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ON THE INTERMEDIATE POINT IN CAUCHY'S MEAN–VALUE THEOREM

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(*communicated by P. Bullen*)

Abstract. If the functions $f, g : I \to \mathbb{R}$ are differentiable on the interval $I \subseteq \mathbb{R}$, then for each $x, a \in I$ there exists a real number $\theta \in]0,1[$ such that

$$(f(x) - f(a))\, g^{(1)}(a + \theta(x-a)) = (g(x) - g(a)) f^{(1)}(a + \theta(x-a)).$$

In this paper we study the behaviour of the number $\theta \in]0,1[$, when x approaches a.

The mean value theorem is a cornerstone of the differential calculus. Cauchy's theorem is one of the generalizations of the mean value theorem.

The purpose of this note is to extend the results by D. I. Duca [5] concerning the mean value theorem to Cauchy's theorem.

Cauchy's theorem is usually presented in the following form:

THEOREM 1. (*A. L. Cauchy*) *Let a and b be real numbers with $a < b$ and $f, g : [a,b] \to \mathbb{R}$. If*

(*i*) *the functions f and g are continuous on $[a,b]$,*

(*ii*) *the functions f and g are differentiable on $]a,b[$,*

then there exists a point $c \in]a,b[$ such that

$$[f(b) - f(a)]\, g^{(1)}(c) = [g(b) - g(a)] f^{(1)}(c). \tag{1}$$

If, in addition,

(*iii*) *$g^{(1)}(x) \neq 0$, for all $x \in]a,b[$,*

then $g(b) \neq g(a)$ and

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^{(1)}(c)}{g^{(1)}(c)}. \tag{2}$$

EXAMPLE 1. For the functions $f, g : [-1,1] \to \mathbb{R}$, defined by $f(x) = x^2$, $g(x) = x$, for all $x \in [-1,1]$, there is a unique point $c \in]-1,1[$, namely $c = 0$, such that (1) holds.

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EXAMPLE 2. For the functions $f,g:[-1,1]\to\mathbb{R}$, defined by $f(x)=x^3$, $g(x)=x$, for all $x\in[-1,1]$, there exist two points $c\in]-1,1[$, namely $c_1=-\sqrt{3}/3$ and $c_2=\sqrt{3}/3$, such that (1) holds.

EXAMPLE 3. If $n\geqslant 1$ is an integer number, then for the functions $f,g:[-n\pi,n\pi]\to\mathbb{R}$, defined by $f(x)=\cos x$, $g(x)=e^x$, for all $x\in[-n\pi,n\pi]$, there exist $2n-1$ points $c\in]-n\pi,n\pi[$, namely

$$c_k=k\pi,\quad k\in]-n,n[\cap\mathbb{Z},$$

such that (1) holds.

The following theorem gives a sufficient condition for the unicity of the real number $c\in]a,b[$ from Theorem 1.

THEOREM 2. *Let a and b be real numbers with $a<b$ and $f,g:[a,b]\to\mathbb{R}$ such that:*

(i) *the functions f and g are continuous on $[a,b]$,*
(ii) *the functions f and g are differentiable on $]a,b[$,*
(iii) $g^{(1)}(x)\neq 0$, *for all $x\in]a,b[$.*

If the function $f^{(1)}/g^{(1)}$ is injective on $]a,b[$, then there exists a unique point $c\in]a,b[$ such that (2) holds.

Proof. By contradiction, we suppose that there exist two points $c_1,c_2\in]a,b[$, $c_1\neq c_2$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^{(1)}(c_1)}{g^{(1)}(c_1)}$$

and

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^{(1)}(c_2)}{g^{(1)}(c_2)}.$$

From this it follows that

$$\frac{f^{(1)}}{g^{(1)}}(c_1)=\frac{f^{(1)}}{g^{(1)}}(c_2).$$

Since $f^{(1)}/g^{(1)}$ is injective, we deduce that $c_1=c_2$, which contradicts $c_1\neq c_2$.

REMARK 1. If $g=1_{[a,b]}$, then Theorem 2 becomes Theorem 5 from $[5]$.

Let now $I\subseteq\mathbb{R}$ be an interval, $a\in I$ and $f,g:I\to\mathbb{R}$ be two differentiable functions on I such that $g^{(1)}(x)\neq 0$, for all $x\in I\setminus\{a\}$. Then, by Theorem 1, for each $x\in I\setminus\{a\}$, there exists a point c_x from the interval with the extremities x and a, such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{(1)}(c_x)}{g^{(1)}(c_x)}. \tag{3}$$

In view of Theorem 2, if $f^{(1)}/g^{(1)}$ is injective on I, then for each $x\in I\setminus\{a\}$ there exists a unique point c_x from the interval with the extremities x and a, such that (3) holds. In this case, we can define the function $c:I\setminus\{a\}\to I\setminus\{a\}$ by

$$c(x)=c_x,\ \text{for all } x\in I\setminus\{a\}. \tag{4}$$

The function c has the property that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{(1)}(c(x))}{g^{(1)}(c(x))},\text{ for all } x\in I\setminus\{a\}. \tag{5}$$

If the function $f^{(1)}/g^{(1)}$ is not injective, then, for some $x\in I\setminus\{a\}$, there exist several points c_x from the interval with the extremities x and a such that (3) is true. If for each $x\in I\setminus\{a\}$ we choose one c_x from the interval with the extremities x and a which satisfies (3), then we can also define the function $c: I\setminus\{a\}\to I\setminus\{a\}$ by formula (4). This function c satisfies (5), too.

