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STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN THE SPACE OF DISTRIBUTIONS

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Abstract. We reformulate a quadratic functional equation of the form

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)=4f(x)+4f(y)+4f(z)$$

and an inequality

$$|f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)-4f(x)-4f(y)-4f(z)|\leqslant\varepsilon$$

in the space of distributions. In view of this fact, we use a mollifier and Gauss transform to show that every distributional solution of the inequality is a tempered distribution and finally the stability problem of the equation in the sense of distributions.

1. Introduction

The concept of the stability for a functional equation arises when the equation is replaced by an inequality which acts as a perturbation of the equation. Here, the stability question is how the solutions of the inequality differ from the solution of the original equation.

This type of problem was first studied by D. H. Hyers [8] in 1941, who solved the stability problem of Cauchy functional equation as follows.

THEOREM 1.1. [8] *Let $f: E_1 \to E_2$ with E_1, E_2 Banach spaces be an $\epsilon-$ additive, that is, f satisfies*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\varepsilon, \tag{1.1}$$

for all $x,\ y\in E_1$. Then there exists a unique additive mapping $g: E_1\to E_2$ such that

$$\|f(x)-g(x)\|\leqslant\varepsilon$$

for all $x\in E_1$. Here, an additive mapping $g: E_1\to E_2$ means the inequality (1.1) *satisfies for $\varepsilon=0$.*

Since the work of Hyers [8], the stability problems of various functional equations, for example: Pexider equation, Jensen equation, D'Alembert equation and so on, have been proposed by many mathematicians.

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Consider the functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0, \tag{1.2}$$

for all x and y in the domain of f. This equation is said to be *quadratic*, for the simple reason that the quadratic functions $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=cx^2$, $c\in\mathbb{R}$, are solutions of the equation (1.2). The stability of the quadratic equation (1.2) was first studied by F. Skof [11], and has been developed by a number of authors, such as P. W. Chilewa [2], S. Czerwik [6], and G. H. Kim [9].

In 2003, J. H. Bae, K. W. Jun and S. M. Jung [1] considered the functional equation

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)-4f(x)-4f(y)-4f(z)=0, \tag{1.3}$$

for all x,y,z in the domain of f, which they also called *quadratic*. In the paper [1], they proved an interesting fact that for given vector spaces X and Y, a function $f:X\to Y$ is a solution of the equation (1.3) if and only if f is a solution of the quadratic equation (1.2), and investigated the stability problem of the equation (1.3). After this result, the estimates of the stability problem of the equation has recently improved by P. Găvruţă and L. Cădariu [7].

This paper deals with the stability problems of the equation (1.3) in the space of distributions. In the papers [3, 4, 5], J. Chung, D. Kim and the third author have proposed methods of how to solve the stability problems in the various spaces of generalized functions. Following the same approaches of them, the equation (1.3) and the inequality

$$|f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)-4f(x)-4f(y)-4f(z)|\leqslant\varepsilon$$

is reformulated in the space of distributions as

$$u\circ A_1+u\circ A_2+u\circ A_3+u\circ A_4+u\circ P_1+u\circ P_2+u\circ P_3=0, \tag{1.4}$$

and

$$\|u\circ A_1+u\circ A_2+u\circ A_3+u\circ A_4+u\circ P_1+u\circ P_2+u\circ P_3\|\leqslant\varepsilon, \tag{1.5}$$

where A_1,A_2,A_3,A_4,P_1,P_2 and P_3 are the functions such that

$$A_1(x,y,z)=x+y+z,\quad A_2(x,y,z)=x-y+z,$$
$$A_3(x,y,z)=x+y-z,\quad A_4(x,y,z)=-x+y+z,$$
$$P_1(x,y,z)=x,\quad P_2(x,y,z)=y \text{ and } P_3(x,y,z)=z,$$

for $x,y,z\in\mathbb{R}^n$. Here, $\circ$ denotes the distributional pullback and $\|v\|\leqslant\varepsilon$ means that $|\langle v,\varphi\rangle|\leqslant\varepsilon\|\varphi\|_{L^1}$ for all test functions φ.

