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ON RADIALLY SYMMETRIC SOLUTIONS OF SECOND AND HIGHER ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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(*communicated by V. Lakshmikantham*)

Abstract. Explicit radially symmetric entire and non-entire solutions are obtained for the equations of the form

$$\Delta^k u + P(r)f(u) = 0, \quad k \geqslant 1 \tag{1.1}$$

where $P(r)$ is a suitable function, Δ denotes n-dimensional Laplace operator and Δ^k is the k^{th} iterate of Δ. In particular, the cases

$$f(u) = \pm bu^{\frac{n+2k}{n-2k}}$$

where b is a positive constant, are considered. For $n = 2$, infinitely many entire solutions of

$$\Delta u + be^u = 0$$

and non-entire solutions of

$$\Delta u = be^u$$

are derived. Explicit solutions of some nonlinear Dirichlet and Neumann problems and some singular nonlinear ordinary differential equations are also determined. These results are consequence of a differential inequality or appropriately chosen form of the solution.

1. Introduction

Many results have appeared in the literature on the subject for the equation

$$\Delta u = f(u) \tag{1.2}$$

or, more generally, the differential inequality

$$\Delta u \geqslant f(u). \tag{1.3}$$

It is well known that the equation (1.2) has no entire solution, i.e. a C^2 function which satisfies (1.2) in all of Euclidean n-space $\mathbb{R}^n$, under various conditions on f. For instance, if f is continuous, positive and increasing and if f satisfies the condition

$$\int_0^\infty \left(\int_0^t f(s)ds\right)^{-\frac{1}{2}} dt < \infty$$

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then (1.3) has no entire solutions [6, 9]. In particular, $\Delta u = e^u$ has no entire solutions for all $n \geqslant 2$ [13, 16]. However, Walter [14] obtained the startling result that when $n \geqslant 3$ and $k \geqslant 2$, there do exist radially symmetric entire solutions of (1.1) when $f(u) = -e^u$ and $P(r) = 1$.

Nehari [8], used a result of Osserman [9] and obtained explicit upper and lower radial bounds for solutions of (1.3) for a certain class of functions f. In addition, he proved the nonexistence of entire solutions for other types of functions f.

In [5], by applying Nehari's technique, explicit radially symmetric solutions for some nonlinear equations of the form

$$\Delta u = P(r)f(u) \tag{1.4}$$

were obtained. In particular, it was found that the function

$$u = \left(\frac{R\sqrt{n(n-2)}}{\sqrt{c}(R^2 - r^2)}\right)^{\frac{n-2}{2}}, n \geqslant 3, c > 0$$

is a non-entire solution of

$$\Delta u = cu^{\frac{n+2}{n-2}}$$

in the ball $B_R(0)$ of radius R and center at the origin and that the function

$$u = \frac{1}{r^2}\left[\frac{16\lambda(\lambda + \frac{1}{2})}{(R^2 - r^2)^2}\right]^{\lambda}, \lambda > 0$$

is a solution of

$$\Delta u = r^{2(1+\frac{1}{\lambda})}u^{(1+\frac{1}{\lambda})}$$

in $B_R(0) - \{0\}$ when $n = 4$. Nonexistence of entire solutions was also proved in the case of

$$\Delta u \geqslant P(x)f(u)$$

where $P(x) = P(x_1, x_2, \dots, x_n) \geqslant 0$ and such that

$$|\operatorname{grad} P|^2 - P\Delta P \geqslant 0.$$

In [10] explicit entire and non-entire solutions of equations

$$\Delta^k u = \pm\beta e^u \quad (\beta > 0)$$

were found when $k \geqslant 1$.

Here, we follow Nehari's method and obtain an inequality which leads to explicit solutions of nonlinear equations of the form

$$\Delta^k u + P(r)f(u) = 0.$$

Explicit entire and non-entire solutions are also derived for the equation (1.1) where $f(u) = \pm\beta u^{\frac{n+2k}{n-2k}}$. These results lead to the solutions of some nonlinear singular ordinary differential equations as well as the explicit solutions of some nonlinear Dirichlet and Neumann problems.

2. Explicit solutions

Let $B_R(0)$ denote the open ball of radius R and center at the origin in Euclidean n-space and let r denote the distance from the origin to an arbitrary point $x=(x_1,x_2,\dots,x_n)$ in $B_R(0)$. In this section we derive explicit entire and non-entire radial solutions of some equations of the form

$$\Delta u + P(r)f(u) = 0 \tag{2.1}$$

in certain Euclidean spaces $\mathbb{R}^n$.

First, we prove the following differential inequality.

THEOREM 2.1. *Let f be a positive C^1 function on $(0,\infty)$ for which*

$$\int_\gamma^\infty \frac{dt}{f(t)} \text{ exists }, (\gamma>0), f'(\gamma)\int_\gamma^\infty \frac{dt}{f(t)} \leqslant 1+\lambda. \ (\lambda \geqslant 0) \tag{2.2}$$

If, for a positive constant $k\geqslant 1, u(r)$ is defined by

$$\frac{c(R^{2k}+r^{2k})^2}{R^{2k}} = \frac{1}{P(r)}\int_u^\infty \frac{dt}{f(t)}, \quad c = \text{ constant} \tag{2.3}$$

where R is any positive constant and $P(r)$ is a positive C^2 function on $(0,\infty)$, then u satisfies the differential inequality

$$\begin{aligned}\Delta u \leqslant P(r)f(u)\Bigg[&-4kc\left\{(n+2k-2)r^{2k-2}+(n-2-4k\lambda)\frac{r^{4k-2}}{R^{2k}}\right\}\\ &+\frac{16kcr^{2k}(R^{2k}+r^{2k})\dot{P}(r)}{P(r)R^{2k}}\lambda\\ &-\frac{2c(R^{2k}+r^{2k})^2}{R^{2k}}\left\{\frac{2r^2P(r)\ddot{P}(r)+nP(r)\dot{P}(r)-2r^2\dot{P}^2(r)(1+\lambda)}{P^2(r)}\right\}\Bigg]\end{aligned} \tag{2.4}$$

where the dot denotes differentiation with respect to $r^2 (r<R)$.

