze the bound of solutions for an integral equation and a partial differential equation, respectively.

EXAMPLE 1. Consider a partial differential equation

$$pu^{p-1}(x,y)\frac{\partial^2 u(x,y)}{\partial x\partial y}+p(p-1)u^{p-2}\frac{\partial u(x,y)}{\partial x}\frac{\partial u(x,y)}{\partial y}=\sum_{i=1}^{p}h_i(x,y,u(x,y)) \quad (4.1)$$

and

$$u(x,0)=\eta(x),u(0,y)=\theta(y),u(0,0)=d, \quad (4.2)$$

where $h : R^+ \times R \to R$, $\eta,\theta : R^+ \to R$ are continuous functions and d is a real constant, $p > 1$ is a constant. Suppose that

$$|h_i(x,y,u)| \leqslant a_i(x,y)|u^i|, \quad (4.3)$$

$$|\eta(x)+\theta(y)-d| \leqslant k(x,y), \quad (4.4)$$

where $a_i(x,y)$ and $k(x,y)$ are nonnegative continuous functions for $x,y \in R^+$. Let $u(x,y)$ be a solution of (4.1) and (4.2) for $x,y \in R^+$. Then for $x,y \in R^+$,

$$u(x,y) \leqslant k^{\frac{1}{m}}(x,y)+\frac{1}{m}k^{\frac{1}{m}-1}(x,y)e(x,y)\exp\Big(\sum_{i=1}^{m}\frac{i}{m}\int_0^{\alpha_i(x)}\int_0^{\beta_i(y)}a_i(s,t)k^{\frac{i}{m}-1}(s,t)dtds\Big). \quad (4.5)$$

Proof. In fact, if $u(x,y)$ is a solution of (4.1) and (4.2), then it can be written as (see [9])

$$u^p(x,y)=\eta(x)+\theta(y)-d+\sum_{i=1}^{p}\int_0^x\int_0^y h_i(s,t,u)dtds \quad (4.6)$$

for $x,y \in R^+$. From (4.3), (4.4) and (4.6), we have

$$u^p(x,y) \leqslant k(x,y)+\sum_{i=1}^{p}\int_0^x\int_0^y a_i(s,t)u^i(s,t)dtds. \quad (4.7)$$

Now, using Theorem 3.1, we can get (4.5).

EXAMPLE 2. Consider the following integral equation

$$u^p(t)=k(t)+\sum_{i=1}^{p}\int_0^t h_i(u(s))ds, \quad (4.8)$$

where $h_i : R^+ \to R, i=1,2\cdots,m$. Suppose that

$$|h_i(u(t))| \leqslant a_i(t)|u^i(t)|, \quad (4.9)$$

where $a_i(t)$ is a nonnegative continuous function for $t \in R^+$ and $i = 1, 2, \cdots, m$. Let $u(t)$ be a solution of (4.8). Then

$$u(t) \leqslant k^{\frac{1}{m}}(t) + \frac{1}{m} k^{\frac{1}{m}-1}(t) e(t) \exp\Big(\sum_{i=1}^{m} \frac{i}{m} \int_0^{\alpha_i(t)} a_i(s) k^{\frac{i}{m}-1}(s) ds\Big) \text{ for } t \in R^+. \quad (4.10)$$

Proof. From (4.9), it is obvious that if $u(t)$ is a solution of (4.8), then

$$u^p(t) \leqslant k(t) + \sum_{i=1}^{p} \int_0^t a_i(s) u^i(s) ds.$$

From Theorem 2.1, we know that (4.10) holds. This completes the proof.

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