Making use of the functions $\delta_t(x):=t^{-n}\delta(\frac{x}{t})$, $x\in\mathbb{R}^n$, $t>0$, where δ is an infinitely differentiable function such that

$$\delta\geqslant 0,\quad supp\,\delta\subset\{x\in\mathbb{R}^n|\,|x|\leqslant 1\},\quad \int\delta=1,$$

we first show that every distribution satisfying the inequality (1.5) is a tempered distribution. This fact enables us to convolve the heat kernel $E(x,t) := (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t})$, $x \in \mathbb{R}^n$, $t > 0$ on the inequality (1.5) so that, as in [3, 4, 5], we may reduce the problem on distributions to that of infinitely differentiable functions defined on $\mathbb{R}^n \times (0,\infty)$.

As a result, we prove that every distribution u with the inequality (1.5) satisfies

$$\|u - \sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}\, x_i x_j\| \leqslant \frac{3}{8}\varepsilon, \quad x = (x_1, \cdots x_n) \in \mathbb{R}^n,$$

for some $a_{ij} \in \mathbb{C},\ 1 \leqslant i \leqslant j \leqslant n$.

2. Preliminaries

We briefly introduce the space $\mathscr{D}'(\mathbb{R}^n)$ of distributions, and the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

We denote by $\mathscr{C}^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ and by $\mathscr{C}_o^\infty$ the set of all functions in $\mathscr{C}^\infty$ which have a compact support.

DEFINITION 2.1. A distribution u is a linear form on $C_o^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $N \in \mathbb{N}_0$ such that

$$|\langle u, \varphi\rangle| \leqslant C \sum_{|\alpha|\leqslant N} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_o^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions is denoted by $\mathscr{D}'(\mathbb{R}^n)$.

DEFINITION 2.2. We denote by $\mathscr{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ satisfying

$$\sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all α, $\beta \in \mathbb{N}_0^n$. A linear form u on $\mathscr{S}(\mathbb{R}^n)$ is said to be tempered distribution if there is a constant $C \geqslant 0$ and a nonnegative integer N such that

$$|\langle u, \varphi\rangle| \leqslant C \sum_{|\alpha|,|\beta|\leqslant N} \sup |x^\alpha \partial^\beta \varphi|,$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The set of all tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^n)$.

The remaining of this section is devoted to introduce regularization and Gauss transform, which play an important role in this paper. We denote by δ the function on $\mathbb{R}^n$ satisfying

$$\delta(x) = \begin{cases} A\exp(-(1-|x|^2)^{-1}), |x| < 1 \\ 0, |x| \geqslant 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp(-(1-|x|^2)^{-1})dx \right)^{-1}.$$

It is easy to see that δ is an infinitely differentiable function supported in the set $\{x : |x| \leqslant 1\}$ satisfying $\int \delta = 1$. For each $t > 0$, let $\delta_t(x) = t^{-n}\delta(x/t)$. Then δ_t has all the properties of δ except that the support of δ_t is contained in the ball of radius t with center at 0. Suppose u belongs to $\mathscr{D}'(\mathbb{R}^n)$. It is well known that for each $t > 0$, $(u * \delta_t)(x) = \langle u_y, \delta_t(x-y)\rangle$ is a smooth function in $\mathbb{R}^n$ and $(u * \delta_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_o^\infty(\mathbb{R}^n)$,

$$\langle u * \delta_t, \varphi\rangle = \int (u * \delta_t)(x)\varphi(x)dx \longrightarrow \langle u, \varphi\rangle \qquad \text{as} \quad t \to 0^+.$$

For each $t > 0$, the function $u * \delta_t$ is called a regularization of u and the transform which maps u to $u * \delta_t$ is called a mollifier.