Proof. To derive (2.4), let x denote one of the variables x_j and differentiate (2.3) twice with respect to x. This yields

$$\frac{4kcx(R^{2k}+r^{2k})r^{2k-2}}{R^{2k}} = -\frac{u_x}{P(r)f(u)} - \frac{2x\dot{P}(r)}{P^2(r)}\int_u^\infty \frac{dt}{f(t)} \tag{2.5}$$

$$\begin{aligned}&\frac{4kc(R^{2k}+r^{2k})r^{2k-2}+8kcx^2(k-1)(R^{2k}+r^{2k})r^{2k-4}+8k^2cx^2r^{4k-4}}{R^{2k}}\\ &= -\frac{u_{xx}}{P(r)f(u)}+\frac{4x\dot{P}(r)u_x}{P^2(r)f(u)}+\frac{f'(u)u_x^2}{P(r)f^2(u)}\\ &\quad+\left\{-\frac{2\dot{P}(r)}{P^2(r)}+\frac{8x^2\dot{P}^2(r)}{P^3(r)}-\frac{4x^2\ddot{P}(r)}{P^2(r)}\right\}\int_u^\infty\frac{dt}{f(t)}\end{aligned} \tag{2.6}$$

Now using (2.3) and (2.5), we can write (2.6) as

$$\begin{aligned}\frac{u_{xx}}{P(r)f(u)} =& -4kcr^{2k-2} - 8kc(k-1)x^2r^{2k-4}\\ &- \left(\frac{4kcr^{4k-2} + 8kc(k-1)x^2r^{4k-4} + 8k^2cx^2r^{4k-4}}{R^{2k}}\right)\\ &- \frac{16kcx^2r^{2k-2}(R^{2k}+r^{2k})\dot{P}(r)}{P(r)R^{2k}}\\ &+ \frac{4c^2x^2}{R^{4k}}f'(u)P(r)(R^{2k}+r^{2k})^2\left\{2kr^{2k-2} + \frac{\dot{P}(r)}{P(r)}(R^{2k}+r^{2k})\right\}^2\\ &- \frac{2}{P^2(r)}\{2x^2\ddot{P}(r) + \dot{P}(r)\}\int_u^\infty \frac{dt}{f(t)}.\end{aligned}$$

Summing over the variables x_j, we obtain

$$\begin{aligned}\frac{\Delta u}{P(r)f(u)} =& -4kc(n+2k-2)r^{2k-2} - 4kc(n+2(k-1)+2k^2)\frac{r^{4k-2}}{R^{2k}}\\ &- \frac{16kcr^{2k}(R^{2k}+r^{2k})\dot{P}(r)}{R^{2k}P(r)}\\ &+ \frac{4c^2r^2}{R^{4k}}f'(u)P(r)(R^{2k}+r^{2k})^2\left\{2kr^{2k-2} + \frac{\dot{P}(r)}{P(r)}(R^{2k}+r^{2k})\right\}^2\\ &- \frac{2}{P^2(r)}\{2r^2\ddot{P}(r) + n\dot{P}(r)\}\int_u^\infty \frac{dt}{f(t)}.\end{aligned}$$

Finally, by (2.2) and (2.3), it reduces to

$$\begin{aligned}\frac{\Delta u}{P(r)f(u)} \leqslant& -4kc(n+2k-2)r^{2k-2} - \{4kc(n-2-4k\lambda)\}\frac{r^{4k-2}}{R^{2k}}\\ &+ \frac{16kcr^{2k}(R^{2k}+r^{2k})\dot{P}(r)}{R^{2k}P(r)}\lambda\\ &- \frac{2c(R^{2k}+r^{2k})^2}{R^{2k}}\left\{\frac{2r^2P(r)\ddot{P}(r) + nP(r)\dot{P}(r) - 2r^2\dot{P}^2(r)(1+\lambda)}{P^2(r)}\right\}.\end{aligned}$$

which completes the proof of (2.4).

It is worth noting that the inequality (2.2) is always satisfied, for $\lambda = 0$, if $\ln f(t)$ is a convex function of t. We also note that the equality holds in (2.4) if equality holds in (2.2). This will occur if $f(t) = t^{1+\frac{1}{\lambda}}$ when $\lambda > 0$ and, for $\lambda = 0$, if $f(t) = e^t$. We now distinguish between two cases depending on λ.

Case I: $\lambda > 0$.

In this case equality holds in (2.2) and (2.4) if

$$f(t) = t^{1+\frac{1}{\lambda}}. \tag{2.7}$$

Letting $P(r) = d$, a positive constant, then (2.4) reduces to

$$\Delta u = \left[-4kc(n+2k-2)r^{2k-2} - 4kc(n-2-4k\lambda)\frac{r^{4k-2}}{R^{2k}}\right] d \;\; u^{1+\frac{1}{\lambda}}.$$

Now if $\lambda = \frac{n-2}{4k}$ and $c = \frac{1}{4k(n+2k-2)}$, then

$$\Delta u + dr^{2k-2}u^{1+\frac{4k}{n-2}} = 0, \quad n \geqslant 3. \tag{2.8}$$

The solution of (2.8) follows from

$$\frac{(R^{2k}+r^{2k})^2}{4k(n+2k-2)R^{2k}} = \frac{1}{d}\int_u^\infty \frac{dt}{t^{1+\frac{4k}{n-2}}}$$

i.e.

$$\frac{d(R^{2k}+r^{2k})^2}{(n-2)(n+2k-2)R^{2k}} = u^{-\frac{4k}{n-2}}.$$

Consequently, we have the non-trivial solution of (2.8) in $\mathbb{R}^n$ is

$$u = \left\{\frac{(n-2)(n+2k-2)R^{2k}}{d(R^{2k}+r^{2k})^2}\right\}^{\frac{n-2}{4k}}, \quad n \geqslant 3 \tag{2.9}$$

where R is any positive constant.

REMARK 1. If $k = 1$ in (2.9) then we have

$$u = \left[\frac{R\sqrt{n(n-2)}}{\sqrt{d}(R^2+r^2)}\right]^{\frac{n-2}{2}}, \quad n \geqslant 3 \tag{2.9$'$}$$

the well-known entire solution of

$$\Delta u + du^{\frac{n+2}{n-2}} = 0 \tag{2.8$'$}$$

which was also obtained in [10] by a different method.

If we let $P(r) = r^{\frac{n-2}{\lambda}}$, then

$$2r^2\ddot{P}(r)P(r) + nP(r)\dot{P}(r) - 2r^2\dot{P}^2(r)(1+\lambda) = 0$$

and (2.4) becomes

$$\Delta u = \left[-4kc(-n+2+2k)r^{2k-2} + 4kc(n-2+4k\lambda)\frac{r^{4k-2}}{R^{2k}}\right] r^{\frac{n-2}{\lambda}} u^{1+\frac{1}{\lambda}}.$$

For $n = 2k+2$ where k is an integer $\geqslant 1$ and $c = \frac{1}{16k^2}$, we get

$$\Delta u = \left(\lambda + \frac{1}{2}\right)\frac{r^{2k(2+\frac{1}{\lambda})-2}}{R^{2k}} u^{1+\frac{1}{\lambda}}.$$

Now by the transformation

$$V = \frac{(\lambda+\frac{1}{2})^\lambda}{R^{2k\lambda}} u$$

we obtain the nonlinear equation

$$\Delta V = r^{2k(2+\frac{1}{\lambda})-2} \;\; V^{1+\frac{1}{\lambda}}, \tag{2.10}$$

and the solution of (2.10) is obtained by determining u from

$$\frac{(R^{2k}+r^{2k})^2}{16k^2R^{2k}} = \frac{1}{r^{\frac{2k}{\lambda}}}\int_u^\infty \frac{dt}{t^{1+\frac{1}{\lambda}}}.$$

Since

$$u = \left[\frac{16k^2\lambda R^{2k}}{r^{\frac{2k}{\lambda}}(R^{2k}+r^{2k})^2}\right]^\lambda$$

we have

$$V = \frac{1}{r^{2k}}\left[\frac{16k^2\lambda(\lambda+\frac{1}{2})}{(R^{2k}+r^{2k})^2}\right]^\lambda \tag{2.11}$$

is the solution of (2.10) in $\mathbb{R}^n - \{0\}$ for $n = 2k+2$.