Consequently, the following statement is true.

THEOREM 3. *Let I be an interval in $\mathbb{R}$, a be a point of I and $f,g: I\to\mathbb{R}$ be two functions. If the functions f and g are differentiable on I and $g^{(1)}(x)\neq 0$, for all $x\in I\setminus\{a\}$, then there exists a function $c: I\setminus\{a\}\to I\setminus\{a\}$ such that (5) is true.*

Furthermore, if, in addition, the function $f^{(1)}/g^{(1)}$ is injective, then the function c is unique.

If $x\in I\setminus\{a\}$ tends to a, because $|c(x)-a|\leqslant|x-a|$, we have

$$\lim_{x\to a}c(x)=a.$$

Then the function $\overline{c}: I\to I$ defined by

$$\overline{c}(x)=\begin{cases} c(x), & \text{if } x\in I\setminus\{a\}\\ a, & \text{if } x=a\end{cases} \tag{6}$$

is continuous at $x=a$.

The purpose of this paper is to establish under which circumstances the function $\overline{c}$ is differentiable at the point $x=a$ and to compute its derivative $\overline{c}^{(1)}(a)$. Does the derivative $\overline{c}^{(1)}(a)$ of the function $\overline{c}$ at the point $x=a$ depend upon the functions f and g? Under which circumstances is the function $\overline{c}$ unique; if there exist several functions $\overline{c}$ which satisfy (5), does the derivative of the function $\overline{c}$ at $x=a$ depend upon the function $\overline{c}$ we choose?

Since for $x\in I\setminus\{a\}$,

$$\frac{\overline{c}(x)-\overline{c}(a)}{x-a}=\frac{c(x)-a}{x-a},$$

if we denote by

$$\theta(x)=\frac{c(x)-a}{x-a},$$

then $\theta(x)\in]0,1[$ and $c(x)=a+(x-a)\theta(x)$ and hence

$$[f(x)-f(a)]\,g^{(1)}(a+(x-a)\theta(x))=[g(x)-g(a)]f^{(1)}(a+(x-a)\theta(x)).$$

Consequently, the following statement is true.

THEOREM 4. *Let I be an interval in $\mathbb{R}$, a be a point of I and $f,g : I \to \mathbb{R}$ be two functions. If the functions f and g are differentiable on I and $g^{(1)}(x) \neq 0$, for all $x \in I \setminus \{a\}$, then there exists a function $\theta : I \setminus \{a\} \to]0,1[$ such that*

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f^{(1)}(a+(x-a)\theta(x))}{g^{(1)}(a+(x-a)\theta(x))}, \text{ for all } x \in I \setminus \{a\}. \tag{7}$$

Furthermore, if, in addition, the function $f^{(1)}/g^{(1)}$ is injective, then the function θ is unique.

REMARK 2. If $g = 1_I$, then Theorem 4 becomes Theorem 7 from [5].

Obviously, the function $\overline{c} : I \to I$, defined by (6) is differentiable at $x = a$ if and only if the function $\theta : I \setminus \{a\} \to]0,1[$ defined by

$$\theta(x) = \frac{\overline{c}(x) - \overline{c}(a)}{x-a} = \frac{c(x)-a}{x-a}, \text{ for all } x \in I \setminus \{a\}$$

has limit at the point $x = a$. Moreover, if the function $\overline{c}$ is differentiable at $x = a$, then

$$\overline{c}^{(1)}(a) = \lim_{x \to a} \theta(x).$$

The following statement is true.

THEOREM 5. *Let I be an interval in $\mathbb{R}$ and a be an interior point of I. Let $f,g : I \to \mathbb{R}$ be two functions which satisfy the following conditions:*
(i) the functions f and g are twice differentiable on I,
(ii) the functions $f^{(2)}$ and $g^{(2)}$ are continuous on I,
(iii) $g^{(1)}(x) \neq 0$, for all $x \in$ int I,
(iv) $f^{(1)}(a)\,g^{(2)}(a) \neq f^{(2)}(a)\,g^{(1)}(a)$.
Then the following statements are true:
1° There exists a real number $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$,

$$f^{(1)}(x)\,g^{(2)}(x) \neq f^{(2)}(x)\,g^{(1)}(x), \text{ for all } x \in]a-\delta, a+\delta[$$

and $f^{(1)}/g^{(1)}$ is injective on $]a-\delta, a+\delta[$.
2° There exists a unique function $c :]a-\delta, a+\delta[\setminus \{a\} \to]a-\delta, a+\delta[\setminus \{a\}$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))}, \text{ for all } x \in]a-\delta, a+\delta[\setminus \{a\}. \tag{8}$$

3° The function $\theta :]a-\delta, a+\delta[\setminus \{a\} \to]0,1[$ defined by

$$\theta(x) = \frac{c(x)-a}{x-a}, \text{ for all } x \in]a-\delta, a+\delta[\setminus \{a\} \tag{9}$$

has the following properties:
$a)$ For all $x \in]a-\delta, a+\delta[\setminus \{a\}$, we have

