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SOME OPTIMIZATION PROBLEMS FOR THE MINIMAL ANNULUS OF A CONVEX SET

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(*communicated by C. Bandle*)

Abstract. In this paper we relate the minimal annulus of a planar convex body K with the six classic geometric measures associated with it. First, we obtain all the possible bounds (upper and lower bounds) for the measures A, p, D, ω, R_K and r_K of a convex body K with given minimal annulus. Then, we solve the problem of maximizing and minimizing the area and the perimeter of convex bodies with given circumradius and minimal annulus. We prove the optimal inequalities for each of those problems, determining also its corresponding extremal sets.

1. Introduction

Let K be a convex body (i.e., a compact convex set) in the Euclidean plane. Associated with K are a number of well-known functionals: the *area* $A = A(K)$ and the *perimeter* $p = p(K)$; the *diameter* $D = D(K)$ and the *minimal width* $\omega = \omega(K)$ (respectively, the maximum and the minimum distance between two parallel support hyperplanes of K); among all balls containing K, there is exactly one with minimal radius, the *circumradius* R_K; respectively, among all balls which are contained in K, those ones whose radii have maximal value give the *inradius* of the body, r_K. Besides, these special balls (named circumball and inballs) have very interesting properties; some of them will be stated and used later.

For many years mathematicians have been interested in inequalities involving these functionals, and moreover, in finding the convex sets for which the equality sign is attained: the extremal sets. Thus, one of the most studied problems is to find the convex sets which maximize or minimize a particular magnitude Z when other two measures, say X and Y, are fixed. The solution of such a problem is always expressed by means of extremal inequalities of the form $Z \leqslant \varphi(X, Y)$ (the bibliography is extremely wide; let us mention, for instance, [3] or [10]). The question becomes more interesting when the equality for a particular inequality is not attained for a single figure, but for a continuous family of sets; in this case, that inequality (that we name *optimal* inequality) says which is the maximum or minimum value of Z for *each pair* of possible values of the magnitudes X and Y.

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But another interesting functional can be consider for a convex body K: the thick of its *minimal annulus*. The minimal annulus of the body K is the annulus (the closed set consisting of the points between two concentric balls) with minimal difference of radii that contains the boundary of K. Of course, the minimal annulus is uniquely determined (Bonnesen, [2], in $\mathbb{R}^2$ and $\mathbb{R}^3$, and Bárány, [1], in higher dimension).

From now on, we are going to denote by $A(c,r,R)$ the minimal annulus of the planar convex body K, where c, r and R represent, respectively, its center, radius of the inner circle, and radius of the outer circle. This object and its properties were studied mainly by Bonnesen for planar convex sets (see [2] and [3]). More recently, very interesting works have arisen, in which, the minimal annulus has been studied in a more general way: for arbitrary dimension, replacing the ball by the boundary of a fixed smooth strictly convex body, in Minkowski space... (see, for instance, [1, 6, 7, 8, 9, 12]).

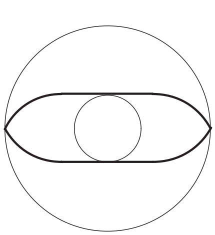

Figure 1.

Bonnesen ([2]) studied some properties of the minimal annulus (which will be stated later) in order to sharp the isoperimetric inequality in $\mathbb{R}^2$. He proved that the minimum of the isoperimetric deficit $p^2/(4\pi) - A$ is attained for the convex set shown in Figure 1.

About the opposite bound, according to Favard ([5]), the isoperimetric deficit is maximized by a certain polygon which is circumscribed about the inner circle of the minimal annulus, and whose vertices, except at most one, lie on the boundary of the outer circle.

In this paper we study some inequalities involving the above six classic geometric measures and the minimal annulus. First, we show some nice properties of the minimal annulus itself, as well as its relation with the inball and circumball of the set; these properties will be very useful in the proofs of the remaining results. Then, we obtain all the possible bounds (upper and lower bounds) for the measures A, p, D, ω, R_K and r_K of a convex body K, when the minimal annulus of K is fixed. Finally, we solve the problem of maximizing and minimizing the area and the perimeter of convex bodies with given circumradius and minimal annulus. We prove the optimal inequalities for each of those problems, determining also their corresponding extremal sets.

2. Some previous results

Before presenting the main results of the paper, let us state some properties of the minimal annulus of a convex body K, which will play a crucial role in the proofs of the results. For, let us denote by c_r and C_R, respectively, the inner and the outer circles of the minimal annulus $A(c,r,R)$ of K. As usual in the literature, ∂ will denote the boundary of a set. The following four properties are well-known, and were obtained by Bonnesen in [2]:

$(P1)$ *Each of the circumferences ∂c_r and ∂C_R touches the boundary ∂K of K in, at least, two points.*

$(P2)$ *The sets $\partial c_r \cap \partial K$ and $\partial C_R \cap \partial K$ can not be separated.*
(Two sets A and B can be separated if there exists a line ℓ such that $A \subset \ell^+$ and $B \subset \ell^-$, where ℓ^+, ℓ^- represent the halfplanes determined by ℓ).

$(P3)$ *The minimal annulus of a convex body* K *is uniquely determined.*

$(P4)$ *The minimal annulus of a convex body* K *is the only annulus that contains* ∂K *and verifies properties* $(P1)$ *and* $(P2)$.

