Mathematical
Inequalities
& Applications
Volume 9, Number 2 (2006), 269–272

ISOPERIMETRIC INEQUALITY IN TAXICAB GEOMETRY

İSMAIL KOCAYUSUFOĞLU

(*communicated by D. Bainov*)

Abstract. The history of the Isoperimetric Inequality goes back to 900 BC. Mainly, the Isoperimetric Theorem states that "*Among all planar shapes in Euclidean geometry, the circle is the best figure in the plane*".

In this paper, we proved the theorem for Taxicab geometry. Surprisingly (there is a reason), "*Among all planar shapes in Taxicab geometry, the (taxicab) square is the best figure in the (taxicab) plane*".

"*The circle is the most simple, and the most perfect figure.*" DANTE.

1. Introduction

The first geometry that any scientist will think of is the Euclidean geometry which is based on the *Euclidean metric*,

$$d_E(A,B) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

for $A=(x_1,y_1), B=(x_2,y_2)$ in R^2. In 1975, E. F. Krause has defined the *Taxicab geometry* by using the *taxicab metric*

$$d_T(A,B) = |x_1-x_2|+|y_1-y_2|$$

for $A=(x_1,y_1), B=(x_2,y_2)$ in R^2.

He mentioned in his book, "*What's taxicab geometry?*", that the taxicab geometry is a non-Euclidean geometry. First of all, it is easy to understand for anyone who has a little bit knowledge about Euclidean geometry. Secondly, the taxicab geometry has significant applications in the real-world and finally, it is very close to Euclidean geometry in its axiomatic structure [1].

One may ask "*What are the differences between Taxicab geometry and Euclidean geometry?*". Some of the differences are given in [2]. Mainly,

(1) The axiom of side-angle-side in Euclidean geometry fails in taxicab geometry.

(2) While Euclidean geometry is a good model of the natural world, taxicab geometry is a better model of urban world that people has built.

(3) While in Euclidean geometry $\pi_E = 3,14285$, in taxicab geometry $\pi_T = 4$.

(4) The graphs, in general, are different.

Mathematics subject classification (2000): 26D20, 26D99, 52A40.
Key words and phrases: Taxicab geometry, isoperimetric inequality.

2. The area in Taxicab geometry

Computing the area in taxicab geometry is based on a unit square. It is proved in [3] that the area formulas for a square, a rectangle and a triangle are the same as in Euclidean case. But, the area formula of the Taxicab circle with radius r is $\frac{1}{2} \cdot \pi_T \cdot r^2$. To see this, consider the taxicab circle with radius r (Figure 1). Clearly, the area of triangle A_1 is half of the area of the square with sides r. And the area of the taxicab circle, A_C, is 4 times the area of A_1. Thus,

$$\begin{aligned} A_C &= 4 \cdot A_1 \\ &= 4 \cdot (\frac{1}{2} r^2); \pi_T = 4 \\ &= \frac{1}{2} \cdot \pi_T \cdot r^2 \end{aligned}$$

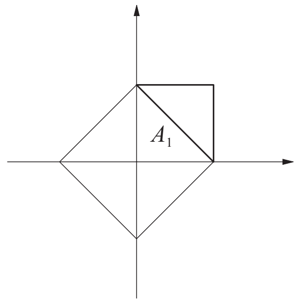

Figure 1.

3. The isoperimetric inequality

The isoperimetric inequality is not new. In fact, the history of the Isoperimetric Theorem goes back to 900 BC. An application is found in the story of Dido. Princess Dido was a daughter a Tyrian king. She arrived in North Africa, at a site later known as Carthage (Tunisia). She wanted to buy some land from the local ruler, King Jambas. They agreed that she could buy all the land she could enclose within a bull's hide. Consequently, Dido had the bull's hide cut into small strips and had the strips stitched together. Dido put the ribbon on the ground in a way that it would enclose the maximum area. So, Dido had to solve the isoperimetric theorem! [4]

The isoperimetric theorem states that "*Among all planar shapes with the same area, the circle has the shortest perimeter.*" This is equivalent to say that "*Among all planar shapes with the same perimeter, the circle has the largest area.*"

Both of the above statements can be expresed as a following inequality: Let L be the perimeter and A be tha area of the planar shape. Then,

$$\frac{L^2}{A} \geqslant 4\pi$$

which is known as the isoperimetric inequality (equality holds iff the planar shape is a circle).

Notice that the taxicab circle is a square in Euclidean geometry. So it is not suprising that the isoperimetric inequality in Taxicab geometry is true for the (taxicab) square.

