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ON AN IMPROVEMENT OF BERNSTEIN'S POLYNOMIAL INEQUALITIES

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Abstract. The following Bernstein inequality

$$\max_{|z|\leqslant 1} |p'(z)| \leqslant n \max_{|z|\leqslant 1} |p(z)|,$$

valid for all complex polynomials p of degree n, has been extended by Ruscheweyh to

$$\max_{|z|\leqslant 1} |p'(z)| \leqslant n \max_{|z|\leqslant 1} |p(z)| - \frac{2n}{n+2}|p(0)|, \quad n \geqslant 2.$$

We prove in this note that two other Bernstein inequalities, i.e.,

$$\max_{-1\leqslant x\leqslant 1} \left|\sqrt{1-x^2}p'(x)\right| \leqslant n \max_{-1\leqslant x\leqslant 1} |p(x)|$$

or

$$\max_{0\leqslant \theta\leqslant 2\pi} |t'(\theta)| \leqslant n \max_{0\leqslant \theta\leqslant 2\pi} |t(\theta)|,$$

where $t(\theta)$ is a complex trigonometric polynomial of degree n do not admit similar extensions. In addition we obtain a new proof of Marcel Riesz interpolation formula.

1. Introduction

Let $\mathcal{P}_n$ denote the set of polynomials $p(z) := \sum_{k=0}^{n} a_k(p)z^k$ with complex coefficients endowed with the norm

$$|p|_{\mathbb{D}} := \max_{z\in\partial\mathbb{D}} |p(z)|,$$

where $\mathbb{D} := \{z \mid |z| < 1\}$ is the unit disc of the complex plane. Let $\mathcal{T}_n$ denote the set of trigonometric polynomials $t(\theta) := \sum_{k=-n}^{n} a_k(t)e^{ik\theta}$ with complex coefficients endowed with the norm

$$|t|_{\mathbb{R}} := \max_{\theta\in\mathbb{R}} |t(\theta)|.$$

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We shall also consider the linear space $\widetilde{\mathcal{P}}_n$ of polynomials

$$p(z) := \sum_{k=0}^{n} A_k(p) T_k(z)$$

(here T_k is the k^{th} Chebyshev polynomial, i.e., $T_k(\cos\theta) = \cos k\theta$ for all $\theta \in \mathbb{R}$) equipped with the norm

$$|p|_{[-1,1]} := \max_{x \in [-1,1]} |p(x)|.$$

The inequalities

(1) $|p'|_{\mathbb{D}} \leqslant n|p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n,$

(2) $|t'|_{\mathbb{R}} \leqslant n|t|_{\mathbb{R}}, \quad t \in \mathcal{T}_n,$

(3) $\left|\sqrt{1-x^2}p'(x)\right|_{[-1,1]} \leqslant n|p|_{[-1,1]}, \quad p \in \widetilde{\mathcal{P}}_n$

are due to Bernstein and have played an important role in the early history of approximation theory in the twentieth century. We refer to the book by Rahman and Schmeisser [7] and by Milovanović, Mitinović and Rassias [6] concerning deatils and historicl matters. It is well known that equality holds in (1) only if $p(z) := Kz^n$ with $K \in \mathbb{C}$. Equality holds in (2) only if $t(\theta) := K_1 e^{-in\theta} + K_2 e^{in\theta}$; finally equality holds in (3) only when p is a multiple of the n^{th} Chebyshev polynomial.

We shall be concerned in this paper by the following refinement of (1).

THEOREM A. *For any $p \in \mathcal{P}_n$ $(n \geqslant 2)$,*

$$|p'|_{\mathbb{D}} \leqslant n|p| - \frac{2n}{n+2}|a_0(p)|.$$

Theorem A is due to Ruscheweyh [8]. Variants of his result (involving $|a_1(p)|$ or $|a_2(p)|$ or higher order coefficients) have appeared in [3, 4]. It has been shown recently [2] that equality holds in Theorem A if and only if p is a monomial of degree n.

We shall prove

THEOREM 1. *Let $n \geqslant 1$ and $-n \leqslant j \leqslant n$; there exists no positive constant $d > 0$ such that*

$$|t'|_{\mathbb{R}} \leqslant n|t|_{\mathbb{R}} - d|a_j(t)|, \quad t \in \mathcal{T}_n.$$

This means that no immediate extension of Theorem A to the class $\mathcal{T}_n$ is possible. Our proof of Theorem 1 relies on the notion of bound-preserving operators over $\mathcal{P}_n$. Let $\mathcal{H}(\mathbb{D})$ denote the class of functions analytic in $\mathbb{D}$; for two members $f(z) := \sum_{n=0}^{\infty} a_n(f) z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g) z^n$ of $\mathcal{H}(\mathbb{D})$, the convolution $f \star g$ is defined as

$$f \star g(z) := \sum_{n=0}^{\infty} a_n(f) a_n(g) z^n.$$

Clearly $f \star g$ also belongs to $\mathcal{H}(\mathbb{D})$. A function $f \in \mathcal{H}(\mathbb{D})$ is called bound-preserving over $\mathcal{P}_n$ if

$$|f \star p|_{\mathbb{D}} \leqslant |p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n.$$

The following theorem gives a characterization of the class $\mathcal{B}_n$ of bound-preserving functions over $\mathcal{P}_n$.