Case II: $\lambda = 0$.

In this case equality holds in (2.2) as well as in (2.4) if

$$f(t) = e^t$$

and then for (2.4), we have

$$\begin{aligned}\frac{\Delta u}{P(r)e^u} = &\left[-4kc(n+2k-2)r^{2k-2} - 4kc(n-2)\frac{r^{4k-2}}{R^{2k}}\right] \\ &- \frac{2c(R^{2k}+r^{2k})^2}{R^{2k}}\left\{\frac{2r^2P(r)\ddot{P}(r)+nP(r)\dot{P}(r)-2r^2\dot{P}^2(r)}{P^2(r)}\right\}\end{aligned} \tag{2.12}$$

For $P(r) = b$, a positive constant, we write (2.12) as

$$\Delta u = -4kc\left[(n+2k-2)r^{2k-2} + (n-2)\frac{r^{4k-2}}{R^{2k}}\right]be^u.$$

Now if $n = 2$ and $c = \frac{1}{8k^2}$, then

$$\Delta u = -br^{2k-2}e^u. \tag{2.13}$$

The solution of (2.13) follows from

$$\frac{(R^{2k}+r^{2k})^2}{8k^2R^{2k}} = \frac{1}{b}\int_u^\infty \frac{dt}{e^t}$$

i.e.

$$\frac{b(R^{2k}+r^{2k})^2}{8k^2R^{2k}} = e^{-u}.$$

Hence, the entire solution of (2.13) in $\mathbb{R}^n$ is

$$u = 2\ln\frac{\sqrt{8}kR^k}{\sqrt{b}(R^{2k}+r^{2k})}. \tag{2.14}$$

for $n = 2$.

Consequently if $u(0,0) = ln\alpha$, then $R = \left(\frac{8k^2}{b\alpha}\right)^{\frac{1}{2k}}$ where α is a positive constant.

REMARK 2. If k, $b = 1$ in (2.13) and (2.14) respectively then we get the well-known solution

$$u = 2\ln \frac{\sqrt{8}R}{R^2 + r^2}$$

of

$$\Delta u + e^u = 0.$$

REMARK 3. Writing (2.13) as

$$\Delta u + be^{u+(2k-2)\ln r} = 0$$

and letting

$$V = u + (2k-2)\ln r$$

we get

$$\Delta V + be^V = 0 \tag{2.13'}$$

and that its solution is

$$V = 2\ln \frac{\sqrt{8}kr^{k-1}R^k}{\sqrt{b}(R^{2k} + r^{2k})}. \tag{2.14'}$$

for $n = 2$.

We observe that the nonlinear equation (2.13′) has infinitely many entire solutions (2.14′) in $\mathbb{R}^n$ for $n = 2$ since $k \geqslant 1$.

REMARK 4. We note that following the method of complex variables it can be easily checked that the function

$$u = 2\ln\left(\frac{\sqrt{8}|f'(z)|}{\sqrt{b}(1 + |f(z)|^2)}\right) \tag{2.14''}$$

if $f(z)$ is regular in a domain and $f'(z) \neq 0$, is also a solution of (2.13′).

REMARK 5. Further, we note that there are no known explicit solutions of

$$\Delta u = e^u$$

when $n \geqslant 3$. But in the case of (2.13′), it can be checked that the function

$$u = \ln\left(\frac{2n-4}{br^2}\right) \tag{2.14'''}$$

is a non-entire solution when $n \geqslant 3$.

Finally, if we let $P(r) = r^{2\delta}$, where δ is a positive arbitrary constant, we have

$$2r^2P(r)\ddot{P}(r) + nP(r)\dot{P}(r) - 2r^2\dot{P}^2(r) = 0$$

and (2.12) reduces to

$$\Delta u = \left[-4kc(n + 2k - 2)r^{2k-2} - 4kc(n-2)\frac{r^{4k-2}}{R^{2k}}\right] r^{2\delta}e^u.$$

If $n = 2$ and $c = \frac{1}{8k^2}$, then

$$\Delta u + r^{2\delta+2k-2}e^u = 0 \tag{2.15}$$

and u is given by

$$\frac{(R^{2k}+r^{2k})^2}{8k^2R^{2k}} = \frac{1}{r^{2\delta}}\int_u^\infty \frac{dt}{e^t}$$

i.e.

$$u = 2\ln\left(\frac{\sqrt{8}kR^k}{r^\delta(R^{2k}+r^{2k})}\right) \tag{2.16}$$

is the non-entire solution of (2.15) in $\mathbb{R}^n - \{0\}$ for $n=2$.

REMARK 6. Writing (2.15) as

$$\Delta u + e^{u+(2k+2\delta-2)\ln r} = 0$$

and letting $V = u + (2k+2\delta-2)\ln r$ we again get infinitely many entire solutions of

$$\Delta V + e^V = 0$$

in the form

$$V = 2\ln\left(\frac{\sqrt{8}kr^{k-1}R^k}{r^{2k}+R^{2k}}\right). \tag{2.16'}$$

which agree with $(2.14')$ if $b=1$ there.

More generally, if $\ln P(r)$ is harmonic and $a>0$ is a constant, then the infinitely many entire solutions of

$$\Delta u + P(r)e^{au} = 0, \quad n=2 \tag{2.17}$$

are given by

$$u = \frac{2}{a}\ln\left(\frac{\sqrt{8}kr^{k-1}R^k}{\sqrt{aP(r)}(R^{2k}+r^{2k})}\right) \tag{2.18}$$

since we can write (2.17) as

$$\Delta u + e^{au+\ln P(r)} = 0$$

and let

$$V = au + \ln P(r)$$

and then use $(2.13')$.