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f^{(1)}(a+(x-a)\theta(x))}{g^{(1)}(a+(x-a)\theta(x))}.$$

b) There exists the limit

$$\lim_{x\to a}\theta(x)=\frac{1}{2}.$$

4° *The function* $\overline{c}:]a-\delta,a+\delta[\to]a-\delta,a+\delta[$ *defined by*

$$\overline{c}(x)=\begin{cases} c(x), & \text{if } x\in]a-\delta,a+\delta[\setminus\{a\}\\ a, & \text{if } x=a\end{cases}$$

is differentiable at $x=a$ *and*

$$\overline{c}^{(1)}(a)=\frac{1}{2}.$$

Proof. 1° Suppose that $f^{(1)}(a)\,g^{(2)}(a)<f^{(2)}(a)\,g^{(1)}(a)$. Then, by the hypothesis (ii) and $a\in\operatorname{int} I$, we deduce that there exists a real number $\delta>0$ such that $]a-\delta,a+\delta[\subseteq I$ and

$$f^{(1)}(x)\,g^{(2)}(x)<f^{(2)}(x)\,g^{(1)}(x),\ \text{for all } x\in]a-\delta,a+\delta[.$$

It follows that

$$\left(\frac{f^{(1)}}{g^{(1)}}\right)^{(1)}(x)=\frac{f^{(2)}(x)\,g^{(1)}(x)-f^{(1)}(x)\,g^{(2)}(x)}{\left(g^{(1)}(x)\right)^2}>0,\ \text{for all } x\in]a-\delta,a+\delta[$$

and hence $f^{(1)}/g^{(1)}$ is strictly increasing on $]a-\delta,a+\delta[$. Consequently, the function $f^{(1)}/g^{(1)}$ is injective on $]a-\delta,a+\delta[$.

If $f^{(1)}(a)\,g^{(2)}(a)>f^{(2)}(a)\,g^{(1)}(a)$, the proof is analogously.

2° It follows from statement 1° above and Theorem 3.

3° $a)$ This follows immediately from (8) and (9).

$b)$ By Taylor's formula, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$, there are two real numbers $\widehat{\theta}_f(x),\widehat{\theta}_g(x)\in]0,1[$ such that

$$f(x)=f(a)+f^{(1)}(a)(x-a)+\frac{1}{2}f^{(2)}(a+(x-a)\widehat{\theta}_f(x))(x-a)^2 \tag{10}$$

and

$$g(x)=g(a)+g^{(1)}(a)(x-a)+\frac{1}{2}g^{(2)}(a+(x-a)\widehat{\theta}_g(x))(x-a)^2. \tag{11}$$

Now, by the mean value theorem applied to the functions $f^{(1)}$ and $g^{(1)}$, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$, there exist two real numbers $\widetilde{\theta}_f(x),\widetilde{\theta}_g(x)\in]0,1[$ such that

$$\begin{aligned} f^{(1)}(c(x)) &= f^{(1)}(a+(x-a)\theta(x))\\ &= f^{(1)}(a)+f^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))(x-a)\theta(x) \end{aligned} \tag{12}$$

and

$$\begin{aligned} g^{(1)}(c(x)) &= g^{(1)}(a+(x-a)\theta(x))\\ &= g^{(1)}(a)+g^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x))(x-a)\theta(x). \end{aligned} \tag{13}$$

Substituting (10)-(13) in (8), we obtain that, for each $x \in]a-\delta, a+\delta[\setminus \{a\}$,

$$\frac{f^{(1)}(a)(x-a)+\frac{1}{2}f^{(2)}(a+(x-a)\widehat{\theta}_f(x))(x-a)^2}{g^{(1)}(a)(x-a)+\frac{1}{2}g^{(2)}(a+(x-a)\widehat{\theta}_g(x))(x-a)^2} = \frac{f^{(1)}(a)+f^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))(x-a)\theta(x)}{g^{(1)}(a)+g^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x))(x-a)\theta(x)},$$

or, equivalent,

$$\begin{aligned}
&\theta(x)\Big\{f^{(1)}(a)g^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x)) - f^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))g^{(1)}(a) \\
&\quad + \frac{1}{2}\Big[f^{(2)}(a+(x-a)\widehat{\theta}_f(x))g^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x)) \\
&\quad - f^{(2)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))g^{(2)}(a+(x-a)\widehat{\theta}_g(x))\Big](x-a)\Big\} \\
&= \frac{1}{2}[f^{(1)}(a)g^{(2)}(a+(x-a)\widehat{\theta}_g(x)) - f^{(2)}(a+(x-a)\widehat{\theta}_f(x))g^{(1)}(a)].
\end{aligned} \tag{14}$$

Since for all $x \in]a-\delta, a+\delta[\setminus \{a\}$, we have that $\theta(x)$, $\widetilde{\theta}_f(x)$, $\widetilde{\theta}_g(x)$, $\widehat{\theta}_f(x)$, $\widehat{\theta}_g(x)$ belong to $]0,1[$, we deduce that

$$\left|(x-a)\widehat{\theta}_f(x)\right| \leqslant |x-a| \text{ and } \left|(x-a)\theta(x)\widetilde{\theta}_f(x)\right| \leqslant |x-a|,$$
$$\left|(x-a)\widehat{\theta}_g(x)\right| \leqslant |x-a| \text{ and } \left|(x-a)\theta(x)\widetilde{\theta}_g(x)\right| \leqslant |x-a|,$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$. The functions $f^{(1)}$, $f^{(2)}$, $g^{(1)}$, $g^{(2)}$ being continuous on I, from (14) it results that there exists

$$\lim_{x \to a} \theta(x) = \frac{1}{2}.$$

$4°$ Statement $4°$ follows from statement $3°$ above.
The theorem is proved.

