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EXISTENCE OF TRIPLE POSITIVE SOLUTIONS FOR A THIRD ORDER GENERALIZED RIGHT FOCAL PROBLEM

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Abstract. We obtain sufficient conditions for the existence of at least three positive solutions for the third-order three-point generalized right focal boundary value problem

$$x''' = q(t)f(t,x,x',x''), t_1 \leqslant t \leqslant t_3,$$
$$x(t_1) = x'(t_2) = 0, \ \eta x(t_3) + \delta x''(t_3) = 0,$$

where $f : [t_1,t_3] \times [0,\infty) \times R^2 \to [0,\infty)$, $q : (t_1,t_3) \to [0,+\infty)$ are nonnegative continuous functions, $\delta > 0$, $\eta \geqslant 0$ are constants. This is an application of a new fixed-point theorem introduced by Avery and Peterson [6].

1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. To identify a few, we refer the reader to [1–11]. The main tools used in above works are fixed-point theorems. Fixed-point theorems and their applications to nonlinear problems have a long history, some of which is documented in Zeidler's book [11], and the recent book by Agarwal, O'Regan and Wong [2] contains an excellent summary of the current results and applications.

An interest in triple solutions evolved from the Leggett-Williams multiple fixed-point theorem [9]. And lately, two triple fixed-point theorems due to Avery [5] and Avery and Peterson [6] have been applied to obtain triple solutions of certain boundary-value problems for ordinary differential equations as well as for their discrete analogues.

Anderson *et al.* [3, 4] obtained some excellent results about the existence and multiplicity of positive solutions for the third-order three-point boundary value problem. Also, a good set of references was included.

However, all the above works were done under the assumption that the lower-order derivative is not involved explicitly in the nonlinear term. More recently, the authors gave an application of a fixed-point theorem due to Avery and Peterson [6] to deal

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with the existence of positive solutions for the second-order two-point boundary value problem with dependence on the first-order derivative [7]. In this paper, we concentrate in getting the existence of positive solutions for the third-order three-point boundary value problem

$$x''' = q(t)f(t,x,x',x''), \quad t_1 \leqslant t \leqslant t_3, \tag{1.1}$$

$$x(t_1) = x'(t_2) = 0, \quad \eta x(t_3) + \delta x''(t_3) = 0. \tag{1.2}$$

In this article, it is assumed that:

$(C1)$ $f \in C([t_1,t_3] \times [0,\infty) \times R^2, [0,\infty))$;

$(C2)$ $q \in C((t_1,t_3),[0,\infty))$ and does not vanish identically on any subinterval of (t_1,t_3). Furthermore, $0 < \int_{t_1}^{t_2} q(s)ds, \int_{t_2}^{t_3} q(s)ds < +\infty$.

$(C3)$ $\eta \geqslant 0, \delta > 0; k := 2\delta + \eta(t_3 - t_1)(t_3 - 2t_2 + t_1) > 0; t_1 < t_2 < t_3$ are real numbers with $t_2 - t_1 > t_3 - t_2$.

Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis is put on the nonlinear term involved with all lower-order derivatives explicitly.

2. Background materials and definitions

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces, these definitions can be found in recent literature.

DEFINITION 2.1. Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and all $a \geqslant 0$ and

(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leqslant y$ if $y - x \in P$.

DEFINITION 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

DEFINITION 2.3. The map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leqslant t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Let γ and θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and ψ a nonnegative continuous functional on P.

For positive real numbers a, b, c and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P \mid b \leqslant \alpha(x), \gamma(x) \leqslant d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P \mid b \leqslant \alpha(x), \theta(x) \leqslant c, \gamma(x) \leqslant d\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P \mid a \leqslant \psi(x), \gamma(x) \leqslant d\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

LEMMA 2.1. ([6]) *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and ψ a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda \psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leqslant \psi(x) \quad \text{and} \quad \|x\| \leqslant M\gamma(x), \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S2) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;

(S3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$\gamma(x_i) \leqslant d \quad \text{for } i = 1, 2, 3;$$
$$b < \alpha(x_1);$$
$$a < \psi(x_2) \quad \text{with } \alpha(x_2) < b;$$
$$\psi(x_3) < a.$$