In a similar manner we obtain non-entire solutions of equations of the form

$$\Delta u = P(r)f(u)$$

as a consequence of the following inequality:

THEOREM 2.2. *Let f be a positive C^1 function on $(0,\infty)$ which satisfies the conditions (2.2). If for a positive constant $k \geqslant 1$, $P(r)$ is defined by*

$$\frac{c(R^{2k}-r^{2k})^2}{R^{2k}} = \frac{1}{P(r)}\int_u^\infty \frac{dt}{f(t)}, \quad c = \text{ constant} \tag{2.19}$$

where $P(r)$ is a positive C^2 function on $(0,\infty)$, then u satisfies the differential inequality

$$\begin{aligned}\Delta u \leqslant P(r)f(u)\Big[&r^{2k-2}4kc(n+2k-2)\\ &-\frac{r^{4k-2}}{R^{2k}}\{4kc(n-2-4k\lambda)\}-\frac{16kcr^{2k}\dot{P}(r)}{P(r)R^{2k}}(R^{2k}-r^{2k})\lambda\\ &-\frac{2c(R^{2k}-r^{2k})^2}{R^{2k}}\left\{\frac{2r^2\ddot{P}(r)+n\dot{P}(r)}{P(r)}-\frac{2r^2\dot{P}^2(r)(1+\lambda)}{P^2(r)}\right\}\Big].\end{aligned} \tag{2.20}$$

Proof is exactly the same as that of Theorem 2.1. As before, we consider the cases depending on λ.

Case I: $\lambda > 0$

Again, the equality holds in (2.2) and (2.20) if

$$f(t) = t^{1+\frac{1}{\lambda}}.$$

Now for $P(r) = \alpha$, a positive constant, $\lambda = \frac{n-2}{4k}$ and $c = \frac{1}{4k(n+2k-2)}$, (2.20) reduces to

$$\Delta u = \alpha r^{2k-2} u^{1+\frac{4k}{n-2}}, \quad n \geqslant 3 \tag{2.21}$$

where the non-entire solution of (2.21) in $\mathbb{R}^n$ is

$$u = \left\{\frac{(n-2)(n-2+2k)R^{2k}}{\alpha(R^{2k}-r^{2k})^2}\right\}^{\frac{n-2}{4k}}, \quad n \geqslant 3. \tag{2.22}$$

REMARK 7. If $k = 1$, then (2.10) and (2.11) in [5] are special cases of (2.21) and (2.22) respectively in $B_R(0)$.

If we take $P(r) = r^{\frac{n-2}{\lambda}}$ then for $n = 2k+2$ where k is an integer $\geqslant 1$ and $c = \frac{1}{16k^2}$ (2.20) reduces to

$$\Delta u = \left(\lambda + \frac{1}{2}\right) r^{2k(2+\frac{1}{\lambda})-2} \ \frac{u^{1+\frac{1}{\lambda}}}{R^{2k}}.$$

Using the change of the variable

$$u = \frac{R^{2k\lambda} V}{(\lambda + \frac{1}{2})^{\lambda}}, \tag{2.23}$$

we get the nonlinear equation

$$\Delta V = r^{2k(2+\frac{1}{\lambda})-2} \ V^{1+\frac{1}{\lambda}}. \tag{2.24}$$

Since

$$u = \left[\frac{16\lambda k^2 R^{2k}}{r^{\frac{2k}{\lambda}}(R^{2k}-r^{2k})^2}\right]^{\lambda}$$

the solution of (2.24) in $\mathbb{R}^n - \{0\}$ is

$$V = \frac{1}{r^{2k}}\left[\frac{16\lambda k^2(\lambda+\frac{1}{2})}{(R^{2k}-r^{2k})^2}\right]^{\lambda}. \tag{2.25}$$

for $n = 2k+2$.

REMARK 8. For $k = 1$, (2.12) and (2.13) in [5] are special cases of (2.24) and (2.25) respectively in $B_R(0) - \{0\}$ for $n = 4$.

Case II: $\lambda = 0$.

In this case equality will hold in (2.2) and (2.20) if

$$f(t) = e^t,$$

and then for (2.20), we have

$$\Delta u = P(r)e^u \left[\left\{ r^{2k-2}(n+2k-2)4kc - \frac{r^{4k-2}}{R^{2k}}(4kc(n-2)) \right\} - \frac{2c(R^{2k}-r^{2k})^2}{R^{2k}} \left\{ \frac{2r^2\ddot{P}(r) + n\dot{P}(r)}{P(r)} - \frac{2r^2\dot{P}^2(r)}{P^2(r)} \right\} \right] \tag{2.26}$$

If, $P(r) = \beta$, where β is an arbitrary positive constant, $n = 2$ and $c = \frac{1}{8k^2}$ then (2.26) reduces to

$$\Delta u = \beta r^{2k-2} e^u \tag{2.27}$$

and u is given by

$$\frac{R^{2k} - r^{2k}}{8k^2 R^{2k}} = \frac{1}{\beta} \int_u^\infty \frac{dt}{e^t}$$

i.e.

$$u = 2\ln\left(\frac{\sqrt{8}kR^k}{\sqrt{\beta}(R^{2k} - r^{2k})} \right) \tag{2.28}$$

is the solution of (2.27) in $\mathbb{R}^n$ for $n = 2$.

REMARK 9. If $k = 1$ and $\beta = 1$ in (2.27) and (2.28) then we get the well-known solution of $\Delta u = e^u$ is

$$u = 2\ln \frac{\sqrt{8}R}{R^2 - r^2}$$

for $n = 2$ in $B_R(0)$.

REMARK 10. Writing (2.27) as

$$\Delta u = \beta e^{u + (2k-2)\ln r}$$

and letting

$$V = u + (2k-2)\ln r$$

we get

$$\Delta V = \beta e^V \tag{2.29}$$

where the solution V is given by

$$V = 2\ln\left(\frac{\sqrt{8}kr^{k-1}R^k}{\sqrt{\beta}(R^{2k} - r^{2k})} \right). \tag{2.30}$$

Thus, (2.29) has infinitely many non-entire solutions given by (2.30) for $k \geqslant 1$ and $n = 2$.

Now if we let $P(r) = r^{2\gamma}$, where γ is a positive arbitrary constant, then (2.20) reduces to

$$\Delta u = \left[r^{2k-2}4kc(n+2k-2) - \frac{r^{4k-2}}{R^{2k}}(4kc(n-2)) \right] r^{2\gamma} e^u.$$

Further, if $n = 2$ and $c = \frac{1}{8k^2}$, we get

$$\Delta u = r^{2\gamma+2k-2} e^u \tag{2.31}$$

and u is given by

$$u = 2\ln\left(\frac{\sqrt{8}kR^k}{r^\gamma(R^{2k} - r^{2k})}\right) \tag{2.32}$$

in $\mathbb{R}^n - \{0\}$.

REMARK 11. If $k = 1$ then (2.15) and (2.16) in [5] are special cases of (2.31) and (2.32) respectively in $B_R(0) - \{0\}$.

REMARK 12. Further, we note that (2.31) can be written as

$$\Delta u = e^{u+(2\gamma+2k-2)\ln r}.$$

Now substituting

$$V = u + (2\gamma + 2k - 2)\ln r$$

we get

$$\Delta V = e^V \tag{2.33}$$

where, from (2.32),

$$V = 2\ln\left(\frac{\sqrt{8}kr^{k-1}R^k}{(R^{2k} - r^{2k})}\right). \tag{2.34}$$

In this case also we get infinitely many non-entire solutions for (2.33) which agree with (2.30) if $\beta = 1$.