REMARK 3. If $g = 1_I$, then Theorem 5 becomes Theorem 8 from [5].

REMARK 4. Theorem 5 remains true if the point a is an extremity of the interval I. The following statement is true.

THEOREM 6. *Let I be an interval in $\mathbb{R}$ and $a \in I$ be the left extremity of I. Let $f, g : I \to \mathbb{R}$ be two functions which satisfy the following conditions:*
(i) the functions f and g are twice differentiable on I,
(ii) the functions $f^{(2)}$ and $g^{(2)}$ are continuous on I,
(iii) $g^{(1)}(x) \neq 0$, for all $x \in$ int I,
(iv) $f^{(1)}(a)\, g^{(2)}(a) \neq f^{(2)}(a)\, g^{(1)}(a)$.
Then the following statements are true:

1° *There exists a real number* $\delta > 0$ *such that* $]a, a+\delta[\subseteq I$,

$$f^{(1)}(x)\,g^{(2)}(x) \neq f^{(2)}(x)\,g^{(1)}(x)\,,\ \textit{for all}\ x \in]a, a+\delta[$$

and $f^{(1)}/g^{(1)}$ *is injective on* $]a, a+\delta[$.

2° *There exists a unique function* $c :]a, a+\delta[\to]a, a+\delta[$ *such that*

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))},$$

for all $x \in]a, a+\delta[$.

3° *The function* $\theta :]a, a+\delta[\to]0,1[$ *defined by*

$$\theta(x) = \frac{c(x)-a}{x-a},\ \textit{for all}\ x \in]a, a+\delta[$$

has the following properties:

a) *For all* $x \in]a, a+\delta[$, *we have*

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f^{(1)}(a+(x-a)\theta(x))}{g^{(1)}(a+(x-a)\theta(x))}.$$

b) *There exists the limit:*

$$\lim_{x \searrow a} \theta(x) = \frac{1}{2}.$$

4° *The function* $\overline{c} : [a, a+\delta[\to \mathbb{R}$ *defined by*

$$\overline{c}(x) = \begin{cases} c(x), & \textit{if}\ x \in]a, a+\delta[\\ a, & \textit{if}\ x = a \end{cases}$$

is differentiable at $x = a$ *and*

$$\overline{c}^{(1)}(a) = \frac{1}{2}.$$

REMARK 5. If $g = 1_I$, then Theorem 6 becomes Theorem 9 from [5].

One can give a similar theorem if a is the right extremity of the interval I.

REMARK 6. If $f^{(1)}(a)\,g^{(2)}(a) = f^{(2)}(a)\,g^{(1)}(a)$, then statement 3° of Theorem 5 might not be true.

Indeed, for the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3,\ g(x) = \exp(x)\,,\quad \text{for all}\ x \in \mathbb{R}$$

we have $f^{(1)}(x) = 3x^2$, $f^{(2)}(x) = 6x$, $g^{(1)}(x) = g^{(2)}(x) = \exp(x)$, for all $x \in \mathbb{R}$ and hence $f^{(1)}(0)\,g^{(2)}(0) = f^{(2)}(0)\,g^{(1)}(0)$. Obviously, $f^{(1)}/g^{(1)}$ is injective on $]-\infty, 0[$. Then, by Theorem 3, there exists a unique function $c_1 :]-\infty, 0[\to]-\infty, 0[$ such that

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f^{(1)}(c_1(x))}{g^{(1)}(c_1(x))},\ \text{for all}\ x \in]-\infty, 0[.$$

Similarly, since $f^{(1)}/g^{(1)}$ is injective on $]0,2[$, by Theorem 3, there exists a unique function $c_2 :]0,2[\to]0,2[$ such that

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f^{(1)}(c_2(x))}{g^{(1)}(c_2(x))}, \text{ for all } x \in]0,2[.$$

Then the function $c :]-\infty,2[\setminus\{0\} \to]-\infty,2[\setminus\{0\}$ defined by

$$c(x) = \begin{cases} c_1(x), & \text{if } x \in]-\infty,0[\\ c_2(x), & \text{if } x \in]0,2[\end{cases}$$

satisfies the equality

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))}, \text{ for all } x \in]-\infty,2[\setminus\{0\},$$

i.e.

$$\frac{x^3}{\exp(x)-1} = \frac{3(c(x))^2}{\exp(c(x))}, \quad \text{for all} \quad x \in]-\infty,2[\setminus\{0\},$$

or, equivalently

$$\frac{x}{\exp(x)-1} = \frac{3(\theta(x))^2}{\exp(x\theta(x))}, \quad \text{for all} \quad x \in]-\infty,2[\setminus\{0\},$$

where $c(x) = x\theta(x)$, for all $x \in]-\infty,2[\setminus\{0\}$. Hence, if there exists $\lim_{x\to 0}\theta(x) \in \mathbb{R}$, then

$$1 = 3\lim_{x\to 0}(\theta(x))^2.$$

It follows that, if there exists $\overline{c}^{(1)}(0) = \lim_{x\to 0}\theta(x) \in \mathbb{R}$, then

$$\overline{c}^{(1)}(0) = \lim_{x\to 0}\theta(x) = \frac{1}{\sqrt{3}} \neq \frac{1}{2}.$$