With the following lemmas, we prove some properties of the minimal annulus of a convex body itself, as well as its relation with the circumradius and the inradius of the convex body. They will be very useful in the proofs of the results. Let us start with some general properties of $A(c,r,R)$:

LEMMA 1. *Let* K *be a convex body with minimal annulus* $A(c,r,R)$. *The following properties hold:*

(a) *There exist two points* $P,Q \in \partial C_R \cap \partial K$ *such that the angle* α *determined by them with respect to* c*, this is,* $\alpha = \sphericalangle(PcQ)$*, verifies*

$$\alpha \geqslant 2\arccos\frac{r}{R}$$

(from now on, we will name such an angle, the central angle *of* P *and* Q*).*

(b) K *contains a 2-cap-body generated by the convex hull of* c_r *and two points of* $\partial C_R \cap \partial K$*, whose minimal annulus is* $A(c,r,R)$ *(a cap-body is the convex hull of a ball and countable many points such that the line segment joining any pair of those points intersects the ball).*

(c) K *is contained in a circular slice of the outer circle* C_R *determined by two support lines to* c_r*, whose minimal annulus is* $A(c,r,R)$ *(a circular slice is the part of a circle bounded by two straight lines, whose intersection point, if it exists, is not interior to it).*

Proof. (a) is a consequence of property $(P2)$: let us notice that any support line to c_r intersects ∂C_R in two points P and Q, being the central angle of those points, precisely, $2\arccos(r/R)$ (see Figure 2); if any two points of $\partial C_R \cap \partial K$ determine a central angle strictly less than $2\arccos(r/R)$, then a suitable straight line PQ will separate c_r and the intersection points $\partial C_R \cap \partial K$, contradicting property $(P2)$.

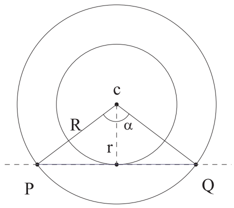

Figure 2. There are points $P,Q \in \partial C_R \cap \partial K$ *whose central angle* α *is greater or equal than* $2\arccos(r/R)$.

(b) is a consequence of item (a), which assures that the convex hull $\operatorname{conv}\{c_r,P,Q\}$ is, indeed, a cap-body.

Finally, we prove (c). By property $(P1)$, there exist, at least, two points $C,D \in \partial c_r \cap \partial K$ such that the support lines ℓ_C, ℓ_D to c_r through those points intersect C_R, determining a convex body K' which contains K. If ℓ_C and ℓ_D are parallel or their intersection point is not interior to C_R, then K' is a circular slice of C_R, and the result is proved. If the intersection point lies in the interior of C_R, let us consider the circular arc $\widehat{AB}$ determined by the support lines ℓ_C and ℓ_D on ∂C_R (see Figure 3). Now, property $(P1)$ assures the existence of two points $S, T \in \partial C_R \cap \partial K$ which, by item (a), determine a central angle $\alpha \geqslant 2 \arccos(r/R)$.

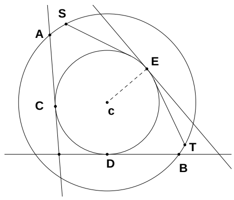

Figure 3. K *is contained in a circular slice of* C_R.

Let us notice that the arcs $\widehat{AB}$ and $\widehat{CD}$ are separated; hence, by property $(P2)$, there exists a point $E \in \partial c_r \cap \partial K$ lying on the portion of the arc determined by the tangent lines to ∂c_r through S and T. The support line to c_r through E determines, joined with ℓ_C (or ℓ_D) the required circular slice.

The following lemma collects some properties relating the minimal annulus of a convex body with its circumradius. From now on, we are going to denote by C_K the circumball of the body K, and by x_0 its circumcenter.

LEMMA 2. *Let* K *be a convex body with minimal annulus* $A(c,r,R)$ *and circumball* C_K. *The following properties hold:*

(i) $R_K \leqslant R$.

(ii) $c_r \subset K \subset C_R \cap C_K$.

(iii) *Either* $C_R \equiv C_K$, *or* $\partial C_K \cap \partial C_R$ *has exactly two points, which will be denoted by* A *and* B.

(iv) *If* $C_K \not\equiv C_R$, *then the points* $\{A,B\} = \partial C_K \cap \partial C_R$ *determine a central angle* α *such that*

$$\alpha \geqslant 2 \arccos \frac{r}{R}. \tag{1}$$

(v) *The circular arc* $\widehat{AB}$ *of* ∂C_K *which lies in* C_R *can not be smaller than a semi-circumference (of* ∂C_K *).*

(vi) *The tangent line to* c_r, *which is parallel and closer to the segment* $\overline{AB}$, *intersects* ∂C_R *in two points* A' *and* B', *such that there exists, at least, one point* $P \in \partial K \cap \partial C_R$ *lying on one of the arcs* $\widehat{AA'}$, $\widehat{BB'}$. *Without loss of generality, let us suppose that* $P \in \widehat{AA'}$. *Then, there exists another point* $Q \neq P$ *lying on the arc* $\widehat{PB}$, *such that the central angle determined by* P *and* Q *verifies* (1), *see Figure 4.*

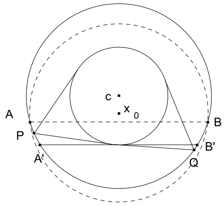

Figure 4. There are, at least, two points $P, Q \in \partial K \cap \partial C_R$.

(vii) K *contains the 2-cap-body* $K^c = \text{conv}\{c_r, P, Q\}$*, where* P *and* Q *are the points obtained in* (vi).

(viii) The 2-cap-body K^c *of the above property* (vii) *determines on the boundary of* c_r *two circular arcs, each one having, at least, one point of* ∂K.

(ix) K *is contained in the intersection of* C_K *with the circular slice of* C_R *determined by the support lines to* c_r *through the two points of* $\partial K \cap \partial c_r$ *given by property* $(viii)$.

Proof. Properties (i) and (ii) are trivial: they can be deduced from the definitions of circumball and minimal annulus. In order to prove (iii), let us suppose that $C_K \not\equiv C_R$. Of course, $C_K \not\subset C_R$, because by property $(P1)$ $\partial K \cap \partial C_R$ contains, at least, two points. Then, we can assure that the intersection $\partial C_K \cap \partial C_R$ has, exactly, two points.