On the other hand, the n-dimensional *heat kernel* is the fundamental solution $E(x,t)$ of the heat operator $\partial_t - \triangle_x$ in $\mathbb{R}^n \times (0,\infty)$ given by

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t) & , \ x \in \mathbb{R}^n, \quad t > 0, \\ 0 & , \ x \in \mathbb{R}^n, \quad t \leqslant 0. \end{cases}$$

For convenience, we sometimes use the notation $E_t(x)$ instead of $E(x,t)$. The semi-group property

$$(E_s * E_t)(x) = E_{s+t}(x), \quad x \in \mathbb{R}^n, \quad s,t > 0 \tag{2.1}$$

of the heat kernel is very useful later.

Since for each $t > 0$, $E(\cdot,t)$ belongs to $\mathscr{S}(\mathbb{R}^n)$, the following transform G of u

$$Gu(x,t) := (u * E)(x,t) = u_y(E(x-y,t)), \qquad x \in \mathbb{R}^n, \ t > 0$$

is well defined for each $u \in \mathscr{S}'(\mathbb{R}^n)$, which is called the Gauss transform of u. It is easy to see that for any $t > 0$, $Gu(\cdot,t)$ is contained in $\mathscr{S}(\mathbb{R}^n)$.

It is shown in [10] that the Gauss transform $Gu(x,t)$ of u is a C^∞ solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$ and $Gu(\cdot,t)$ converges to u as $t \to 0^+$ in the following sense that for each $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle Gu(\cdot,t), \varphi\rangle = \int Gu(x,t)\varphi(x)dx \longrightarrow \langle u, \varphi\rangle \qquad \text{as} \quad t \to 0^+.$$

3. Main theorem

In this section, we solve the stability problem of the quadratic equation (1.3) in the space of distributions. First, we show that every distribution satisfies the inequality (1.5) is a tempered distribution. To see this, we first show the following lemma.

LEMMA 3.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function satisfying*

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)-4f(x)-4f(y)-4f(z) = 0 \tag{3.1}$$

Then

$$f(x) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j, \quad x = (x_1, \cdots x_n) \tag{3.2}$$

for some $a_{ij} \in \mathbb{C}, \quad 1 \leqslant i \leqslant j \leqslant n$.

Proof. To prove the lemma, we use the induction. For $n = 1$, it is obvious. Now, assume that the lemma holds for $n = l$, and let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function satisfying (3.1). Then (3.1) can be rewritten as

$$\begin{aligned} &f(x+y+z, \xi+\eta+\zeta) + f(x-y+z, \xi-\eta+\zeta) + f(x+y-z, \xi+\eta-\zeta) \\ &\quad + f(-x+y+z, -\xi+\eta+\zeta) - 4f(x,\xi) - 4f(y,\eta) - 4f(z,\zeta) = 0, \end{aligned} \tag{3.3}$$

for $x, y, z \in \mathbb{R}^n$ and $\xi, \eta, \zeta \in \mathbb{R}$. Now, let $h(x,\xi) = f(x,\xi) - f(0,\xi) - f(x,0)$. Then it is easily seen that h satisfies

$$h(0,\xi) = 0, \quad \xi \in \mathbb{R}, \tag{3.4}$$

$$h(x,0) = 0, \quad x \in \mathbb{R}^n \tag{3.5}$$

and

$$\begin{aligned} &h(x+y+z, \xi+\eta+\zeta) + h(x-y+z, \xi-\eta+\zeta) + h(x+y-z, \xi+\eta-\zeta) \\ &\quad + h(-x+y+z, -\xi+\eta+\zeta) - 4h(x,\xi) - 4h(y,\eta) - 4h(z,\zeta) = 0, \end{aligned} \tag{3.6}$$

for $x, y, z \in \mathbb{R}^n$ and $\xi, \eta, \zeta \in \mathbb{R}$.
Putting $y = z = 0$ and $\eta = \zeta = 0$ in (3.6), we have

$$h(x,\xi) = h(-x,-\xi), \quad x \in \mathbb{R}^n, \quad \xi \in \mathbb{R}. \tag{3.7}$$