One asks: If $f^{(1)}(a)g^{(2)}(a) = f^{(2)}(a)g^{(1)}(a)$, then $\overline{c}^{(1)}(a) = \lim_{x\to a}\theta(x) = \frac{1}{\sqrt{3}}$? A partial answer is given by the following theorem:

THEOREM 7. *Let I be an interval in $\mathbb{R}$ and a be an interior point of I. Let $f,g : I \to \mathbb{R}$ be two functions which satisfy the following conditions:*

(i) *the functions f and g are three times differentiable on I,*

(ii) *the functions $f^{(3)}$ and $g^{(3)}$ are continuous on I,*

(iii) $g^{(1)}(x) \neq 0$, *for all $x \in$ int I,*

(iv) $f^{(1)}(a)g^{(2)}(a) = f^{(2)}(a)g^{(1)}(a)$, $f^{(1)}(a)g^{(3)}(a) \neq f^{(3)}(a)g^{(1)}(a)$.

Then the following statements are true:

1° *There exists a real number $\delta > 0$ such that $]a-\delta,a+\delta[\subseteq I$ and*

$$f^{(1)}(x)g^{(2)}(x) \neq f^{(2)}(x)g^{(1)}(x), \text{ for all } x \in]a-\delta,a+\delta[\setminus\{a\}.$$

2° *The function $f^{(1)}/g^{(1)}$ is injective on each of the intervals $]a-\delta,a[$ and $]a,a+\delta[$.*

3° *There exist a unique function* $c_1 :]a-\delta, a[\to]a-\delta, a[$ *and a unique function* $c_2 :]a, a+\delta[\to]a, a+\delta[$ *such that the function* $c :]a-\delta, a+\delta[\setminus \{a\} \to]a-\delta, a+\delta[\setminus \{a\}$ *defined by*

$$c(x) = \begin{cases} c_1(x), & \text{if } x \in]a-\delta, a[\\ c_2(x), & \text{if } x \in]a, a+\delta[\end{cases}$$

satisfies the equality

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(c(x))}{g^{(1)}(c(x))}, \tag{15}$$

for all $x \in]a-\delta, a+\delta[\setminus \{a\}$.

4° *The function* $\theta :]a-\delta, a+\delta[\setminus \{a\} \to]0,1[$ *defined by*

$$\theta(x) = \frac{c(x) - a}{x - a}, \text{ for all } x \in]a-\delta, a+\delta[\setminus \{a\}, \tag{16}$$

has the following properties:

$a)$ *For all* $x \in]a-\delta, a+\delta[\setminus \{a\}$, *we have*

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(a + (x-a)\theta(x))}{g^{(1)}(a + (x-a)\theta(x))}.$$

$b)$ *There exists the limit*

$$\lim_{x \to a} \theta(x) = \frac{1}{\sqrt{3}}.$$

5° *The function* $\overline{c} :]a-\delta, a+\delta[\to]a-\delta, a+\delta[$ *defined by*

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in]a-\delta, a+\delta[\setminus \{a\} \\ a, & \text{if } x = a \end{cases}$$

is differentiable at $x = a$ *and*

$$\overline{c}^{(1)}(a) = \frac{1}{\sqrt{3}}.$$

Proof. $1^\circ - 2^\circ$ Suppose that $f^{(1)}(a)g^{(3)}(a) < f^{(3)}(a)g^{(1)}(a)$. From (ii) and $a \in \operatorname{int} I$, it follows that there exists a real number $\delta > 0$ such that $]a-\delta, a+\delta[\subseteq I$ and

$$f^{(1)}(x)g^{(3)}(x) < f^{(3)}(x)g^{(1)}(x), \text{ for all } x \in]a-\delta, a+\delta[.$$

It follows that

$$\left(f^{(2)}g^{(1)} - f^{(1)}g^{(2)}\right)^{(1)}(x) = f^{(3)}(x)g^{(1)}(x) - f^{(1)}(x)g^{(3)}(x) > 0,$$

for all $x \in]a-\delta, a+\delta[$ and hence the function $f^{(2)}g^{(1)} - f^{(1)}g^{(2)}$ is strictly increasing on $]a-\delta, a+\delta[$. Since $f^{(1)}(a)g^{(2)}(a) = f^{(2)}(a)g^{(1)}(a)$, we deduce that

$$f^{(2)}(x)g^{(1)}(x) - f^{(1)}(x)g^{(2)}(x) < 0, \text{ for all } x \in]a-\delta, a[$$

and

$$f^{(2)}(x)g^{(1)}(x) - f^{(1)}(x)g^{(2)}(x) > 0, \text{ for all } x \in]a, a+\delta[.$$

Then

$$\left(\frac{f^{(1)}}{g^{(1)}}\right)^{(1)}(x)=\frac{f^{(2)}(x)\,g^{(1)}(x)-f^{(1)}(x)\,g^{(2)}(x)}{\left(g^{(1)}(x)\right)^{2}}<0,\ \text{for all } x\in]a-\delta,a[$$

and

$$\left(\frac{f^{(1)}}{g^{(1)}}\right)^{(1)}(x)=\frac{f^{(2)}(x)\,g^{(1)}(x)-f^{(1)}(x)\,g^{(2)}(x)}{\left(g^{(1)}(x)\right)^{2}}>0,\ \text{for all } x\in]a,a+\delta[.$$

It follows that the function $f^{(1)}/g^{(1)}$ is strictly decreasing on $]a-\delta,a[$ and strictly increasing on $]a,a+\delta[$. Consequently $f^{(1)}/g^{(1)}$ is injective on each of the intervals $]a-\delta,a[$ and $]a,a+\delta[$.