Let us see (iv). Since $K \subset C_K \cap C_R$, the set $\partial K \cap \partial C_R$ (not empty) is contained in the circular arc of ∂C_R determined by A and B (the points obtained in the previous item). Thus, if the central angle determined by them is less than $2\arccos(r/R)$, we obtain a contradiction with property (a) of Lemma 1.

Item (v) is a consequence of the well-known property of the convex sets which assures that the circumball C_K contains either two diametrically opposite points of the boundary of K, or three points of ∂K that form the vertices of an acute-angled triangle (see, for instance, [3]); since it holds $\partial K \cap \partial C_K \subset \overset{\frown}{AB} \subset \partial C_K \subset C_R$, both cases imply that the circular arc $\overset{\frown}{AB}$ is greater or equal than a semi-circumference of ∂C_K.

Items (vi) and (vii) are direct consequences of (iv) and Lemma 1. Hence, we have to see $(viii)$. By property $(P1)$ there are, at least, two points in the intersection $\partial c_r \cap \partial K$; and this set will be contained in the two circular arcs of ∂c_r which form the boundary of the cap-body K^c. Let us suppose first that the above intersection is contained just in the circular arc of $\partial c_r \cap \partial K^c$ that lies over the line segment $\overline{AB}$; then, the straight line ℓ, parallel to $\overline{AB}$, through c, separates $\partial c_r \cap \partial K$ from $\partial C_R \cap \partial K$, a contradiction with property $(P2)$ (see Figure 5 (a)).

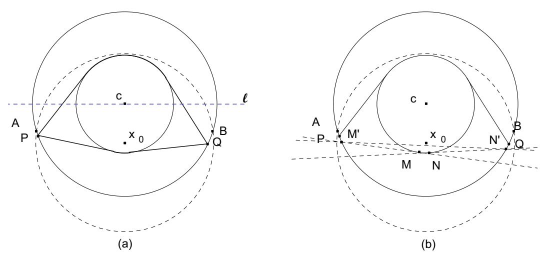

Figure 5. There are points of ∂K *on the circular arcs of* ∂K^c.

Finally, we suppose that the set $\partial c_r \cap \partial K$ is contained just in the circular arc of $\partial c_r \cap \partial K^c$ that lies under the line segment $\overline{AB}$; let M, N be the extreme points of the circular arc of ∂c_r containing $\partial c_r \cap \partial K$ (let us notice that $M \neq N$ since there are, at least, two different points in $\partial c_r \cap \partial K$). Then, K is contained in the part of $C_K \cap C_R$ determined by the support lines to c_r through M and N. We denote by M' and N' the intersection points of these support lines with ∂C_R, respectively (see Figure 5 (b)). The line segment $\overline{MN}$ lies under the segment $\overline{M'N'}$; hence, the straight line

$M'N'$ separates $\partial C_R \cap \partial K$ from $\widehat{MN}$, and thus, from $\partial c_r \cap \partial K$, again a contradiction. Finally, item (ix) is obvious from property $(viii)$.

From this moment on, we will follow the notation of the above Lemma 2: A, B will represent the intersection points of ∂C_K and ∂C_R; besides, we will denote by A' and B' the intersection points of ∂C_R with the parallel line to AB which is tangent to ∂c_r (see Figure 4).

As in the circumradius case, the following lemma collects some properties relating the minimal annulus of a convex body with its inradius. From now on, we are going to denote by c_K an inball of the body K, and by y_0 one of its incenters.

LEMMA 3. *Let K be a convex body with minimal annulus $A(c,r,R)$ and inball c_K. The following properties hold:*

(i) $r \leqslant r_K$.

(ii) $\mathrm{conv}(c_r \cup c_K) \subset K \subset C_R$.

(iii) *c_r can not be strictly contained in c_K, being the possible relative positions between them the following (see Figure 6):*

- (a) $c_r \equiv c_K$.
- (b) *$\partial c_r \cap \partial c_K$ contains, exactly, two points.*
- (c) *The boundaries ∂c_r, ∂c_K touch (from outside) in one point.*
- (d) *There are no common points.*

(iv) *The boundary of the set $\mathrm{conv}(c_r \cup c_K)$ is formed by two line segments $\overline{PS}$ and $\overline{QT}$, and the corresponding circular arcs $\widehat{PQ} \subset \partial c_r$ and $\widehat{ST} \subset \partial c_K$. Then, each of those arcs has, at least, one point of ∂K.*

(v) *The circular arc $\widehat{PQ} \subset \partial c_r$ contains two points $P', Q' \in \partial K$ (which can coincide with P and Q), determining a central angle $\alpha \geqslant 2\,\mathrm{arc}\cos(r/R)$.*

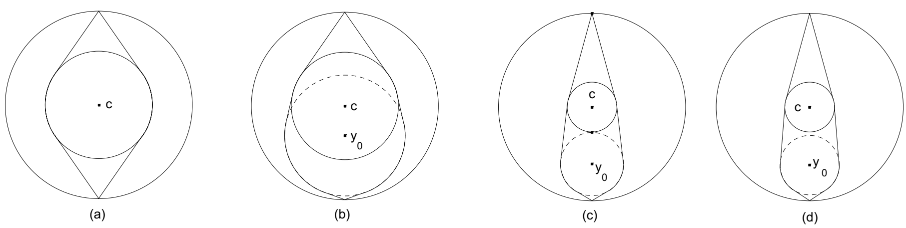

Figure 6. Some examples for the relative positions of c_r and c_K.

Proof. Properties (i) and (ii) are trivial. Let us see (iii). Obviously, c_r can not be contained in c_K, because property $(P1)$ states that $\partial c_r \cap \partial K$ contains, at least, two points. It is easy to find examples of the different possibilities for the relative positions between c_r and c_K (see Figure 6); just one remark: in order to obtain the cases (c) and (d), it is a necessary condition that $R \geqslant 3r$.