Now, let $y = z = 0$ and $\zeta = 0$ in (3.6). Then from (3.7) we have

$$h(x,\xi+\eta) + h(x,\xi-\eta) - 2h(x,\xi), \quad x \in \mathbb{R}^n, \quad \xi, \eta \in \mathbb{R},$$

which is, from (3.5), equivalent to the equation

$$h(x,\xi+\eta) = h(x,\xi) + h(x,\eta), \quad x \in \mathbb{R}^n, \quad \xi, \eta \in \mathbb{R}.$$

This implies that for each $x \in \mathbb{R}^n$, the function $h(x,\cdot)$ satisfies Cauchy equation, and hence $h(x,\xi) = h(x,1)\xi$, for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$.

But putting $z = 0$, $\eta = \zeta = 0$ and $\xi = 1$ in (3.6), it is easily seen that there exist $b_1, \cdots, b_l \in \mathbb{C}$ such that

$$h(x,1) = b_1 x_1 + \cdots + b_l x_l, \quad x = (x_1, \cdots, x_l) \in \mathbb{R}^l.$$

Since $f(x,\xi) = f(x,0) + h(x,\xi) + f(0,\xi)$, it follows from the induction assumption that we complete the proof. □

THEOREM 3.2. *Let* $u \in \mathscr{D}'(\mathbb{R}^n)$ *satisfy the inequality (1.5). Then* u *is an element in* $\mathscr{S}'(\mathbb{R}^n)$.

Proof. Convolving $\delta_t(x)\delta_s(y)\delta_r(z)$ in the left-hand side of (1.5) we have

$$\begin{aligned}|\,(u*\delta_t*\delta_s*\delta_r)(x+y+z)+(u*\delta_t*\delta_s*\delta_r)(x-y+z)\\ +(u*\delta_t*\delta_s*\delta_r)(x+y-z)+(u*\delta_t*\delta_s*\delta_r)(-x+y+z)\\ -4(u*\delta_t)(x)-4(u*\delta_s)(y)-4(u*\delta_r)(z)\,|\leqslant\varepsilon\end{aligned} \tag{3.8}$$

for all $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$. In view of (3.8) it is easy to see that

$$f(x):=\limsup_{t\to 0^+}(u*\delta_t)(x)$$

exists, for all $x\in\mathbb{R}^n$.
Putting $x=y=z=0$ and $t=s=r=s_n\to 0^+$ so that $(u*\delta_{s_n})(0)\to f(0)$ in (3.8), we have

$$|f(0)|\leqslant\frac{\varepsilon}{8}. \tag{3.9}$$

Now, let $y=z=0$ and, for any $x\in\mathbb{R}^n$, put $t=t_n\to 0^+$ so that $(u*\delta_{t_n})(x)\to f(x)$, and $s=s_n\to 0^+$ so that $(u*\delta_{s_n})(0)\to f(0)$ in (3.8). Then from (3.9), we have

$$|-(u*\delta_t)(x)+(u*\delta_t)(-x)|\leqslant 2\varepsilon,\quad x\in\mathbb{R}^n. \tag{3.10}$$

On the other hand, if we put $y=z=0$, $t=t_n\to 0^+$ so that $(u*\delta_{t_n})(x)\to f(x)$ and $r=r_n\to 0^+$ so that $(u*\delta_{r_n})(0)\to f(0)$ in (3.8) then we have

$$|-(u*\delta_s)(x)+(u*\delta_s)(-x)+4(u*\delta_s)(x)-4f(x)-4(u*\delta_s)(0)-4f(0)|\leqslant\varepsilon,$$

and from (3.10), we get the inequality

$$|(u*\delta_s)(x)-f(x)-(u*\delta_s)(0)|\leqslant\frac{3}{4}\varepsilon+|f(0)|\leqslant\frac{7}{8}\varepsilon, \tag{3.11}$$