If $f^{(1)}(a)g^{(3)}(a)>f^{(3)}(a)\,g^{(1)}(a)$, the proof is analogously.

3° From Theorem 3 and statement 2° above, it follows that there exists a unique function $c_1:]a-\delta,a[\to]a-\delta,a[$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{(1)}(c_1(x))}{g^{(1)}(c_1(x))},\ \ \text{for all } x\in]a-\delta,a[$$

and a unique function $c_2:]a,a+\delta[\to]a,a+\delta[$ such that

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{(1)}(c_2(x))}{g^{(1)}(c_2(x))},\ \ \text{for all } x\in]a,a+\delta[.$$

Then the function $c:]a-\delta,a+\delta[\setminus\{a\}\to]a-\delta,a+\delta[\setminus\{a\}$ defined by

$$c(x)=\begin{cases} c_1(x), & \text{if } x\in]a-\delta,a[\\ c_2(x), & \text{if } x\in]a,a+\delta[\end{cases}$$

satisfies equality (15).

4° $a)$ Statement $a)$ follows from (15) and (16).

$b)$ By Taylor's formula, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$ there exist two real numbers $\widehat{\theta}_f(x),\widehat{\theta}_g(x)\in]0,1[$ such that

$$f(x)-f(a)=f^{(1)}(a)(x-a)+\frac{1}{2!}f^{(2)}(a)(x-a)^2+\frac{f^{(3)}(a+(x-a)\widehat{\theta}_f(x))}{3!}(x-a)^3, \quad (17)$$

and

$$g(x)-g(a)=g^{(1)}(a)(x-a)+\frac{1}{2!}g^{(2)}(a)(x-a)^2+\frac{g^{(3)}(a+(x-a)\widehat{\theta}_g(x))}{3!}(x-a)^3. \quad (18)$$

On the other hand, by Taylor's formula applied to the functions $f^{(1)}$ and $g^{(1)}$, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$, there exists two real numbers $\widetilde{\theta}_f(x),\widetilde{\theta}_g(x)\in]0,1[$ such that

$$\begin{aligned}&f^{(1)}(a+(x-a)\theta(x))=f^{(1)}(c(x))\\&=f^{(1)}(a)+f^{(2)}(a)(x-a)\,\theta(x)+\frac{f^{(3)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))}{2}(x-a)^2(\theta(x))^2\end{aligned} \quad (19)$$

and

$$g^{(1)}(a+(x-a)\theta(x)) = g^{(1)}(c(x)) \\ = g^{(1)}(a)+g^{(2)}(a)(x-a)\theta(x)+\frac{g^{(3)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x))}{2}(x-a)^2(\theta(x))^2. \tag{20}$$

Substituting $(19)-(20)$ in (15), we obtain that, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$,

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f^{(1)}(a)+f^{(2)}(a)(x-a)\theta(x)+\frac{f^{(3)}(a+(x-a)\theta(x)\widetilde{\theta}_f(x))}{2}(x-a)^2(\theta(x))^2}{g^{(1)}(a)+g^{(2)}(a)(x-a)\theta(x)+\frac{g^{(3)}(a+(x-a)\theta(x)\widetilde{\theta}_g(x))}{2}(x-a)^2(\theta(x))^2}.$$

From this and $(17)-(20)$, we obtain that, for each $x\in]a-\delta,a+\delta[\setminus\{a\}$,

$$\begin{aligned}
&\frac{1}{2}\Big(f^{(1)}(a)\,g^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_g(x)\Big)\\
&-f^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_f(x)\Big)g^{(1)}(a)\Big)(\theta(x))^2\\
&+\frac{1}{4}\Big(f^{(2)}(a)\,g^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_g(x)\Big)\\
&-f^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_f(x)\Big)g^{(2)}(a)\Big)(x-a)^2(\theta(x))^2\\
&+\frac{1}{6}\Big(f^{(3)}\Big(a+(x-a)\,\widehat{\theta}_f(x)\Big)g^{(1)}(a)-f^{(1)}(a)\,g^{(3)}\Big(a+(x-a)\,\widehat{\theta}_g(x)\Big)\Big)\\
&+\frac{1}{6}\Big(f^{(3)}\Big(a+(x-a)\,\widehat{\theta}_f(x)\Big)g^{(2)}(a)-f^{(2)}(a)\,g^{(3)}\Big(a+(x-a)\,\widehat{\theta}_g(x)\Big)\Big)(x-a)\,\theta(x)\\
&+\frac{1}{12}\Big(f^{(3)}\Big(a+(x-a)\,\widehat{\theta}_f(x)\Big)g^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_g(x)\Big)\\
&-f^{(3)}\Big(a+(x-a)\,\theta(x)\,\widetilde{\theta}_f(x)\Big)g^{(3)}\Big(a+(x-a)\,\widehat{\theta}_g(x)\Big)\Big)(x-a)^2(\theta(x))^2=0.
\end{aligned}$$