The circular arc $\widehat{PQ}$ is the only part of ∂c_r lying outside $\mathrm{conv}(c_r \cup c_K)$; hence, the points of $\partial c_r \cap \partial K$ (whose existence is assured by $(P1)$) must lie on that arc. The same happens with the arc $\widehat{ST}$ with respect to the inball: we just have to use the well-known property of the convex sets which assures that the incircle c_K of K meets the boundary of K either in two diametrically opposite points, or in three points that form the vertices of an acute-angled triangle (see [3]). This shows (iv).

Finally, Lemma 1 (c) establishes that K is contained in a circular slice of C_R determined by support lines to c_r through two points, say P' and Q', of $\partial c_r \cap \partial K$; clearly, these two points P', Q' lie necessarily on the arc $\overset{\frown}{PQ}$. Besides, these support lines either intersect on the boundary ∂C_R, or outside C_R (they can be parallel): in the first case, the central angle determined by P' and Q' is, precisely, $2 \arccos(r/R)$; in the second case, the central angle will be strictly greater. It shows (v).

3. The minimal annulus and the six classical geometric magnitudes

In this section we are going to state the relation between the minimal annulus and the other six classic geometric magnitudes. More precisely, we are going to obtain the best bounds (upper and lower bounds) for A, p, D, ω, R_K and r_K when we suppose that the minimal annulus of the convex body is fixed, determining also the extremal sets in each case. Let us say that the area and perimeter cases were studied and solved by Favard in [4]. We are going to state them for completeness, but without proof.

The area and the perimeter

PROPOSITION 1. *Let K be a convex body with minimal annulus $A(c,r,R)$. Then:*

$$A \geqslant 2r\left(\sqrt{R^2 - r^2} + r \arcsin \frac{r}{R}\right), \tag{2}$$

$$p \geqslant 4\left(\sqrt{R^2 - r^2} + r \arcsin \frac{r}{R}\right). \tag{3}$$

The equality holds, in both inequalities, for any cap-body given by the convex hull of c_r and two points of $\partial C_R \cap \partial K$.

$$A \leqslant 2\left(r\sqrt{R^2 - r^2} + R^2 \arcsin \frac{r}{R}\right), \tag{4}$$

$$p \leqslant 4\left(\sqrt{R^2 - r^2} + R \arcsin \frac{r}{R}\right). \tag{5}$$

The equality holds, in both inequalities, for the circular slices of C_R determined by two support lines to c_r.

The diameter and the minimal width

Before stating the corresponding theorem for these magnitudes, let us establish some notation: Lemma 1 (c) assures that K is contained in a circular slice of C_R determined by two support lines to c_r; then, following the notation of Figure 7,we denote by C and D the contact points of these support lines with c_r, by P, Q, S and T the intersection points of these support lines with ∂C_R, and by N the “north pole” of C_R.

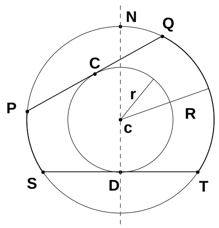

Figure 7. Some notation for the circular slice of C_R *.*

PROPOSITION 2. *Let* K *be a convex body with minimal annulus* $A(c,r,R)$*. Then:*

$$\omega \geqslant 2r. \tag{6}$$

The equality holds, for instance, for the circular symmetric slice of C_R *determined by two parallel support lines to* c_r*.*

$$\omega \leqslant \begin{cases} R+r & \text{if } R \leqslant 2r, \quad (7.a) \\ \dfrac{4r}{R^2}(R^2-r^2) & \text{if } R \geqslant 2r. \quad (7.b) \end{cases} \tag{7}$$

The equality holds, for instance, for the circular slices of C_R *determined by two support lines to* c_r *which intersect on the boundary of* C_R*.*

Proof. Inequality (6) is trivial, since $c_r \subset K$. Hence, we prove inequality (7). Since Lemma 1 (c) states that K is contained in a circular slice K^s of C_R, which has also minimal annulus $A(c,r,R)$, $\omega(K) \leqslant \omega(K^s)$, and we just have to maximize the minimal width for this family of sets. Thus, let us consider the circular symmetric slice of C_R, and let us move the point C continuously on ∂c_r in the counter-clockwise. In this way, we obtain all the possible circular slices of C_R (all of them with the same minimal annulus) till the limit case when $Q \equiv T$, see Figure 8 (a).

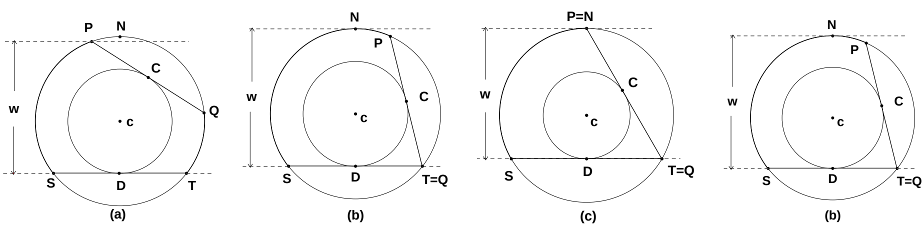

Figure 8. Convex sets with minimal annulus $A(c,r,R)$ *and maximum (minimal) width.*

The minimal width is given by the distance from P to the segment $\overline{ST}$ if P does not pass through N. In the opposite case, the minimal width will be the distance from N to $\overline{ST}$, i.e., $R+r$; this is the case when ω attains the maximum possible value. This change of situation depends on the relation between R and r: if $R=2r$, then $P \equiv N$ (see Figure 8 (c)); when $R \leqslant 2r$, P lies beyond N (see Figure 8 (b)), being the minimal width $\omega = R+r$; finally, if $R \geqslant 2r$, then P has not passed through N, (see Figure 8 (d)), which gives $\omega = 4r(R^2-r^2)/R^2$.