REMARK 13. It can be checked that

$$V = 2\ln\frac{2}{r(D - \sqrt{2}\ln r)}$$

where D is an arbitrary constant is also a non-entire solution of (2.33).

More generally, if $\ln P(r)$ is harmonic, then the infinitely many non-entire solutions of

$$\Delta u = P(r)e^{au}, \quad n = 2 \tag{2.35}$$

are given by

$$u = \frac{2}{a}\ln\left(\frac{\sqrt{8}kr^{k-1}R^k}{\sqrt{aP(r)}(R^{2k} - r^{2k})}\right). \tag{2.36}$$

REMARK 14. If k, $a = 1$ then (2.17) and (2.18) in [5] are special cases of (2.35) and (2.36) respectively.

We now determine some explicit radial solutions of

$$\Delta^k u \pm bu^{\frac{n+2k}{n-2k}} = 0 \quad (k \geqslant 2)$$

which are not entire solutions as well as some which are entire solutions in certain Euclidean spaces $\mathbb{R}^n$.

First, we have the following Lemma:

LEMMA 2.1.

$$\Delta^k(r^{-m}) = \frac{m(m+2)(m+4)\ldots(m+(2k-2))(m+2-n)(m+4-n)\ldots(m+2k-n)}{r^{m+2k}},$$

which can be easily proved by the method of Induction. As an immediate consequence we have the following theorem.

THEOREM 2.3. *The radial function*

$$u = \left\{ \frac{m(m+2)(m+4)\dots(m+(2k-2))(m+2-n)(m+4-n)\dots(m+2k-n)}{b} \right\}^{\frac{m}{2k}} \frac{1}{r^m} \tag{2.37}$$

is a positive non-entire solution of

$$\Delta^k u = bu^{\frac{m+2k}{m}}$$

where $k \geqslant 1$ *is an integer,* b *and* m *are positive constants and that* $m+2>n$*.*

Proof. Let

$$u = \frac{A_m}{r^m}$$

where A_m is a constant to be determined. It follows by the Lemma 2.1 that

$$\Delta^k u = A_m \frac{m(m+2)(m+4)\dots(m+(2k-2))(m+2-n)(m+4-n)\dots(m+2k-n)}{r^{m+2k}}.$$

If we choose

$$A_m = \left\{ \frac{m(m+2)(m+4)\dots(m+(2k-2))(m+2-n)(m+4-n)\dots(m+2k-n)}{b} \right\}^{\frac{m}{2k}}$$

then, we get

$$\Delta^k u = \left(b \left\{ \frac{m(m+2)(m+4)\dots(m+(2k-2))(m+2-n)(m+4-n)\dots(m+2k-n)}{b} \right\}^{\frac{m}{2k}} \frac{1}{r^m} \right)^{\frac{m+2k}{m}}$$
$$= bu^{\frac{m+2k}{m}}.$$

In particular, from Theorem 2.3, the function

$$u = \left\{ \frac{m(m+2)(m+2-n)(m+4-n)}{b} \right\}^{\frac{m}{4}} \cdot \frac{1}{r^m}$$

is a positive non-entire solution of

$$\Delta^2 u = bu^{\frac{m+4}{m}}$$

where $m+2>n$.

THEOREM 2.4. *The radial function*

$$u = \left[\frac{R^k \sqrt{(n-2k)(n-2(k-1))(n-2(k-2))\dots(n+2(k-2))(n+2(k-1))}}{\sqrt{b}(R^2-r^2)^k} \right]^{\frac{n-2k}{2k}} \tag{2.38}$$

is a non-entire solution of

$$\Delta^k u = bu^{\frac{n+2k}{n-2k}}, (b>0 \text{ is a constant}) \tag{2.39}$$

in $B_R(0)$ *in* $\mathbb{R}^n$ *when* $n \geqslant 2k+1$*.*

Proof. To prove the result we use an induction argument. First, we take u to be of the form

$$u = \frac{a}{(R^2 - r^2)^c}$$

where a and c are positive constants to be determined. Then, we compute

$$\Delta u = \frac{2ca}{(R^2 - r^2)^{c+2}}[nR^2 + r^2(2(c+1) - n)]$$

$$\Delta^2 u = \frac{4c(c+1)a}{(R^2 - r^2)^{c+4}} \big[n(n+2)R^4 + 2(n+2)(2(c+2) - n)R^2 r^2 + \{(2(c+2) - n)(2(c+2) - (n+2))\}r^4\big]$$

$$\Delta^3 u = \frac{8c(c+1)(c+2)a}{(R^2 - r^2)^{c+6}} \big[n(n+2)(n+4)R^6 + 3(n+2)(n+4)(2(c+3) - n)R^4 r^2 + 3(n+4)(2(c+3) - n)(2(c+3) - (n+2))R^2 r^4 + (2(c+3) - n)(2(c+3) - (n+2))(2(c+3) - (n+4))r^6\big]$$

$$\Delta^4 u = \frac{16c(c+1)(c+2)(c+3)a}{(R^2 - r^2)^{c+8}} \big[n(n+2)(n+4)(n+6)R^8 + 4(n+2)(n+4)(n+6)(2(c+4) - n)R^6 r^2 + 6(n+4)(n+6)(2(c+4) - n)(2(c+4) - (n+2))R^4 r^4 + 4(n+6)(2(c+4) - n)(2(c+4) - (n+2))(2(c+4) - (n+4))R^2 r^6 + (2(c+4) - n)(2(c+4) - (n+2))(2(c+4) - (n+4))(2(c+4) - (n+6))r^8\big]$$

Now, in view of the pattern that Δu, $\Delta^2 u$, $\Delta^3 u$ and $\Delta^4 u$ appear to follow, we have the induction hypothesis

$$\Delta^k u = \frac{2^k c(c+1)...(c+k-1)a}{(R^2 - r^2)^{c+2k}} \Bigg[\binom{k}{o}\{(n+2(k-1))(n+2(k-2)) \dots (n+4)(n+2)n\}R^{2k} + \binom{k}{1}\{(n+2(k-1))(n+2(k-2))...(n+4)(n+2)\}\{(2(c+k)-n)\}R^{2(k-1)}r^2 + \binom{k}{2}\{(n+2(k-1))(n+2(k-2))...(n+6)(n+4)\}\times \times\{(2(c+k)-n)(2(c+k)-(n+2))\}R^{2(k-2)}r^4 + \cdots + \binom{k}{p}\{(n+2(k-1))...(n+2p)\}\times \times\{(2(c+k)-n)...(2(c+k)-(n+2(p-1))\}R^{2(k-p)}r^{2p} + \cdots + \binom{k}{k}\{(2(c+k)-n)(2(c+k)-(n+2))...(2c-(n-2))\}r^{2k}\Bigg]$$