THEOREM B. *$f \in \mathcal{B}_n$ if and only if*

$$f(z) = \int_{\partial\mathbb{D}} \frac{1}{1-\xi z}\, d\mu(\xi) + o(z^n), \quad z \in \mathbb{D},$$

where μ is a complex Borel measure over $\partial\mathbb{D}$ with $\int_{\partial\mathbb{D}} |d\mu(\xi)| \leqslant 1$.

Here $o(z^n)$ means a function of the type $z^{n+1}g(z)$ where $g \in H(\mathbb{D})$. It follows in particular that for $f \in \mathcal{B}_n$ with $f(0) = 1$, the measure μ in Theorem B may be chosen as a probability measure and the class $\{f \in \mathcal{B}_n \mid f(0) = 1\}$ is identical with the class

$$P_{1/2}(n) = \left\{f \in \mathcal{H}(\mathbb{D}) \mid f(0) = 1 \text{ and } \operatorname{Re}\big(f(z) + o(z^n)\big) > \tfrac{1}{2}, z \in \mathbb{D}\right\}.$$

This is a consequence of the celebrated formula of Herglotz. Further facts concerning $\mathcal{B}_n$ may be found in Ruscheweyh's lecture notes [8] or in the more recent book by Sheil–Small [9].

Proof of Theorem 1

Let $t(\theta) := \sum_{k=-n}^{n} a_k(t)e^{ik\theta}$. It follows from Theorem A, when applied to $p(z) := \sum_{k=0}^{2n} a_{k-n}(t)z^k \in \mathcal{P}_{2n}$, that

$$|t'(\theta) \pm int(\theta)| \leqslant 2n|t|_{\mathbb{R}} - \frac{2n}{n+1}|a_{\mp n}(t)|, \quad \theta \in \mathbb{R}.$$

This last inequality, together with (2) , seems to suggest that a statement of the type

$$|t'|_{\mathbb{R}} \leqslant n|t|_{\mathbb{R}} - d|a_{\mp n}(t)|$$

might be valid for some $d > 0$. Our Theorem 1 shows that this is not the case.

For a given $t \in \mathcal{T}_n$, the following inequalities are easily seen to be equivalent:

$$|t'|_{\mathbb{R}} + d|a_j(t)| \leqslant n|t|_{\mathbb{R}},$$

$$\left|\frac{t'(\theta)}{n} + \frac{d}{n}a_j(t)e^{ij\theta}e^{i\psi}\right| \leqslant n|t|_{\mathbb{R}}, \quad \theta, \psi \in \mathbb{R},$$

$$\left|\sum_{k=-n}^{n} \frac{k}{n}a_k(t)e^{i(n+k)\theta} + \frac{d}{n}a_j(t)e^{i(n+j)\theta}e^{i\psi}\right| \leqslant n|t|_{\mathbb{R}}, \quad \theta, \psi \in \mathbb{R},$$

$$\left|\sum_{k=0}^{2n} a_{k-n}(t)z^k \star \left(\sum_{k=0}^{2n} \frac{n-k}{n}z^k + e^{i\psi}\frac{d}{n}z^{j+n}\right)\right| \leqslant \max_{u\in\mathbb{D}} \left|\sum_{k=0}^{2n} a_{k-n}(t)u^k\right|$$

this last inequality being valid for all $z \in \mathbb{D}$ and $\psi \in \mathbb{R}$. It follows in particular that the statement of Theorem 1 amounts to the fact that for each $d > 0$,

$$F_{d,\psi,j}(z) := 1 + \sum_{k=1}^{2n} \frac{n-k}{n}z^k + e^{i\psi}\frac{d}{n}z^{j+n},$$

is not bound preserving for at least one real value ψ_0 of ψ, i.e., $F_{d,\psi_0,j}$ does not belong to $\mathcal{B}_{2n}$. By Theorem B, the Taylor coefficients of any function in $\mathcal{B}_{2n}$ must belong to the closed unit disc; since $a_0(F_{d,\psi,-n}) = 1 + e^{i\psi}\frac{d}{n}$ and $a_{2n}(F_{d,\psi,n}) = -1 + e^{i\psi}\frac{d}{n}$, we only have to prove Theorem 1 for $-n < j < n$.