PROPOSITION 3. *Let K be a convex body with minimal annulus $A(c,r,R)$. Then:*

$$D \leqslant 2R. \tag{8}$$

The equality holds for any set containing two diametrically opposite points of ∂C_R.

$$D \geqslant \begin{cases} R+r & \text{if } R \leqslant 5r/3, \qquad (9.a) \\ 2\sqrt{R^2-r^2} & \text{if } R \geqslant 5r/3. \qquad (9.b) \end{cases} \tag{9}$$

The equality holds, in both inequalities, for the 2-cap-bodies $\mathrm{conv}\{c_r,P,Q\}$.

Proof. Again, inequality (8) is trivial, since $K \subset C_R$. In order to prove inequality (9), we use property (b) of Lemma 1: K contains a 2-cap-body $K^c = \mathrm{conv}\{c_r,P,Q\}$, where $P,Q \in \partial C_R \cap \partial K$. It leads to $D(K) \geqslant D(K^c)$, and since the minimal annulus of K^c is also $A(c,r,R)$, it is enough to prove the result for this particular family of cap-bodies.

The diameter of a 2-cap-body is attained in the distance between, either the vertices P and Q, or any of these two points, say P, and the tangent line to ∂c_r which is orthogonal to the line segment $\overline{Qc}$; it depends on the position of P and Q with respect to c_r, and hence, on the relation between R and r (see Figure 9).

Thus, let us consider the symmetric 2-cap-body K^c, in which the vertices P and Q are symmetric with respect to c: in this case, since K^c has two diametrically opposite points, the diameter attains its maximum possible value: $2R$. Now, let us move the point Q continuously on ∂C_R in the counter-clockwise, till the limit case, when $\overline{PQ}$ is tangent to ∂c_r. This construction generates all the possible 2-cap-bodies (up to congruences) with minimal annulus $A(c,r,R)$.

Of course, if $\overline{PQ}$ is tangent to ∂c_r, the distance $d(P,Q)$ between P and Q will be the smallest one. Besides, this distance will be the diameter of the set if it is greater than $R+r$; if not, the value $R+r$ will be the diameter. It is easy to see that $d(P,Q) = R+r$ if, and only if, $R = 5r/3$. Therefore, when $R \geqslant 5r/3$, the diameter will be $D = d(P,Q) = 2\sqrt{R^2-r^2} \geqslant R+r$; otherwise, $D = R+r$ (see Figure 9).

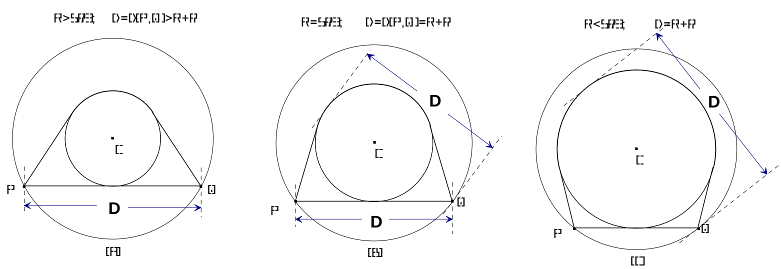

Figure 9. Convex sets with minimal annulus $A(c,r,R)$ and minimum diameter.

The circumradius

The upper and lower bounds for the circumradius of a convex body K are stated in the following result:

PROPOSITION 4. *Let K be a convex body with minimal annulus $A(c,r,R)$. Then:*

$$R_K \leqslant R. \tag{10}$$

The equality holds for any set containing two diametrically opposite points of ∂C_R.

$$R_K \geqslant \begin{cases} \dfrac{R^2+3r^2}{4r} & \text{if } R \leqslant \sqrt{5}\,r \qquad (11.a) \\ \sqrt{R^2-r^2} & \text{if } R \geqslant \sqrt{5}\,r \qquad (11.b) \end{cases} \tag{11}$$

The equality holds when $\overline{AB}$ is tangent to ∂c_r and: in inequality (11.a), for the part of C_K (whose boundary touches ∂c_r) determined by $\overline{AB}$, see Figure 10 (a); in inequality $(11.b)$, for the circular slice of C_K determined by $\overline{AB}$ and the parallel line tangent to ∂c_r, see Figure 10 (b).

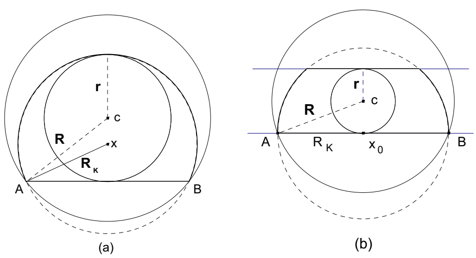

Figure 10. Convex sets with minimal annulus $A(c,r,R)$ and minimum circumradius.

Inequality (10) is trivial. The proof of inequality (11) will be made in two steps, corresponding to the following propositions; then, inequality (11) will be a direct consequence of them.

PROPOSITION 5. *For a given annulus $A(c,r,R)$, let $s \in [r,R]$ be the least real number such that*

(1) *C_s is a circle with radius s containing c_r, and*

(2) *∂C_s intersects ∂C_R in two points A and B such that the straight line determined by them is tangent to ∂c_r.*

Then, it holds:

(a) *If $R \leqslant \sqrt{5}\,r$, then $s = \dfrac{R^2+3r^2}{4r}$.*

(b) *If $R \geqslant \sqrt{5}\,r$, then $s = \sqrt{R^2-r^2}$.*

Proof. Let us fix the points A and B, and first, let us consider $C_s \equiv C_R$. If we move the center x_0 of C_s on the orthogonal line to $\overline{AB}$ which passes through c, since the segment $\overline{AB}$ is a chord of C_s, the radius s will decrease till the limit case when the distance $d(A,B) = D(C_s)$. But C_s must contain c_r, which implies that the minimum possible value for s will be attained when, either ∂C_s touches ∂c_r, or the segment $\overline{AB}$

determines, precisely, a diameter of C_s (which depends on the relation between r and R, see Figure 11).