where $\binom{k}{p}$ denotes the binomial coefficient and $p = 1,2,\ldots,(k-1)$. Choosing $c = \frac{n-2k}{2}$ and

$$a = \left\{ \frac{R^{2k}(n-2k)(n-2(k-1))\ldots(n-2)n(n+2)\ldots(n+2(k-1))}{b} \right\}^{\frac{(n+2k)}{4k}}$$

we get

$$\Delta^k u = b\left(\frac{R^k\sqrt{(n-2k)(n-2(k-1))\ldots(n-2)n(n+2)\ldots(n+2(k-1))}}{\sqrt{b}(R^2-r^2)^k} \right)^{\frac{(n+2k)}{2k}}$$
$$= bu^{\frac{n+2k}{n-2k}}$$

where

$$u = \left(\frac{R^k\sqrt{(n-2k)(n-2(k-1))\ldots(n-2)n(n+2)\ldots(n+2(k-1))}}{\sqrt{b}(R^2-r^2)^k} \right)^{\frac{n-2k}{2k}}.$$

In particular, the function

$$u = \left(\frac{R^2\sqrt{(n-4)(n-2)n(n+2)}}{\sqrt{b}(R^2-r^2)^2} \right)^{\frac{n-4}{4}}$$

solves

$$\Delta^2 u = bu^{\frac{n+4}{n-4}} \quad (n \geqslant 5)$$

whereas the solution of

$$\Delta^3 u = bu^{\frac{n+6}{n-6}} \quad (n \geqslant 7)$$

is given by

$$u = \left(\frac{R^3\sqrt{(n-6)(n-4)(n-2)n(n+2)(n+4)}}{\sqrt{b}(R^2-r^2)^3} \right)^{\frac{n-6}{6}}$$

In the same way, we obtain an entire solution to the equation

$$\Delta^k u = (-1)^k bu^{\frac{n+2k}{n-2k}}. \tag{2.40}$$

THEOREM 2.5. *The radial function*

$$u = \left(\frac{R^k\sqrt{(n-2k)(n-2(k-1))\ldots(n-2)n(n+2)\ldots(n+2(k-1))}}{\sqrt{b}(R^2+r^2)^k} \right)^{\frac{n-2k}{2k}} \tag{2.41}$$

is an entire solution of (2.40) *in* $\mathbb{R}^n$ *if* $n > 2k$.

Proof. Again, the result follows by an induction argument. Suppose u to be of the form

$$u = \frac{a}{(R^2 + r^2)^c}$$

where c and a are positive constants to be determined. We compute

$$\Delta u = -\frac{2ca}{(R^2 + r^2)^{c+2}}[nR^2 + r^2(n - 2(c+1))]$$

$$\Delta^2 u = \frac{2^2 c(c+1)a}{(R^2 + r^2)^{c+4}}[n(n+2)R^4 + 2(n+2)(n-2(c+2))R^2 r^2 + \{(n-2(c+2))((n+2)-2(c+2))\}r^4]$$

$$\Delta^3 u = -\frac{2^3 c(c+1)(c+2)a}{(R^2 + r^2)^{c+6}}[n(n+2)(n+4)R^6 + 3(n+2)(n+4)(n-2(c+3))R^4 r^2 + 3(n+4)(n-2(c+3))((n+2)-2(c+3))R^2 r^4 + (n-2(c+3))((n+2)-2(c+3))((n+4)-2(c+3))r^6].$$

Again, based on the pattern that Δu, $\Delta^2 u$ and $\Delta^3 u$ appear to follow, we have the induction hypothesis

$$\begin{aligned}\Delta^k u = \frac{(-1)^k 2^k c(c+1)...(c+k-1)a}{(R^2+r^2)^{c+2k}} \Bigg[&\binom{k}{0}\{(n+2(k-1))(n+2(k-2))...(n+2)n\}R^{2k} \\ &+ \binom{k}{1}\{(n+2(k-1))(n+2(k-2))...(n+4)(n+2)\}(n-2(c+k))R^{2(k-1)}r^2 \\ &+ \binom{k}{2}\{(n+2(k-1))(n+2(k-2))...(n+6)(n+4)\}(n-2(c+k))((n+2)-2(c+k))R^{2(k-2)}r^4 \\ &+ \cdots + \binom{k}{p}\{(n+2(k-1))(n+2(k-2))...(n+2p)\}\times \\ &\times (n-2(c+k))...((n+2(p-1))-2(c+k))R^{2(k-p)}r^{2p} \\ &+ \cdots + \binom{k}{k}\{(n-2(c+k))((n+2)-2(c+k))...((n-2)-2c)\}r^{2k} \Bigg]\end{aligned}$$

If we let $c = \frac{n-2k}{2}$ and

$$a = \left(\frac{R^{2k}(n-2k)(n-2(k-1))\dots(n-2)n(n+2)\dots(n+2(k-2))(n+2(k-1))}{b}\right)^{\frac{n-2k}{4k}}$$

then

$$\begin{aligned}\Delta^k u &= (-1)^k b\left(\frac{R^k\sqrt{(n-2k)(n-2(k-1))...(n-2)n(n+2)...(n+2(k-2))(n+2(k-1))}}{\sqrt{b}(R^2+r^2)^k}\right)^{\frac{n+2k}{2k}} \\ &= (-1)^k b u^{\frac{n+2k}{n-2k}}\end{aligned}$$

where

$$u=\left(\frac{R^k\sqrt{(n-2k)(n-2(k-1))...(n-2)n(n+2)...(n+2(k-2))(n+2(k-1))}}{\sqrt{b}(R^2+r^2)^k}\right)^{\frac{n-2k}{2k}}$$

This completes the proof of Theorem 2.5.