for $x\in\mathbb{R}^n$ and $s>0$. We also have

$$\begin{aligned}|(u*\delta_s)(x+y+z)+(u*\delta_s)(x-y+z)+(u*\delta_s)(x+y-z)\\ +(u*\delta_s)(-x+y+z)-4f(x)-4(u*\delta_s)(y)-4f(z)|\leqslant\varepsilon,\end{aligned} \tag{3.12}$$

for $x,y,z\in\mathbb{R}^n$ and $s>0$, if we put $t=t_n\to 0^+$ so that $(u*\delta_{t_n})(x)\to f(x)$ and $r=r_n\to 0^+$ so that $(u*\delta_{r_n})(z)\to f(z)$ in (3.8).

From the inequality (3.11), (3.12) and the triangle inequality we have

$$|f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)-4f(x)-4f(y)-4f(z)|\leqslant 8\varepsilon$$

for all $x,y,z\in\mathbb{R}^n$.

In view of the Corollary 2.6 in [7], there exists a unique function $q:\mathbb{R}^n\to\mathbb{C}$ satisfying

$$q(x+y+z)+q(x-y+z)+q(x+y-z)+q(-x+y+z)-4q(x)-4q(y)-4q(z)=0, \tag{3.13}$$

for all $x,y,z\in\mathbb{R}^n$ such that

$$|f(x)-q(x)|\leqslant\varepsilon \tag{3.14}$$

for all $x\in\mathbb{R}^n$.

From (3.11) and (3.14) we have

$$|(u*\delta_s)(x)-q(x)-(u*\delta_s)(0)|\leqslant \frac{15}{8}\varepsilon, \tag{3.15}$$

for all $x\in\mathbb{R}^n$ and $s>0$. Putting $s=s_n\to 0^+$ so that $(u*\delta_{s_n})(0)\to f(0)$ in (3.15) we have

$$\|u-q(x)\|\leqslant 2\varepsilon. \tag{3.16}$$

On the other hand, as we see the proof of Theorem 2.1 in [7], the function q inherits its measurability from f and it follows from Lemma 3.1 that every measurable solution of the equation (3.13) has the form

$$q(x)=\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}\,x_ix_j,\quad x=(x_1,\cdots,x_n).$$

Thus it follows from (3.16) that $h(x):=u-q(x)$ belongs to $(L^1)'=L^\infty$ and $u=q(x)+h(x)\in\mathscr{S}'(\mathbb{R}^n)$. □

The following lemma is necessary to prove the main theorem.

LEMMA 3.3. *Let $f:\mathbb{R}^n\times(0,\infty)\to\mathbb{C}$ be a continuous function satisfying*

$$\begin{aligned}&f(x+y+z,t+s+r)+f(x-y+z,t+s+r)+f(x+y-z,t+s+r)\\&\quad+f(-x+y+z,t+s+r)-4f(x,t)-4f(y,s)-4f(z,r)=0.\end{aligned} \tag{3.17}$$

for all $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$. Then there exist a_{ij} and b in $\mathbb{C}$ such that

$$f(x,t)=\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}x_ix_j+bt,$$

for $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $t>0$.

Proof. Let $h(x,t):=f(x,t)-f(0,t)$. Then h satisfies

$$\begin{aligned}&h(x+y+z,t+s+r)+h(x-y+z,t+s+r)+h(x+y-z,t+s+r)\\&\quad+h(-x+y+z,t+s+r)-4h(x,t)-4h(y,s)-4h(z,r)=0\end{aligned} \tag{3.18}$$

for all $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$, and

$$h(0,t)=0,\quad t>0. \tag{3.19}$$

Put $y=z=0$ and $s=r\to 0^+$ in (3.18). Then from (3.19), we have

$$h(x,t)=h(-x,t),\quad x\in\mathbb{R}^n,\quad t>0. \tag{3.20}$$

If we take $y=z=0$ in (3.18), then by virtue of (3.19) and (3.20), we have

$$h(x,t+s+r)=h(x,t),$$

which implies that $h(x,t)$ is independent of $t>0$. Thus by the Lemma 3.1, there exist $a_{ij}\in\mathbb{C}$, $1\leqslant i\leqslant j\leqslant n$ such that

$$h(x,t)=\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}x_ix_j,\quad x=(x_1,\cdots,x_n).$$