The corresponding Green's function $G(t, s)$ for the homogeneous problem

$$x''' = 0, \ \ t_1 \leqslant t \leqslant t_3,$$
$$x(t_1) = x'(t_2) = 0, \ \ \eta x(t_3) + \delta x''(t_3) = 0,$$

is established by Anderson [3]:

$$G(t, s) = \begin{cases} s \in [t_1, t_2] : \begin{cases} u_1(t, s), t \leqslant s; \\ v_1(t, s), s \leqslant t; \end{cases} \\ s \in [t_2, t_3] : \begin{cases} u_2(t, s), t \leqslant s; \\ v_2(t, s), s \leqslant t, \end{cases} \end{cases}$$

for $t,s\in[t_1,t_3]$, where

$$u_1(t,s) := \frac{t-t_1}{2}(2s-t-t_1)+\frac{\eta(t-t_1)}{2k}(s-t_1)^2(2t_2-t-t_1),$$
$$v_1(t,s) := u_1(t,s)+\frac{1}{2}(t-s)^2,$$
$$u_2(t,s) := \frac{t-t_1}{2k}(2t_2-t-t_1)\left[2\delta+\eta(t_3-s)^2\right],$$
$$v_2(t,s) := u_2(t,s)+\frac{1}{2}(t-s)^2.$$

LEMMA 2.2. ([3]) *Assume* $k=2\delta+\eta(t_3-t_1)(t_3-2t_2+t_1)>0$. *If* $t_2-t_1>t_3-t_2$, *then the Green's function* $G(t,s)$ *satisfies*

$$G(t,s)>0,\quad \frac{\partial^2}{\partial t^2}G(t,s)>0,\ \ for\ \ (t,s)\in(t_1,t_3]\times(t_1,t_3]. \tag{2.2}$$

3. Existence results of positive solutions

In this section, we impose growth conditions on f which allow us to apply Lemma 2.1 to establish the existence of triple positive solutions of Problem (1.1), (1.2).

Let $X=C^2[t_1,t_3]$ be endowed with the ordering $x\leqslant y$ if $x(t)\leqslant y(t)$ for all $t\in[t_1,t_3]$, and the maximum norm,

$$\|x\|=\max\left\{\max_{t_1\leqslant t\leqslant t_3}|x(t)|,\ \max_{t_1\leqslant t\leqslant t_3}|x'(t)|,\ \max_{t_1\leqslant t\leqslant t_3}|x''(t)|\right\}.$$

From the fact $x'''=q(t)f(t,x,x',x'')\geqslant 0, x(t)\geqslant 0$ and assumption $(C3)$, we know that x is concave on $[t_1,t_3]$. So, define the cone $P\subset X$ by

$$P=\{x\in X{:}x(t)\geqslant 0, x(t_1)=x'(t_2)=0, \eta x(t_3)+\delta x''(t_3)=0, x \text{ is concave on } [t_1,t_3]\}.$$

Given $h\in(0,t_3-t_2)$, let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ,γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(x)=\max_{t_1\leqslant t\leqslant t_3}|x''(t)|,\quad \psi(x)=\theta(x)=\max_{t_1\leqslant t\leqslant t_3}|x(t)|,\quad \alpha(x)=\min_{t_2-h\leqslant t\leqslant t_2+h}|x(t)|.$$

Note that for $x\in P$, there is

$$\max_{t_1\leqslant t\leqslant t_3}|x(t)|\leqslant(t_3-t_1)\cdot\max_{t_1\leqslant t\leqslant t_3}|x'(t)|, \tag{3.1}$$

$$\max_{t_1\leqslant t\leqslant t_3}|x'(t)|\leqslant(t_2-t_1)\cdot\max_{t_1\leqslant t\leqslant t_3}|x''(t)|. \tag{3.2}$$

Consequently, combining with the concavity of x, the functionals defined above satisfy:

$$\frac{t_3-t_2-h}{t_3-t_2}\theta(x)\leqslant\alpha(x)\leqslant\theta(x)=\psi(x), \tag{3.3}$$

$$\|x\| \leqslant \max\{(t_3-t_1)(t_2-t_1),1\}\cdot\gamma(x), \tag{3.4}$$

for all $x \in \overline{P(\gamma,d)} \subset P$. Therefore, condition (2.1) is satisfied.

Let

$$B = \min\Big\{\int_{t_2-h}^{t_2+h} G(t_2-h,s)q(s)ds, \int_{t_2-h}^{t_2+h} G(t_2+h,s)q(s)ds\Big\},$$

$$M = \max\Big\{\int_{t_1}^{t_2}[1+\frac{\eta}{k}(s-t_1)^2]q(s)ds + \frac{\eta}{k}\int_{t_2}^{t_3}[2\delta+\eta(t_3-s)^2]q(s)ds\ ,$$
$$\frac{\eta}{k}\int_{t_1}^{t_2}(s-t_1)^2q(s)ds + \frac{1}{k}\int_{t_2}^{t_3}[k+2\delta+\eta(t_3-s)^2]q(s)ds\Big\},$$

$$N = \max_{t_1\leqslant t\leqslant t_3}\int_{t_1}^{t_3} G(t,s)q(s)ds.$$

To present our main result, we assume there exist $0<a<b\leqslant \frac{(t_2-t_1)(t_3-t_1)(t_3-t_2-h)}{t_3-t_2}d$ such that