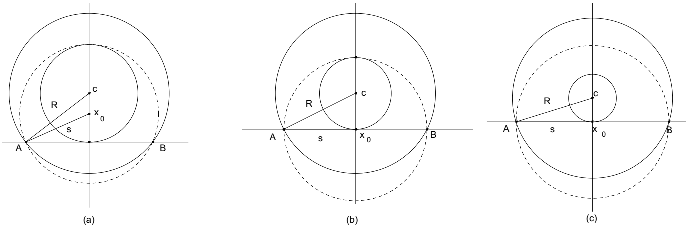

Figure 11. Cases (a) $R \leqslant \sqrt{5}\,r$*, (b)* $R = \sqrt{5}\,r$*, (c)* $R \geqslant \sqrt{5}\,r$*.*

Some easy computations assure that these two possibilities correspond, respectively, to the cases (a) and (b) in the proposition.

PROPOSITION 6. *For a given annulus* $A(c,r,R)$*, it holds:*

(a) *If* $R \leqslant \sqrt{5}\,r$*, then there exists a convex body* K *with minimal annulus* $A(c,r,R)$ *and circumradius* $R_K = (R^2+3r^2)/(4r)$*.*

(b) *If* $R \geqslant \sqrt{5}\,r$*, then there exists a convex body* K *with minimal annulus* $A(c,r,R)$ *and circumradius* $R_K = \sqrt{R^2-r^2}$*.*

Besides, in both cases, the corresponding value R_K *is the least value that can be attained by the circumradius of a convex body* K *with minimal annulus* $A(c,r,R)$*.*

Proof. If $R \leqslant \sqrt{5}\,r$, let us consider the circle C_s with radius $s = (R^2+3r^2)/(4r)$ obtained in the previous proposition (see Figure 11 (a)). The part of C_s determined by the line segment $\overline{AB}$ is a convex body with minimal annulus $A(c,r,R)$ (by property $(P4)$), and circumradius s (its boundary contains two diametrically opposite points of ∂C_s).

When $R \geqslant \sqrt{5}\,r$, we get the case (b) of Proposition 5: if we consider the circle C_s so obtained, then the circular slice of C_s determined by $\overline{AB}$ and the tangent line to ∂c_r which is parallel to this segment, verifies the required conditions (see Figures 11 (c) and 10).

The minimality of both values, $\sqrt{R^2-r^2}$ and $(R^2+3r^2)/(4r)$, respectively in each case, is again a consequence of Proposition 5. In fact, if $R \leqslant \sqrt{5}\,r$, let us suppose that K is a convex body with minimal annulus $A(c,r,R)$ and circumradius $R_K < (R^2+3r^2)/(4r)$, and let x_0 be its circumcenter. We consider the points $A', B' \in \partial C_R$ as defined previously. Since $C_K \supset c_r$, Proposition 5 (for C_K) assures us that ∂C_K intersects the line segment $\overline{A'B'}$ strictly in the interior of C_R; therefore, the central angle determined by the intersection points $\{A,B\} = \partial C_K \cap \partial C_R$ is strictly less than $2\arccos(r/R)$, which contradicts Lemma 2, property (iv).

The minimality for the value $\sqrt{R^2-r^2}$ is analogous.

The inradius

The following result establishes the lower and upper bounds for the inradius of a convex body K with given minimal annulus:

PROPOSITION 7. *Let K be a convex body with minimal annulus $A(c,r,R)$. Then:*

$$r_K \geqslant r. \tag{12}$$

The equality holds, for instance, for the circular symmetric slice of C_R determined by two parallel support lines to c_r.

$$r_K \leqslant \frac{2rR}{R+r}. \tag{13}$$

The equality holds for the circular slice of C_R determined by two support lines to c_r which meet on the boundary ∂C_R.

Proof. Inequality (12) is obvious, so, we have to prove the upper bound (13). By Lemma 1 (c), K is contained in a circular slice of C_R determined by two support lines to c_r through points $P,Q \in \partial c_r \cap \partial K$ whose minimal annulus is also $A(c,r,R)$. Hence, it is enough to maximize the inradius for this family of sets.

Let K^s be a circular slice of C_R and ℓ_1, ℓ_2 the support lines to c_r which determine it. Then, the inradius of K^s is the radius of the circle whose center lies on the angle bisector determined by ℓ_1 and ℓ_2, which is tangent to ∂C_R (see Figure 12).

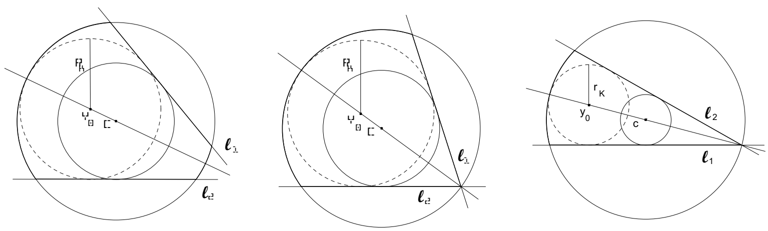

Figure 12. Convex sets with minimal annulus $A(c,r,R)$ and maximum inradius.

Clearly, the inradius will increase with the angle, and therefore, the maximum value for r_K will be attained in the limit case, when $\ell_1 \cap \ell_2 \in \partial C_R$ (let us recall that the intersection point $\ell_1 \cap \ell_2$ can not lie in the interior of C_R). It is an easy computation to check that, for that precise situation, $r_K = 2Rr/(R+r)$, which shows the result.