Let us now assume that $F_{d,\psi,j} \in \mathcal{B}_{2n}$, for all $\psi \in \mathbb{R}$ and some $d \geqslant 0$; then $F_{d,\psi,j} \in P_{1/2}(2n)$. We shall need the following lemma, whose proof can be found in ([5, Chapters 4 and 7]):

LEMMA A. *Let* $F(z) := 1 + \sum_{k=1}^{2n} b_k(F) z^k \in P_{1/2}(2n)$. *Then for* $1 \leqslant k \leqslant 2n$,

$$-1 \leqslant Re\big(b_k(F)\big)$$

with equality for some k *iff*

$$F(z) = \sum_{m=1}^{k} \frac{\ell_m}{1 - \xi_m z} + o(z^{2n})$$

where $\{\xi_m\}_{m=1}^{k}$ *is the set of distinct complex* k*-roots of* -1 *and* $\ell_m \geqslant 0$, $1 \leqslant m \leqslant k$.

Of course $b_{2n}(F_{d,\psi,j}) = -1$ and by Lemma A we obtain, for any real ψ,

$$F_{d,\psi,j}(z) = 1 + \sum_{k=1}^{2n} \frac{n-k}{n} z^k + e^{i\psi}\frac{d}{n} z^{j+n} = \sum_{k=1}^{2n} \frac{\ell_k}{1 - \xi_k z} + o(z^{2n}) \tag{1}$$

with $\ell_k = \ell_k(\psi) \geqslant 0$, $1 \leqslant k \leqslant 2n$ and $W(z) := \prod_{k=1}^{2n}(z - \xi_k) = z^{2n} + 1$. By comparing coefficients in (1) we are led to the linear system

$$\sum_{k=1}^{2n} \ell_k \xi_k^m = \frac{n-m}{n} + \delta_{m,n+j}\frac{d}{n} e^{i\psi}, \quad m = 0, 1, \ldots, 2n-1, \tag{2}$$

where δ is the Kronecker's symbol. We wish to solve this system explicitly for $\{\ell_1, \ell_2, \ldots, \ell_{2n}\}$. We write (2) as

$$V^T L = C$$

where V is the $2n \times 2n$ Vandermonde matrix associated with the set $\{\xi_k\}_{k=1}^{2n}$, $L = (\ell_1, \ell_2, \ldots, \ell_{2n})^T$ and C is the column-matrix formed with the constants to the right of the equations in the system (2). Then of course

$$L = (V^T)^{-1} C = (V^{-1})^T C$$

and if

$$\frac{W(z)}{(z - \xi_k) W'(\xi_k)} = \frac{z^{2n} + 1}{-2n\bar{\xi}_k (z - \xi_k)} := \sum_{m=0}^{2n-1} W_{m,k} z^m, \tag{3}$$

we obtain from a well known property of Vandermonde matrices ([1, p. 64])

$$\ell_k = \ell_k(\psi) = \sum_{m=0}^{2n-1} \frac{n-m}{n} W_{m,k} + W_{n+j,k}\frac{d e^{i\psi}}{n}, \quad \psi \in \mathbb{R}. \tag{4}$$

Now by comparing coefficients in (3) we obtain

$$W_{m,k} = \frac{\xi_k^{-m-2}}{2n}, \quad 0 \leqslant m \leqslant 2n-1$$

and in particular $W_{m,k} \neq 0$ for all admissible values of m. Since $\ell_k \geqslant 0$, we must now deduce from (4) that $d = 0$. This completes our proof of Theorem 1.

Further computations yield

$$\sum_{m=0}^{2n} \frac{n-m}{n} z^m = \frac{1}{4n^2} \sum_{k=1}^{2n} \frac{\csc^2\left(\frac{(2k-1)\pi}{4n}\right)}{1-\xi_k z} + o(z^{2n}), \quad z \in \mathbb{D}, \tag{5}$$

and convoluting both sides of (5) with $q \in \mathcal{P}_{2n}$ we easily obtain

$$q(z) - \frac{zq'(z)}{n} = \sum_{k=1}^{2n} \lambda_k q(\xi_k z), \quad z \in \mathbb{C} \tag{6}$$

with $\lambda_k := \frac{\csc^2\left(\frac{(2k-1)\pi}{4n}\right)}{4n^2} > 0$, $\sum_{k=1}^{2n} \lambda_k = 1$ and $\xi_k = e^{(2k-1)i\pi/2n}$. We also wish to point out the following known consequence of (5). Our contribution here concerns the case of equality.

COROLLARY 11. *Let $p \in \mathcal{P}_{2n}$. Then*

$$\left| p(z) - \frac{zp'(z)}{n} \right|_{\mathbb{D}} \leqslant |p|_{\mathbb{D}}$$

with equality iff $p(z) \equiv A + Bz^{2n}$, $A, B \in \mathbb{C}$.