THEOREM 1. (ISOPERIMETRIC THEOREM IN TAXICAB GEOMETRY) *Among all planar shapes with the same area in taxicab plane R_T^2, the (taxicab) square has the shortest perimeter.*

Proof. The proof will be in several steps. In first step, we will prove that $\frac{L_S^2}{A_S} = 4\pi_T$ for (taxicab) square. Then we will prove that for a rectangle $\frac{L_R^2}{A_R} \geqslant 4\pi_T$. Finally, in third step, we will prove that for any planar shape γ, $\frac{L_\gamma^2}{A_\gamma} \geqslant \frac{L_R^2}{A_R}$. □

Step 1. Let L_S be the perimeter and A_S be tha area of the square S with sides a. Then, $L_S = 4a$, and $A_S = a^2$. Hence,

$$\frac{L_S^2}{A_S} = \frac{(4a)^2}{a^2} = 16 = 4\pi_T$$

Step 2. Let L_R be the perimeter and A_R be tha area of the rectangle R with sides a, b. Then, $L_R = 2(a+b)$, and $A_R = ab$. Thus,

$$\begin{aligned} \frac{L_R^2}{A_R} &= \frac{(2(a+b))^2}{ab} \\ &= \frac{4(a+b)^2}{ab} \quad ; (a+b)^2 \geqslant 4ab \\ &\geqslant \frac{4.4ab}{ab} \\ &= 4\pi_T \end{aligned}$$

Step 3. Let L_γ be the perimeter and A_γ be tha area of the planar region γ. We can always choose a rectangle R such that γ stays in R (Figure 2). Obviously, $L_R \leqslant L_\gamma$ and $A_R \geqslant A_\gamma$. Thus,

$$\frac{L_\gamma^2}{A_\gamma} \geqslant \frac{L_R^2}{A_R}$$

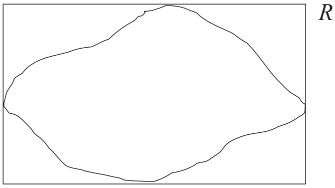

Figure 2.

As a result, we have

$$\frac{L_\gamma^2}{A_\gamma} \geqslant \frac{L_R^2}{A_R} \geqslant \frac{L_S^2}{A_S} = 4\pi_T$$

as desired.

Let us now prove as a problem that "when the areas of the taxicab circle and the square are equal, the perimeter of the square is less then the perimeter of the taxicab circle".

THEOREM 2. *Let the area of the taxicab square with sides a be equal to the area of the taxicab circle with radius r. Then the perimeter of the taxicab square is less then the perimeter of the taxicab circle.*

Proof. Since $A_S = a^2$, $A_C = \frac{1}{2}\pi_T r^2$, it follows that

$$A_S = A_C \Rightarrow a^2 = \frac{1}{2}\pi_T r^2.$$

Then

$$L_S = 4a = 4\sqrt{2}r < 8r = L_C$$

as claimed. □

REMARK 1. The above problem follows from the fact that for taxicab circle with radius a, it is clear that $L_C = 8a$ and $A_C = \frac{1}{2}\pi_T a^2 = 2a^2$. Hence,

$$\frac{L_C^2}{A_C} = \frac{(8a)^2}{2a^2} = 32 > 4\pi_T$$

REMARK 2. For the isoperimetric theorem, it is equivalent to say that "*Among all planar shapes with the same perimeter in taxicab plane* R_T^2, *the square has the largest area*".

The proof is similar and left to the reader.

REMARK 3. When the perimeters of the taxicab circle and the (taxicab) square are equal, the area of the (taxicab) square is greater then the area of taxicab circle. So the buildings are in (taxicab) square or (taxicab) rectangular shapes.

(Is it because the Taxicab geometry is a better model of urban world that people has built and the (taxicab) squre is the best model for isoperimetric inequality? Maybe!).

Acknowledgement. The author would like to thank Prof. David L. Johnson for his valuable suggestions on this problem.

REFERENCES

[1] E. F. KRAUSE, *Taxicab Geometry*, Addison-Wesley, Menlo Park, NJ, (1975).
[2] İ. KOCAYUSUFOĞLU, Z. AKCA AND C. EKICI, *The Inner-Product in Taxicab Geometry*, Mathematics, Mechanics and Informatics, (1998).
[3] İ. KOCAYUSUFOĞLU, *Some Area Problems in Taxicab Geometry*, Jour. of Inst. of Math and Comp. Sci., **12**, (1999), 95–98.
[4] J. BOGAERT, *Isoperimetric Geometry*, hPUMAS, ID 01_22_03_1, (2003).

(Received April 25, 2005)

İsmail Kocayusufoğlu
Osmangazi University
Department of Mathematics
26480 Eskisehir
Turkey
e-mail: ikyusuf@ogu.edu.tr