Now, since for each $x\in]a-\delta,a+\delta[\setminus\{a\}$, we have that $\theta(x)$, $\widetilde{\theta}_f(x)$, $\widetilde{\theta}_g(x)$, $\widehat{\theta}_f(x)$, $\widehat{\theta}_g(x)$ belong to $]0,1[$, we deduce that

$$\left|(x-a)\widehat{\theta}_f(x)\right|\leqslant|x-a|,\quad \left|(x-a)\widehat{\theta}_g(x)\right|\leqslant|x-a|,$$
$$\left|(x-a)\theta(x)\,\widetilde{\theta}_f(x)\right|\leqslant|x-a| \text{ and } \left|(x-a)\theta(x)\,\widetilde{\theta}_g(x)\right|\leqslant|x-a|,$$

for all $x\in]a-\delta,a+\delta[\setminus\{a\}$.

Now, from (ii), it results that there exists

$$\lim_{x\to a}(\theta(x))^2=\frac{f^{(1)}(a)\,g^{(3)}(a)-f^{(3)}(a)\,g^{(1)}(a)}{3\left(f^{(1)}(a)\,g^{(3)}(a)-f^{(3)}(a)\,g^{(1)}(a)\right)}=\frac{1}{3}.$$

5° Statement 5° follows from statement 4° above.

The theorem is proved.

REMARK 7. If $g=1_I$, then Theorem 7 becomes Theorem 11 from [5].

REMARK 8. Theorem 7 remains true if the point a is an extremity of the interval I.

The following theorem answers the question: which is the limit of the function θ at the point $x=a$, when $f^{(1)}(a)g^{(k)}(a)=f^{(k)}(a)g^{(1)}(a)$, for all $k\in\{2,...,n-1\}$, and $f^{(1)}(a)g^{(n)}(a)\neq f^{(n)}(a)g^{(1)}(a)$?

THEOREM 8. *Let I be an interval in $\mathbb{R}$ and a be a point of I. Let $f,g:I\to\mathbb{R}$ be functions which satisfy the following conditions:*

(i) *the functions f and g are $n\geqslant 2$ times differentiable on I,*

(ii) *the functions $f^{(n)}$ and $g^{(n)}$ are continuous on I,*

(iii) $f^{(1)}(a)g^{(k)}(a)=f^{(k)}(a)g^{(1)}(a)$, *for all $k\in\{2,...,n-1\}$,*

(iv) $f^{(1)}(a)g^{(n)}(a)\neq f^{(n)}(a)g^{(1)}(a)$.

Let $\theta: I\setminus\{a\}\to]0,1[$ be a function such that

$$(f(x)-f(a))\,g^{(1)}(a+(x-a)\theta(x))=(g(x)-g(a))f^{(1)}(a+(x-a)\theta(x)), \tag{21}$$

for all $x\in I\setminus\{a\}$.

Then there exists the limit:

$$\lim_{x\to a}\theta(x)=\frac{1}{\sqrt[n-1]{n}}. \tag{22}$$

Proof. By Taylor's formula, for each $x\in I\setminus\{a\}$ there exist $\theta_1(x)$, $\theta_2(x)$, $\theta_3(x)$, $\theta_4(x)\in]0,1[$ such that

$$f(x)=f(a)+\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(a+(x-a)\theta_1(x)), \tag{23}$$

$$g(x)=g(a)+\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}g^{(k)}(a)+\frac{(x-a)^n}{n!}g^{(n)}(a+(x-a)\theta_2(x)), \tag{24}$$

$$\begin{aligned}&f^{(1)}(a+(x-a)\theta(x))\\&=\sum_{i=1}^{n-1}\frac{((x-a)\theta(x))^{i-1}}{(i-1)!}f^{(i)}(a)+\frac{((x-a)\theta(x))^{n-1}}{(n-1)!}f^{(n)}(a+(x-a)\theta(x)\theta_3(x)),\end{aligned} \tag{25}$$

$$\begin{aligned}&g^{(1)}(a+(x-a)\theta(x))\\&=\sum_{i=1}^{n-1}\frac{((x-a)\theta(x))^{i-1}}{(i-1)!}g^{(i)}(a)+\frac{((x-a)\theta(x))^{n-1}}{(n-1)!}g^{(n)}(a+(x-a)\theta(x)\theta_4(x)).\end{aligned} \tag{26}$$

Substituting (23) – (26) in (21), we obtain

$$\left(\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}f^{(k)}(a)+\frac{(x-a)^n}{n!}f^{(n)}(a+(x-a)\theta_1(x))\right)\times$$
$$\times\left(\sum_{i=1}^{n-1}\frac{((x-a)\theta(x))^{i-1}}{(i-1)!}g^{(i)}(a)+\frac{((x-a)\theta(x))^{n-1}}{(n-1)!}g^{(n)}(a+(x-a)\theta(x)\theta_4(x))\right)$$
$$=\left(\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}g^{(k)}(a)+\frac{(x-a)^n}{n!}g^{(n)}(a+(x-a)\theta_2(x))\right)\times$$
$$\times\left(\sum_{i=1}^{n-1}\frac{((x-a)\theta(x))^{i-1}}{(i-1)!}f^{(i)}(a)+\frac{((x-a)\theta(x))^{n-1}}{(n-1)!}f^{(n)}(a+(x-a)\theta(x)\theta_3(x))\right),$$