In particular, the function $u=\left(\frac{R^2\sqrt{(n-4)(n-2)n(n+2)}}{\sqrt{b}(R^2+r^2)^2}\right)^{\frac{n-4}{4}}$ is a solution of

$$\Delta^2 u = bu^{\frac{n+4}{n-4}}, \quad n \geqslant 5$$

and the function $u=\left(\frac{R^3\sqrt{(n-6)(n-4)(n-2)n(n+2)(n+4)}}{\sqrt{b}(R^2+r^2)^3}\right)^{\frac{n-6}{6}}$ solves the equation

$$\Delta^3 u + bu^{\frac{n+6}{n-6}} = 0, \quad n \geqslant 7.$$

3. Solutions of ordinary differential equations

It is well known that if $n=3$ then

$$\Delta^m u = \frac{d^{2m}u}{dr^{2m}} + \frac{2m}{r}\quad\frac{d^{2m-1}u}{dr^{2m-1}}$$

where u is a function of r only. Further, if

$$u=\frac{v}{r}$$

then

$$\Delta^m u = \frac{1}{r}\frac{d^{2m}v}{dr^{2m}}.$$

Hence, the equation

$$\Delta^m u = u^{\frac{k+2m}{k}}$$

is equivalent to

$$\frac{d^{2m}v}{dr^{2m}} = r^{-\frac{2m}{k}}v^{\frac{k+2m}{k}}, \tag{3.1}$$

and because of the Theorem 2.3 its solution is

$$v=\frac{\{k(k+2)(k+4)\dots(k+2m-2)(k-1)(k+1)\dots(k+2m-3)\}^{\frac{k}{2m}}}{r^{k-1}}. \tag{3.2}$$

In the same way, since, the solution of

$$\Delta u + cr^{2k-2}u^{1+\frac{4k}{n-2}} = 0 \quad n \geqslant 3, k \geqslant 1$$

by (2.9) is

$$u = \left[\frac{R^k\sqrt{(n-2)(n+2k-2)}}{\sqrt{c}(R^{2k}+r^{2k})}\right]^{\frac{n-2}{2k}}$$

therefore the function

$$v = r\left(\frac{R^k\sqrt{1+2k}}{\sqrt{c}(R^{2k}+r^{2k})}\right)^{\frac{1}{2k}} \tag{3.3}$$

solves the equation

$$\frac{d^2v}{dr^2} + \frac{c}{r^{2k+2}}v^{1+4k} = 0 \tag{3.4}$$

where $c > 0$ is a constant.

Further, as by (2.22)

$$u = \left(\frac{R^k\sqrt{(n-2)(n+2k-2)}}{\sqrt{c}(R^{2k}-r^{2k})}\right)^{\frac{n-2}{2k}} \quad n \geqslant 3, k \geqslant 1$$

is a solution of

$$\Delta u = cr^{2k-2}u^{1+\frac{4k}{n-2}}$$

hence the function

$$v = r\left(\frac{R^k\sqrt{1+2k}}{\sqrt{c}(R^{2k}-r^{2k})}\right)^{\frac{1}{2k}} \tag{3.5}$$

solves the equation

$$\frac{d^2v}{dr^2} - \frac{c}{r^{2k+2}}\,v^{1+4k} = 0. \tag{3.6}$$

Finally, since by $(2.14''')$ the solution of

$$\Delta u + P(r)e^{au} = 0, a > 0 \text{ is a constant} \tag{3.7}$$

is the function

$$u = \frac{1}{a}\ln\left(\frac{2n-4}{ar^2P(r)}\right) \tag{3.8}$$

provided $\ln P(r)$ is harmonic and $n \geqslant 3$, consequently the function

$$v = \frac{r}{a}\ln\left(\frac{2}{ar^2P(r)}\right) \tag{3.9}$$

is the solution of the equation

$$\frac{d^2v}{dr^2} + rP(r)e^{a\frac{v}{r}} = 0 \tag{3.10}$$

if $\frac{d^2}{dr^2}\ln P(r) + \frac{2}{r}\frac{d}{dr}\ln P(r) = 0$.

4. Solutions of some Dirichlet and Neumann problems

There is a vast literature [see e.g. [3] and [7]] dealing with existence, nonexistence, uniqueness and nonuniqueness of solutions to problems of the form

$$\begin{aligned} \Delta u + P(r) f(u) &= 0 \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

where D is a bounded domain in $\mathbb{R}^n$ such as $D = B_R(0)$. However, there do not seem to be any 'explicit' solution results when a solution does exist. For example, a number of mathematicians, in particular, Bieberbach [2] have considered solving the Dirichlet problem for the equation

$$\Delta u - F(x) e^u = f(x) \tag{4.1}$$

in a bounded domain D in $\mathbb{R}^n$. The method of Bieberbach only shows that for any bounded functions f and F there is at least one solution of the Dirichlet problem for the equation (4.1). In [4] the explicit solution of the problem

$$\begin{aligned} \Delta u - F(x) e^{au} &= f(x) \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned} \tag{4.2}$$

where D is a bounded domain in $\mathbb{R}^n$ and $a > 0$ is a constant, was obtained as

$$u = \frac{1}{a} \ln \left(\frac{f(x)}{F(x)} \right)$$

provided the function $f(x)$ and $F(x)$ are positive and such that $\ln\left(\frac{F(x)}{f(x)}\right)$ is harmonic in D and $f(x) = F(x)$ on ∂D. Here, we give some more explicit solution results:

In view of equation (2.8) and (2.9) one easily concludes

THEOREM 4.1. *The solution of the nonlinear Dirichlet problem*

$$\begin{aligned} \Delta u + c r^{2k-2} u^{1+\frac{4k}{n-2}} &= 0 \textit{ in } D_R \\ u &= \phi \textit{ on } \partial D_R \end{aligned} \tag{4.3}$$

is given by

$$u = \left(\frac{R^k \sqrt{(n-2)(n+2k-2)}}{\sqrt{c}(R^{2k} + r^{2k})} \right)^{\frac{n-2}{2k}}, \quad n \geqslant 3, k \geqslant 1$$

where $R^k = \sqrt{\frac{(n-2)(n+2k-2)}{c}} \cdot \frac{1}{2\phi^{\frac{2k}{n-2}}}$. *If* $\phi = 0$ *when* $r = R$ *then the problem* (4.3) *has no solution except* $u = 0$.

Again, because of equations (2.17) and (2.18) with $k = 1$, we have

THEOREM 4.2. *If* $a > 0$ *is an arbitrary constant and* $\ln(P(r))$ *is harmonic, the solution of the problem*

$$\begin{aligned} \Delta u + P(r) e^{au} &= 0 \quad \textit{in } B_R(0) \\ u &= \psi \quad \textit{on } \partial B_R(0) \end{aligned} \tag{4.4}$$

is given by

$$u = \frac{2}{a}\ln\left(\frac{\sqrt{8}R}{\sqrt{aP(r)}(R^2+r^2)}\right), \quad n = 2$$

where $R^2 = \frac{2}{aP(R)e^{a\psi}}$. *If* $\psi = 0$ *when* $r = R$ *then* $R^2 = \frac{2}{aP(R)}$.

Bandle [1] also treated this problem by a method of complex variables when $P(r)$ is a positive constant and $a = 1$. See also [10] for the case $a = 1$.

We now consider a more general problem

$$\begin{aligned} \Delta u + P(r)f(u) &\geqslant 0 \text{ in } B_R(0) \\ u &= \phi \text{ on } \partial B_R(0) \end{aligned} \tag{4.5}$$

We first establish the following differential inequality which is a special case of Theorem 2 in [5] in a manner similar to Theorem 2.1.