$(A1)$ $f(t,u,v,w)\leqslant \frac{d}{M}$, for $(t,u,v,w)\in[t_1,t_3]\times[0,(t_2-t_1)(t_3-t_1)d]\times[(t_1-t_2)d,(t_2-t_1)d]\times[-d,d]$;

$(A2)$ $f(t,u,v,w)>\frac{b}{B}$, for $(t,u,v,w)\in[t_2-h,t_2+h]\times[b,b(t_3-t_2)/(t_3-t_2-h)]\times[(t_1-t_2)d,(t_2-t_1)d]\times[-d,d]$;

$(A3)$ $f(t,u,v,w)<\frac{a}{N}$, for $(t,u,v,w)\in[t_1,t_3]\times[0,a]\times[(t_1-t_2)d,(t_2-t_1)d]\times[-d,d]$.

THEOREM 3.1. *Under assumptions $(A1)-(A3)$, the boundary value problem (1.1), (1.2) has at least three positive solutions x_1, x_2 and x_3 satisfying*

$$\max_{t_1\leqslant t\leqslant t_3}|x_i''(t)|\leqslant d, \quad for\ \ i=1,2,3;$$
$$\max_{t_1\leqslant t\leqslant t_3}|x_i'(t)|\leqslant (t_2-t_1)d, \quad for\ \ i=1,2,3;$$
$$b<\min_{t_2-h\leqslant t\leqslant t_2+h}|x_1(t)|;$$
$$a<\max_{t_1\leqslant t\leqslant t_3}|x_2(t)|\leqslant \frac{b(t_3-t_2)}{t_3-t_2-h}, \quad with \quad \min_{t_2-h\leqslant t\leqslant t_2+h}|x_2(t)|<b;$$
$$\max_{t_1\leqslant t\leqslant t_3}|x_3(t)|<a.$$

Proof. Problem (1.1), (1.2) has a solution $x=x(t)$ if and only if x solves the operator equation

$$x(t)=Tx(t):=\int_{t_1}^{t_3}G(t,s)q(s)f(s,x(s),x'(s),x''(s))ds.$$

Clearly, with the definition of $G(t,s)$, assumptions $(C1)-(C3)$ and Lemma 2.2, $T:P\to P$, moreover, a standard argument can proof T is completely continuous. We now show that all the conditions of Lemma 2.1 are satisfied.

If $x \in \overline{P(\gamma, d)}$, then $\gamma(x) = \max_{t_1 \leqslant t \leqslant t_3} |x''(t)| \leqslant d$. From (3.1), (3.2), one has $\max_{t_1 \leqslant t \leqslant t_3} |x'(t)| \leqslant (t_2 - t_1)d, \max_{t_1 \leqslant t \leqslant t_3} |x(t)| \leqslant (t_3 - t_1)(t_2 - t_1)d$, then assumption $(A1)$ implies $f(t, x(t), x'(t), x''(t)) \leqslant \frac{d}{M}$. Note that for $t \in [t_1, t_2]$,

$$\begin{aligned}(Tx)''(t) &= \left(\int_{t_1}^{t_2} + \int_{t_2}^{t_3}\right) \frac{\partial^2}{\partial t^2} G(t,s) q(s) f(s, x(s), x'(s), x''(s)) ds \\ &= -\frac{\eta}{k} \int_{t_1}^{t_2} (s - t_1)^2 q(s) f(s, x(s), x'(s), x''(s)) ds \\ &\quad - \frac{\eta}{k} \int_{t_2}^{t_3} [2\delta + \eta(t_3 - s)^2] q(s) f(s, x(s), x'(s), x''(s)) ds \\ &\quad + \int_{t_2}^{t} q(s) f(s, x(s), x'(s), x''(s)) ds;\end{aligned}$$

for $t \in [t_2, t_3]$,

$$\begin{aligned}(Tx)''(t) = &- \frac{\eta}{k} \int_{t_1}^{t_2} (s - t_1)^2 q(s) f(s, x(s), x'(s), x''(s)) ds \\ &- \frac{1}{k} \int_{t_2}^{t_3} [2\delta + \eta(t_3 - s)^2] q(s) f(s, x(s), x'(s), x''(s)) ds \\ &+ \int_{t_2}^{t} q(s) f(s, x(s), x'(s), x''(s)) ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma(Tx) &= \max_{t_1 \leqslant t \leqslant t_3} |(Tx)''(t)| \\ &\leqslant \frac{d}{M} \cdot \max \left\{ \int_{t_1}^{t_2} [1 + \frac{\eta}{k}(s - t_1)^2] q(s) ds + \frac{\eta}{k} \int_{t_2}^{t_3} [2\delta + \eta(t_3 - s)^2] q(s) ds \right., \\ &\quad \left. \frac{\eta}{k} \int_{t_1}^{t_2} (s - t_1)^2 q(s) ds + \frac{1}{k} \int_{t_2}^{t_3} [k + 2\delta + \eta(t_3 - s)^2] q(s) ds \right\} \\ &= \frac{d}{M} \cdot M = d.\end{aligned}$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

To check condition $(S1)$ of Lemma 2.1, we choose $x(t) = b(t_3 - t_2)/(t_3 - t_2 - h)$, $t_1 \leqslant t \leqslant t_3$.