or equivalent

$$\sum_{k=1}^{n-1}\sum_{i=1}^{n-1}\frac{(x-a)^{k+i-1}}{k!(i-1)!}(\theta(x))^{i-1}\left(f^{(k)}(a)g^{(i)}(a)-f^{(i)}(a)g^{(k)}(a)\right)$$
$$+\frac{(x-a)^{n-1}(\theta(x))^{n-1}}{(n-1)!}\sum_{k=1}^{n-1}\frac{(x-a)^k}{k!}\left(f^{(k)}(a)g^{(n)}(a+(x-a)\theta(x)\theta_4(x))\right.$$
$$\left.-f^{(n)}(a+(x-a)\theta(x)\theta_3(x))g^{(k)}(a)\right)$$
$$+\frac{(x-a)^n}{n!}\sum_{i=1}^{n-1}\frac{(x-a)^{i-1}(\theta(x))^{i-1}}{(i-1)!}\left(f^{(n)}(a+(x-a)\theta_1(x))g^{(i)}(a)\right.$$
$$\left.-f^{(i)}(a)g^{(n)}(a+(x-a)\theta_2(x))\right)$$
$$+\frac{(x-a)^{2n-1}(\theta(x))^{n-1}}{(n-1)!n!}\left(f^{(n)}(a+(x-a)\theta_1(x))g^{(n)}(a+(x-a)\theta(x)\theta_4(x))\right.$$
$$\left.-f^{(n)}(a+(x-a)\theta(x)\theta_3(x))g^{(n)}(a+(x-a)\theta_2(x))\right)=0.$$

From this and (iii) we deduce that

$$(\theta(x))^{n-1}\sum_{k=1}^{n-1}\frac{(x-a)^{k-1}}{k!}\left(f^{(k)}(a)g^{(n)}(a+(x-a)\theta(x)\theta_4(x))\right.$$
$$\left.-f^{(n)}(a+(x-a)\theta(x)\theta_3(x))g^{(k)}(a)\right)+\frac{1}{n}\sum_{i=1}^{n-1}\frac{(x-a)^{i-1}(\theta(x))^{i-1}}{(i-1)!}\times$$
$$\times\left(f^{(n)}(a+(x-a)\theta_1(x))g^{(i)}(a)\ -f^{(i)}(a)g^{(n)}(a+(x-a)\theta_2(x))\right)$$
$$+\frac{(x-a)^{n-1}(\theta(x))^{n-1}}{n!}\left(f^{(n)}(a+(x-a)\theta_1(x))g^{(n)}(a+(x-a)\theta(x)\theta_4(x))\right.$$
$$\left.-f^{(n)}(a+(x-a)\theta(x)\theta_3(x))g^{(n)}(a+(x-a)\theta_2(x))\right)=0,$$

for all $x \in I \setminus \{a\}$.

Now, since

$$|(x-a)\theta_1(x)| \leqslant |x-a| \ , \ |(x-a)\theta_2(x)| \leqslant |x-a| \ ,$$
$$|(x-a)\theta(x)\,\theta_3(x)| \leqslant |x-a| \text{ and } |(x-a)\theta(x)\,\theta_4(x)| \leqslant |x-a|\,,$$

for all $x \in I \setminus \{a\}$, from (ii), it results that there exists

$$\lim_{x\to a} (\theta(x))^{n-1} = \frac{1}{n}\frac{f^{(1)}(a)g^{(n)}(a) - f^{(n)}(a)g^{(1)}(a)}{f^{(1)}(a)g^{(n)}(a) - f^{(n)}(a)g^{(1)}(a)} = \frac{1}{n}.$$

The theorem is proved.

REMARK 9. If $g = 1_I$, then Theorem 8 becomes Theorem 12 from [5].

THEOREM 9. *Let I be an interval in $\mathbb{R}$ and a be an interior point of I. Let $f, g : I \to \mathbb{R}$ be functions which satisfy the following conditions:*

(i) *the functions f and g are $n \geqslant 2$ times differentiable on I,*

(ii) *the functions $f^{(n)}$ and $g^{(n)}$ are continuous on I,*

(iii) $g^{(k)}(a) \neq 0$, *for all $k \in \{1, ...n\}$,*

(iv)

$$\frac{f^{(1)}(a)}{g^{(1)}(a)} = \frac{f^{(2)}(a)}{g^{(2)}(a)} = ... = \frac{f^{(n-1)}(a)}{g^{(n-1)}(a)},$$

(v)

$$\frac{f^{(1)}(a)}{g^{(1)}(a)} \neq \frac{f^{(n)}(a)}{g^{(n)}(a)},$$

(vi)

$$g^{(1)}(x) \neq 0, \ \textit{for all } x \in I.$$

Let $\theta : I \setminus \{a\} \to]0,1[$ be a function such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f^{(1)}(a + (x-a)\theta(x))}{g^{(1)}(a + (x-a)\theta(x))}, \textit{ for all } x \in I \setminus \{a\}.$$

Then there exists the limit:

$$\lim_{x\to a} \theta(x) = \frac{1}{\sqrt[n-1]{n}}.$$

Proof. Apply Theorem 8 above.

REMARK 10. If $g = 1_I$, then Theorem 9 becomes Theorem 12 from [5].

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