On the other hand, putting $x=y=z=0$ and $r\to 0^+$ in (3.17), we have

$$f(0,t+s)=f(0,t)+f(0,s),\quad t,s>0,$$

for it is easy to see that $\lim_{r\to 0^+} f(0,r)=0$, by putting $x=y=z=0$ and $s=r$ in (3.17). Then, since $f(x,t)=h(x,t)+f(0,t)$, we have the result. □

Now we are ready to state and prove the main theorem of this paper.

THEOREM 3.4. *Let $u\in\mathscr{D}'(\mathbb{R}^n)$ satisfy the inequality* (1.5). *Then there exists a unique quadratic function*

$$q(x)=\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}\,x_ix_j,\quad x=(x_1,\cdots,x_n)$$

such that

$$\|u-q(x)\|\leqslant\frac{3}{8}\varepsilon. \tag{3.21}$$

Proof. From Theorem 3.2, without loss of generality, we may assume that u belongs to $\mathscr{S}'(\mathbb{R}^n)$. Now we employ the n-dimensional heat kernel $E_t(x),\ t>0$. Convolving $E_t(x)E_s(y)E_s(z)$ in the left-hand side of (1.5) we get the stability of the functional equation of type

$$\begin{aligned}|Gu(x{+}y{+}z,t{+}s{+}r)+Gu(x{-}y{+}z,t{+}s{+}r)+Gu(x{+}y{-}z,t{+}s{+}r)\\ +Gu(-x{+}y{+}z,t{+}s{+}r)-4Gu(x,t)-4Gu(y,s)-4Gu(z,r)|\leqslant\varepsilon\end{aligned} \tag{3.22}$$

for $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$, where Gu is the *Gauss transform* of u given by

$$Gu(x,t)=\langle\, u_\xi,\, E_t(x-\xi)\rangle\,.$$

Putting $x=y=z=0$ and $s=t$ in (3.22), we have

$$\left|\frac{1}{2}Gu(0,2t+r)-Gu(0,t)-Gu(0,r)\right|\leqslant\frac{\varepsilon}{8},\quad t,r>0. \tag{3.23}$$

From (3.23), it is easy to see that $\limsup_{t\to 0^+} Gu(0,t)$ exists. Then, since Gu is continuous on $\mathbb{R}^n\times(0,\infty)$, for any $r>0$ we have,

$$\left|\limsup_{t\to 0^+} Gu(0,t)\right|=\left|\limsup_{t\to 0^+}\left(\frac{1}{2}Gu(0,2t{+}r)-\frac{1}{2}Gu(0,r)-Gu(0,t)\right)\right|\leqslant\frac{\varepsilon}{8}. \tag{3.24}$$

On the other hand, putting $x=y=z=0,\ r=s$ and $t=t_n\to 0^+$ so that $Gu(0,t_n)$ converges to $\limsup_{t\to 0^+} Gu(0,t)$ as $n\to\infty$ in (3.22) and dividing the result by 8, we have

$$\left|\frac{1}{2}Gu(0,t_n+2s)-\frac{1}{2}Gu(0,t_n)-Gu(0,s)\right|\leqslant\frac{\varepsilon}{8}.$$

for all $n\in\mathbb{N}$ and $s>0$.Then it follow from (3.24) that

$$\left|\frac{1}{2}Gu(0,2s)-Gu(0,s)\right|\leqslant\frac{\varepsilon}{8}+\frac{\varepsilon}{16}=\frac{3}{16}\varepsilon,$$

for all $s > 0$. By the induction argument we have

$$\left|\frac{1}{2^n}Gu(0,2^n s) - Gu(0,s)\right| \leqslant \frac{3}{8}\varepsilon \tag{3.25}$$