It is easy to see that

$$x(t) = b(t_3 - t_2)/(t_3 - t_2 - h) \in P(\gamma, \theta, \alpha, b, b(t_3 - t_2)/(t_3 - t_2 - h), d)$$

and

$$\alpha(x) = \alpha(b(t_3 - t_2)/(t_3 - t_2 - h)) = b(t_3 - t_2)/(t_3 - t_2 - h) > b,$$

and so

$$\{x \in P(\gamma, \theta, \alpha, b, b(t_3 - t_2)/(t_3 - t_2 - h), d) \mid \alpha(x) > b\} \neq \emptyset.$$

Hence, if $x \in P(\gamma, \theta, \alpha, b, b(t_3 - t_2)/(t_3 - t_2 - h), d)$, then $b \leqslant x(t) \leqslant b(t_3 - t_2)/(t_3 - t_2 - h)$, $|x'(t)| \leqslant (t_2 - t_1)d, |x''(t)| \leqslant d$ for $t_2 - h \leqslant t \leqslant t_2 + h$.

From assumption $(A2)$, we have $f(t,x(t),x'(t),x''(t)) > b/B$ for $t_2 - h \leqslant t \leqslant t_2 + h$, and by the conditions of α and the cone P, we have to distinguish two cases, (i) $\alpha(Tx) = (Tx)(t_2 - h)$ and (ii) $\alpha(Tx) = (Tx)(t_2 + h)$.

In case (i), we have

$$\begin{aligned}\alpha(Tx) &= (Tx)(t_2 - h)\\ &= \int_{t_1}^{t_3} G(t_2 - h, s)q(s)f(s,x(s),x'(s),x''(s))ds\\ &> \frac{b}{B}\cdot\int_{t_2-h}^{t_2+h} G(t_2 - h, s)q(s)ds \geqslant b\,.\end{aligned}$$

In case (ii), we have

$$\begin{aligned}\alpha(Tx) &= (Tx)(t_2 + h)\\ &= \int_{t_1}^{t_3} G(t_2 + h, s)q(s)f(s,x(s),x'(s),x''(s))ds\\ &> \frac{b}{B}\cdot\int_{t_2-h}^{t_2+h} G(t_2 + h, s)q(s)ds \geqslant b\,.\end{aligned}$$

i.e.,

$$\alpha(Tx) > b,\ \text{for all } x \in P(\gamma,\theta,\alpha,b,b(t_3 - t_2)/(t_3 - t_2 - h),d).$$

This shows that condition $(S1)$ of Lemma 2.1 is satisfied.

Secondly, with (3.3) we have

$$\alpha(Tx) \geqslant \frac{t_3 - t_2 - h}{t_3 - t_2}\theta(Tx) > \frac{t_3 - t_2 - h}{t_3 - t_2}\cdot\frac{(t_3 - t_2)b}{t_3 - t_2 - h} = b,$$

for all $x \in P(\gamma,\alpha,b,d)$ with $\theta(Tx) > b(t_3 - t_2)/(t_3 - t_2 - h)$. Thus, condition $(S2)$ of Lemma 2.1 is satisfied.

Finally we show that $(S3)$ of Lemma 2.1 holds, too. Clearly, as $\psi(0) = 0 < a$, we have that $0 \notin R(\gamma,\psi,a,d)$. Suppose that $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$. Then, by the assumption $(A3)$,

$$\begin{aligned}\psi(Tx) &= \max_{t_1\leqslant t\leqslant t_3} |(Tx)(t)|\\ &= \max_{t_1\leqslant t\leqslant t_3} \int_{t_1}^{t_3} G(t,s)q(s)f(s,x(s),x'(s),x''(s))ds\\ &< \frac{a}{N}\cdot\max_{t_1\leqslant t\leqslant t_3}\int_{t_1}^{t_3} G(t,s)q(s)ds = a.\end{aligned}$$

So, condition $(S3)$ of Lemma 2.1 is also satisfied. Therefore, an application of Lemma 2.1 ends the proof.

REMARK 3.1. To apply Lemma 2.1, we only need $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, therefore, condition $(C1)$ can be substituted with a weaker condition

$(C1)'$ $f \in C([t_1,t_3]\times[0,(t_2-t_1)(t_3-t_1)d]\times[(t_1-t_2)d,(t_2-t_1)d]\times[-d,d],[0,\infty))$

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