4. Fixing the minimal annulus and the circumradius

In this section we are going to state the relation between the minimal annulus, the circumradius and both, the area and the perimeter of a convex body K. More precisely, we are going to obtain the best bounds (upper and lower bounds) for A and p, when we suppose that the minimal annulus of the convex body and its circumradius are fixed, determining also the extremal sets in each case. The following theorem states when the minimum area and perimeter are obtained:

THEOREM 1. *Let K be a convex body with minimal annulus $A(c,r,R)$ and circumradius R_K. Then:*

$$A \geqslant 2r\left(\sqrt{R^2-r^2} + r\arcsin\frac{r}{R}\right), \tag{14}$$

$$p \geqslant 4\left(\sqrt{R^2-r^2}+r\arcsin\frac{r}{R}\right). \tag{15}$$

The equality holds, in both inequalities, if and only if the set K is the 2-cap-body $K^c = \operatorname{conv}\{c_r, A, B\}$, where as usual, $\{A,B\} = \partial C_K \cap \partial C_R$.

Proof. Let us recall that, by Lemma 1 (b), if $A(c,r,R)$ is the minimal annulus of K, then it contains a 2-cap-body K^c which is the convex hull of c_r and two points of ∂C_R. These sets have, all of them, the same area and perimeter (those given by the right-hand side of inequalities (14) and (15), respectively); hence, we have

$$A \geqslant A(K^c) \quad \text{and} \quad p \geqslant p(K^c),$$

and in order to conclude the proof, we just have to show the following result:
"If $A(c,r,R)$ is an annulus and s a positive real number satisfying that

(a) $R \geqslant s \geqslant \dfrac{R^2+3r^2}{4r}$ *when* $R \leqslant \sqrt{5}\,r$*, and*

(b) $R \geqslant s \geqslant \sqrt{R^2-r^2}$ *when* $R \geqslant \sqrt{5}\,r$*,*

then, there exists a 2-cap-body $K^c = \operatorname{conv}\{c_r, A, B\}$ for suitable points $A, B \in \partial C_R$, whose minimal annulus is $A(c,r,R)$ and whose circumradius is $R_K = s$".

In both cases, if $s = R$ there is nothing to prove. First, let us suppose that condition (a) holds, and let C_s be the circle with radius s containing c_r, such that ∂C_s touches ∂c_r in a certain point T. Then, as a consequence of Proposition 5, we know that the intersection points $\{A,B\} = \partial C_s \cap \partial C_R$ determine a central angle α verifying $\alpha \geqslant 2\arccos(r/R)$. This allows to assure that the set $K^c = \operatorname{conv}\{c_r, A, B\}$ is a cap-body, which clearly has minimal annulus $A(c,r,R)$. But, what about the circumradius?

If the distance between the centers of C_K and C_R verifies $d(x_0,c) \leqslant d(x_0,\overline{AB})$, then the triangle $\triangle(ATB)$ is an isosceles and acute-angled triangle, and the circumference ∂C_s circumscribes it; hence, s is its circumradius, and as a consequence, also $R_{K^c} = s$ (see Figure 13, left).

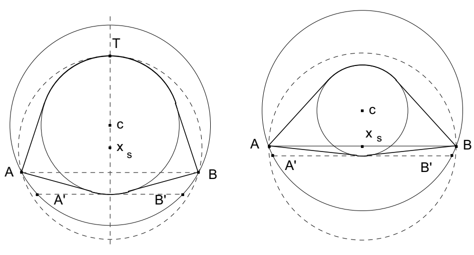

Figure 13. A 2-cap-body with given minimal annulus and circumradius.

However, it can also happen that $d(x_0,c) > d(x_0,\overline{AB})$; in this case, we can move the circle C_s, bringing x_0 nearer c, till the corresponding line AB passes through the circumcenter x_0. Thus, we obtain an isosceles right-angled triangle $\triangle(ATB)$ which is inscribed in C_s; therefore, it has circumradius s, as well as $K^c = \operatorname{conv}\{c_r, A, B\}$ (see Figure 13, left).

Now, we suppose that condition (b) is satisfied. Then, we can choose C_s such that the intersection points $\{A,B\} = \partial C_s \cap \partial C_R$ determine a diameter of the circle C_s

(see Proposition 6). Now, by Proposition 5, we know that this segment $\overline{AB}$ intersects c_r. Hence, the 2-cap-body $K^c = \mathrm{conv}\{c_r, A, B\}$ has circumradius $R_{K^c} = s$ (since K^c contains diametrically opposite points of ∂C_s) and minimal annulus $A(c,r,R)$ (property $(P4)$), see Figure 13, right.

We conclude this work stating the upper bounds for the area and the perimeter of a convex body with prescribed minimal annulus and circumradius. For, we define the following particular convex body: given R_K and $A(c,r,R)$, we take the circle C_K with radius R_K such that

(i) $\partial C_K \cap \partial C_R = \{A, B\}$ and

(ii) the straight line AB is tangent to ∂c_r;

let ℓ be the parallel line to AB also tangent to ∂c_r; we define the *circular trapezium*, and we denote it by K^T, as the intersection of the circle C_K with the circular symmetric slice of C_R determined by the lines AB and ℓ (see Figure 14).

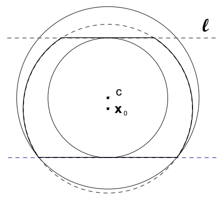

Figure 14. The circular trapezium K^T.