for all $n \in \mathbb{N},\ s > 0$. Now let $h : (0,\infty) \to \mathbb{C}$ be defined by

$$h(s) = \lim_{m\to\infty} \frac{1}{2^m}Gu(0,2^m s). \tag{3.26}$$

Then from (3.25), it can be seen that the righthand side of (3.26) converges uniformly and h is the unique function satisfying

$$|Gu(0,t) - h(t)| \leqslant \frac{\varepsilon}{2}, \quad t > 0. \tag{3.27}$$

Moreover, if we put $x = y = z = 0$ and $r = t_n$ in (3.22), then by the inequality (3.24) we have

$$h(t+s) = h(t) + h(s), \tag{3.28}$$

for all $t,s > 0$.
Now, putting $y = x, z = 0, t = t_n$ and $s = r$ in (3.22), we have

$$|2Gu(2x,t_n+2s) + 2Gu(0,t_n+2s) - 8Gu(x,s) - 4Gu(0,t_n)| \leqslant \varepsilon,$$

for all $n \in \mathbb{N}$ and $s > 0$, and so, from (3.24), it is seen that

$$|2Gu(2x,2t) + 2Gu(0,2t) - 8Gu(0,t)| \leqslant \varepsilon + \frac{\varepsilon}{2} = \frac{3}{2}\varepsilon, \tag{3.29}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Dividing (3.29) by 8 and using the induction argument, we have

$$\left|\frac{1}{4^n}Gu(2^n x,\ 2^n t) - Gu(x,t)\sum_{k=1}^{n}\frac{1}{4}Gu(0,2^k t)\right| \leqslant \frac{\varepsilon}{4}, \tag{3.30}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $t > 0$. It follows from (3.27) and (3.28) that

$$\left|\sum_{k=1}^{n}\frac{1}{4^k}Gu(0,2^k t) - (1-\frac{1}{2^n})h(t)\right| \leqslant \frac{\varepsilon}{8}. \tag{3.31}$$

for all $n \in \mathbb{N}$, $t > 0$. From (3.30) and (3.31), letting $F(x,t) := Gu(x,t) - h(t)$ we have

$$\left|F(x,t) - \frac{1}{4^n}F(2^n x,\ 2^n t)\right| \leqslant \frac{\varepsilon}{8} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{8}, \tag{3.32}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Using similar method we have done, it can be seen that the function $g : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ defined by

$$g(x,t) = \lim_{m\to\infty} \frac{1}{4^m}F(2^m x, 2^m t)$$

is the unique function satisfying

$$|F(x,t) - g(x,t)| \leqslant \frac{3\varepsilon}{8} \tag{3.33}$$

and

$$
\begin{aligned}
&g(x+y+z,t+s+r)+g(x-y+z,t+s+r)+g(x+y-z,t+s+r)\\
&\qquad +g(-x+y+z,t+s+r)-4g(x,t)-4g(y,s)-4g(z,r)=0,
\end{aligned} \tag{3.34}
$$

for all $x,y,z\in\mathbb{R}^n$ and $t,s,r>0$.

Let $q(x,t):=g(x,t)+h(t)$. Then $q(x,t)$ is a continuous function satisfying the equation(3.34) and, by the lemma 3.3, has the form

$$
q(x,t)=\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}\,x_ix_j+bt,
$$

for $x=(x_1,\cdots x_n)\in\mathbb{R}^n$, $t>0$. Thus in view of the equation (3.33) we have

$$
|Gu(x,t)-\sum_{1\leqslant i\leqslant j\leqslant n} a_{ij}\,x_ix_j-bt|\leqslant\frac{3}{8}\varepsilon. \tag{3.35}
$$

for $x=(x_1,\cdots x_n)\in\mathbb{R}^n$, $t>0$. Letting $t\to 0^+$ in (3.35) we get (3.21). This completes the proof. □

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