The identity (5) has a wider scope; for example an application of (6) to $q \in \mathcal{P}_{2n}$ defined by

$$q(e^{i\theta}) = e^{in\theta} t(\theta)$$

where $t \in \mathcal{T}_n$ yields the well-known Marcel Riesz interpolation formula for trigonometric polynomials. Indeed, (6) is equivalent to the Riesz formula.

Conclusion

An argument rather similar to the previous one leads to the following result:

THEOREM 2. *Let $n \geqslant 1$ and $-n \leqslant j \leqslant n$; there exists no positive constant $d > 0$ such that*

$$\left| \sqrt{1-x^2} p'(x) \right|_{[-1,1]} \leqslant n|p|_{[-1,1]} - d|A_j(p)|, \quad p \in \widetilde{\mathcal{P}}_n \tag{7}$$

Indeed, it is possible to give a simple class of counterexample to (7). It is clearly enough to assume that $0 \leqslant j < n$. We consider $p \in \widetilde{\mathcal{P}}_n$ defined by

$$p(x) := T_n(x) + i\varepsilon T_j(x) \tag{8}$$

where ε is real. Then we have $|p|_{[-1,1]} = \sqrt{1+\varepsilon^2}$ and

$$|\sin(\theta)p'(\cos\theta)| = n\sqrt{1+\varepsilon^2\frac{j^2}{n^2}\sin^2(j\theta)}$$

if θ is chosen such that $\sin(n\theta) = 1$. Assuming now that (7) holds we readily obtain

$$n\sqrt{1+\varepsilon\frac{j^2}{n^2}\sin^2(j\theta)} \leqslant n\sqrt{1+\varepsilon^2} - d|\varepsilon|$$

i.e., $d \leqslant n|\varepsilon|\big(1-\frac{j^2}{n^2}\sin^2(j\theta)\big)$ and the conclusion follows by letting $\varepsilon \to 0$. A similar example related to Theorem 1 can be obtained by considering

$$t(\theta) := \cos(n\theta) + i\varepsilon\cos(j\theta) \tag{9}$$

with ε real. The fact that the polynomials in (8) and (9) have imaginary coefficients is irrelevant: the trigonometric polynomials

$$t_n(\theta) := -e^{-in\theta} + \varepsilon + e^{in\theta}, \quad \varepsilon > 0$$

may be also used instead of (9). An interesting problem is the following: does there exist a constant $d_n' > 0$ such that

$$|t'|_{\mathbb{R}} \leqslant n|t|_{\mathbb{R}} - d_n'|A_0(t)|$$

for all trigonometric polynomials $t(\theta)$ such that

$$t(\theta) := \sum_{k=-n}^{n} A_k(t)e^{ik\theta}, \quad A_{-k}(t) = \bar{A_k}(t), \qquad (0 \leqslant k \leqslant n)?$$

We are unable to decide this question which is equivalent to the following conjectural variant of a famous inequality of Szëgo:

$$\big|\mathrm{Im}\big(zp'(z)\big)\big|_{\mathbb{D}} \leqslant n\frac{\beta_p-\alpha_p}{2} - d_n'\left|\frac{\beta_p+\alpha_p}{2} - \mathrm{Re}\big(p(0)\big)\right|, \quad (p \in \mathcal{P}_n)$$

with

$$\alpha_p = \min_{z\in\partial\mathbb{D}} \mathrm{Re}\big(p(z)\big) \quad \text{and} \quad \beta_p = \max_{z\in\partial\mathbb{D}} \mathrm{Re}\big(p(z)\big).$$

Finally we have the following explanation concerning the sharp contrast between the conclusion of Theorem A and Theorem 1: the “Fejer” polynomials

$$F_n(z) := \sum_{k=0}^{n-1}\frac{n-k}{n}z^k$$

satisfy as well known

$$\mathrm{Re}F_n(z) > \tfrac{1}{2}, \quad z \in \mathbb{D}$$

but they in some sense do not live on the natural boundary of the function set $P_{1/2}(n)$, i.e., we may perturb them to the point that

$$F_n(z) + \varepsilon z^j \in P_{1/2}(n), \quad 0 < j \leqslant n,\ |\varepsilon| < d_{n,j},$$

for strictly positive constants $d_{n,j}$. The known proofs of Theorem A and its generalizations are based on that plain fact. The negative conclusion of Theorem 1 is a consequence of the fact that

$$G_{2n}(z) := \sum_{k=0}^{2n} \frac{n-k}{n} z^k$$

belongs to the boundary of $P_{1/2}(2n)$ (remark that $a_{2n}(G_{2n}) = -1$ and recall Lemma A) and for that reason cannot be perturbed as above.

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