THEOREM 2. *Amongst all convex bodies with circumradius R_K and minimal annulus $A(c,r,R)$, the set with maximum both, area and perimeter, is the circular trapezium K^T shown in Figure 14. Or equivalently: if K is a convex body with circumradius R_K and minimal annulus $A(c,r,R)$, then*

$$A \leqslant R_K^2 \left(\arcsin \frac{\rho}{R_K} + \arcsin \frac{2r-\rho}{R_K} \right) + \rho\sqrt{R^2 - r^2} + (2r-\rho)\sqrt{R_K^2 - (2r-\rho)^2}, \tag{16}$$

$$p \leqslant 2R_K \left(\arcsin \frac{\rho}{R_K} + \arcsin \frac{2r-\rho}{R_K} \right) + 2\sqrt{R^2 - r^2} + 2\sqrt{R_K^2 - (2r-\rho)^2}, \tag{17}$$

where $\rho = \sqrt{R_K^2 - R^2 + r^2}$, with equality if, and only if, $K = K^T$.

Proof. We know (Lemma 2, property (vii)) that K contains a 2-cap-body $K^c = \mathrm{conv}\{c_r, P, Q\}$, for suitable $P, Q \in \partial C_R$. Besides, it determines on the boundary ∂c_r two circular arcs, each one having, at least, one point of ∂K (same Lemma, property $(viii)$). We denote by D and E such points. Then, K lies in a set K_1 formed by the intersection of C_K with the circular slice of C_R determined by the support lines ℓ_D and ℓ_E to K through D and E, respectively (see Figure 15, left). Therefore, $A \leqslant A(K_1)$ and $p \leqslant p(K_1)$, and since K_1 has also circumradius R_K and minimal annulus $A(c,r,R)$, it is enough to maximize the area and the perimeter for this type of sets.

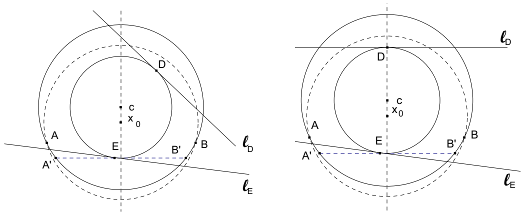

Figure 15. Maximizing the area and the perimeter.

Let us fix the line ℓ_E (notice that it can not intersect ∂C_R over the points A and B). If we consider the case when ℓ_D is orthogonal to the line cx_0 (see Figure 15, right), and we move the point D continuously on ∂c_r in the counter-clockwise (the reasoning is analogous for the opposite movement), all the possible sets K_1 are obtained, till the limit case when ℓ_D and ℓ_E intersect on the boundary ∂C_R. For each fixed line ℓ_E, the area (resp., the perimeter) of the sets K_1 will be greater, as the area (resp., the arc length) of the part of the circle C_K determined by ℓ_D is smaller; and it will happen when the length of the chord $\ell_D \cap C_K$ is as small as possible. Since c_r and C_K are not (necessarily) concentric circles, the minimum of the lengths $l(\ell_D \cap C_K)$ is attained when the distance from D to ∂C_K is minimal; this is, when ℓ_D is orthogonal to the line cx_0 joining both centers.

Thus, if we denote by K_2 the sets of this type (i.e., sets K_1 for which ℓ_D is orthogonal to cx_0), it holds $A \leqslant A(K_1) \leqslant A(K_2)$ and $p \leqslant p(K_1) \leqslant p(K_2)$, and then, we just have to consider this kind of sets. The above construction is feasible for any position (of course, amongst all the possible ones) of the point E, and hence, of the line ℓ_E. Let us also notice that, for any of such positions, all the possible sets K_2 have the same area and perimeter: indeed, since ℓ_D can not pass over the points A and B, it always intersects $\partial(C_R \cap C_K)$ on the circumference ∂C_R, which is concentric to ∂c_r. Hence, we can consider without loss of generality, as our body K_2, the one where both lines ℓ_D and ℓ_E are orthogonal to cx_0 (see Figure 16 (a)).

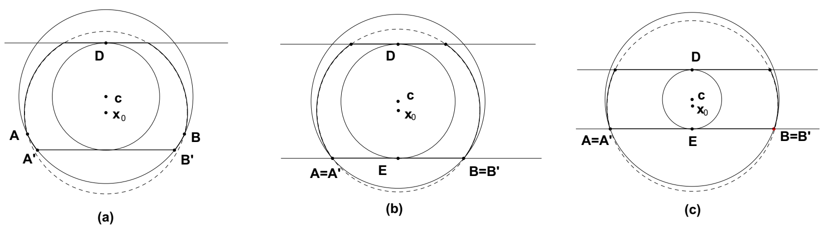

Figure 16. The maximum area and perimeter are attained when $\ell_D \perp \ell_E$ and $A \equiv A'$, $B \equiv B'$.

Now then, for a fixed circumradius R_K, the area of the body K_2 depends on the possible positions of the circumcenter x_0: (i) if $R \leqslant \sqrt{5}\,r$, the limit positions for x_0 are attained when $\partial C_K \cap \partial c_r = \{D\}$, and when A, B coincide with A', B', respectively; (ii) if $R \geqslant \sqrt{5}\,r$, the limit positions for x_0 are attained when the line segment $\overline{AB}$ determines a diameter of C_K, and again when A, B coincide with A', B', respectively.

In both cases, $A(K_2)$ is the area of the intersection of C_K with the circular symmetric slice of C_R determined by the lines ℓ_D and ℓ_E. Since $R_K \leqslant R$, the area increases as the part of C_K intersecting C_R is greater; thus, the maximum will be attained when the circumcenter x_0 lies as closer as possible to the center c of the minimal annulus. But the closest possible position of x_0 to c is given if A,B coincide with A',B', respectively, this is, when K_2 is the circular trapezium K^T. Hence, $A \leqslant A(K_2) \leqslant A(K^T)$. Analogously, it can be deduced that the body K^T is also the set with maximum perimeter (see Figure 16, cases (b) and (c)). A tedious computation leads to the formula of the area and the perimeter of the circular trapezium, which are the ones given in inequalities (16) and (17). This concludes the proof.

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