LEMMA 4.1. *Let* $f(t)$ *and* $g(t)$ *be positive, increasing and differentiable functions and such that*

$$\int_0^\gamma \frac{dt}{f(t)} \text{ and } \int_0^\delta \frac{dt}{g(t)}$$

exist for $\gamma, \delta > 0$ *and let*

$$u = u(x_1, x_2, \ldots, x_n) \text{ and } v = v(x_1, x_2, \ldots, x_n)$$

be two functions related by the identity

$$\int_0^u \frac{dt}{f(t)} = \int_0^v \frac{dt}{g(t)} \tag{4.6}$$

then

$$\frac{\Delta u}{f(u)} \geqslant \frac{\Delta v}{g(v)} \tag{4.7}$$

provided $f'(u) \geqslant g'(v)$.

Proof. We write x as one of the variables $x_1, x_2, \ldots, x_n$ and differentiate (4.6) twice with respect to x. This results in

$$\frac{u_x}{f(u)} = \frac{v_x}{g(v)} \tag{4.8}$$

$$\frac{u_{xx}}{f(u)} - \frac{f'(u)u_x^2}{f^2(u)} = \frac{v_{xx}}{g(v)} - \frac{g'(v)v_x^2}{g^2(v)} \tag{4.9}$$

With the help of (4.8), (4.9) becomes

$$\frac{u_{xx}}{f(u)} = \frac{v_{xx}}{g(v)} + (f'(u) - g'(v))\frac{v_x^2}{g^2(v)}.$$

Since $f'(u) \geqslant g'(v)$, we get, by summing over all x_k

$$\frac{\Delta u}{f(u)} \geqslant \frac{\Delta v}{g(v)}$$

which completes the proof of the lemma.

THEOREM 4.3. *Under the conditions of the lemma* (4.1) *on* f, *the solution of the problem* (4.5), *if* $\ln P(r)$ *is harmonic, is given by*

$$1-\frac{P(r)(R^2+r^2)^2}{8R^2}=\int_0^u\frac{dt}{f(t)},\quad (n=2) \tag{4.10}$$

where $R^2=\frac{2}{P(R)}\left(1-\int_0^{\phi}\frac{dt}{f(t)}\right)$. *If* $\phi=0$ *when* $r=R$ *then* $R^2=\frac{2}{P(R)}$.

Proof. If, in (4.7) $g(v)=e^v$, then

$$\Delta v+P(r)e^v=0 \tag{4.11}$$

implies

$$\Delta u+P(r)f(u)\geqslant 0 \tag{4.12}$$

where the solution of (4.12) is determined by

$$\int_0^u\frac{dt}{f(t)}=\int_0^v\frac{dt}{e^t}. \tag{4.13}$$

Since in $B_R(0)$

$$v=\ln\left(\frac{8R^2}{P(r)(R^2+r^2)^2}\right),\quad (n=2)$$

u is obtained, with the help of (4.13), from

$$1-\frac{P(r)(R^2+r^2)^2}{8R^2}=\int_0^u\frac{dt}{f(t)}. \tag{4.14}$$

If $u=\phi$ when $r=R$, equation (4.14) yields

$$R^2=\frac{2}{P(R)}\left(1-\int_0^{\phi}\frac{dt}{f(t)}\right).$$

However, if $\phi=0$ when $r=R$, then

$$R^2=\frac{2}{P(R)}$$

which proves the assertion.

Next result is concerning the Dirichlet problem for the Yamabe equation

$$\Delta u+k(x)u^{\frac{n+2}{n-2}}=\frac{n-2}{4(n-1)}k_0(x)u. \tag{4.15}$$

The existence of a positive solution for (4.15) if $k(x)$ is a positive constant was proved by Yamabe [17], Trudinger [12] and Schoen [11]. Here, we give an explicit solution for (4.15) and the corresponding Dirichlet problem by an elementary method.

THEOREM 4.4. *The solution of the Dirichlet problem*

$$\begin{aligned}\Delta u+k(x)u^{\frac{n+2}{n-2}}&=\frac{n-2}{4(n-1)}k_0(x)u \quad \text{in } D\\ u&=\phi \quad \text{on } \partial D\end{aligned} \tag{4.16}$$

where the function $k(x)$ and $k_0(x)$ are positive, is given by

$$u = \left(\frac{n-2}{4(n-1)}\frac{k_0(x)}{k(x)}\right)^{\frac{n-2}{4}} \quad n \geqslant 3 \tag{4.17}$$

If $\left(\frac{k_0(x)}{k(x)}\right)^{\frac{n-2}{4}}$ *is harmonic in D and* $k_0(x) = \frac{4(n-1)k(x)}{(n-2)}\phi^{\frac{4}{n-2}}$ *on* ∂D.

Proof. First, we write (4.15) as

$$\Delta u + k(x)u\left(u^{\frac{4}{n-2}} - \frac{n-2}{4(n-1)}\frac{k_0(x)}{k(x)}\right) = 0.$$

Clearly, if

$$\Delta\left(\frac{k_0(x)}{k(x)}\right)^{\frac{n-2}{4}} = 0$$

then

$$u = \left(\frac{n-2}{4(n-1)}\frac{k_0(x)}{k(x)}\right)^{\frac{n-2}{4}}$$

is a solution of (4.15).

Hence (4.17) is the solution of the problem (4.16) if

$$k_0(x) = \frac{4(n-1)}{n-2}k(x)\phi^{\frac{4}{n-2}}$$

on ∂D.

In a manner similar to Theorems 4.1 and 4.2, we obtain explicit solutions for the following Neumann problems:

THEOREM 4.5. *The solution of the Neumann problem*

$$\begin{aligned} \Delta u + cu^{\frac{n+2}{n-2}} &= 0 \quad \text{in } B_R(0) \\ \frac{\partial u}{\partial \nu} &= \phi \quad \text{on } \partial B_R(0) \end{aligned} \tag{4.18}$$

is given by

$$u = \left(\frac{R\sqrt{n(n-2)}}{\sqrt{c}(R^2+r^2)}\right)^{\frac{n-2}{2}}, \quad n \geqslant 3$$

provided

$$\phi = -\left(\sqrt{\frac{n(n-2)}{c}}\right)^{\frac{n-2}{2}} \cdot \frac{n-2}{(2R)^{n/2}} \tag{4.19}$$

when $r = R$. If $\phi = 0$ on the boundary then the only possible solution is $u \equiv 0$.

Thus the problem (4.18) has no solution if ϕ is positive.

THEOREM 4.6. *If $a > 0$ is a constant and $\ln P(r)$ is harmonic, then the solution of*

$$\begin{aligned} \Delta u + P(r)e^{au} &= 0 \quad \text{in } B_R(0) \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B_R(0) \end{aligned} \tag{4.20}$$

is the function

$$u = \frac{2}{a} \ln \left(\frac{\sqrt{8}R}{\sqrt{aP(r)}(R^2 + r^2)} \right), \quad n = 2$$

if $P(R) = \frac{B}{R^4}$ *on the boundary, where* B *is an arbitrary constant.*

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