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SECOND–ORDER LINEAR RECURRENCES WITH RESTRICTED COEFFICIENTS AND THE CONSTANT $(1/3)^{1/3}$

KENNETH S. BERENHAUT AND EVA G. GOEDHART

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Abstract. This paper provides bounds for second-order linear recurrences with restricted coefficients. It is determined that whenever the coefficients of the associated monic equation are less than the constant $(1/3)^{1/3}$, all solutions tend to zero at an exponential rate. This constant is optimal. Explicit inequalities are also provided, and some residue class structure is revealed.

1. Introduction

This paper provides bounds for second-order linear recurrences with restricted coefficients. In particular, we are concerned with solutions to the homogeneous equation

$$b_n + \alpha_n b_{n-1} + \beta_n b_{n-2} = 0, \quad (n \geqslant 2) \tag{1}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences satisfying

$$\alpha_n, \beta_n \in [0, A], \quad (n \geqslant 2), \tag{2}$$

for some $A > 0$.

We are interested in the structure of the bounding sequence $\{U_j\}_{j=2}^{\infty}$ defined by

$$U_n = U_n(A, b_0, b_1) = \max\{|b_n| : \{b_i\}, \{\alpha_i\} \text{ and } \{\beta_i\} \text{ satisfy } (1) \text{ and } (2)\}, \tag{3}$$

for $n \geqslant 2$.

Bounds for linear recurrence sequences and those for zeroes of power series are closely related (cf. Berenhaut and Morton [1]).

Note that if b_n is viewed as a function of b_0, b_1 and $\{(\alpha_i, \beta_i)\}$, then

$$b_n(b_0, b_1, \{(\alpha_i, \beta_i)\}) = b_0\, b_n(1, 0, \{(\alpha_i, \beta_i)\}) + b_1\, b_n(0, 1, \{(\alpha_i, \beta_i)\}), \tag{4}$$

for $n \geqslant 2$. Thus, in terms of $\{U_j(A, 0, 1)\}$ and $\{U_j(A, 1, 0)\}$, we have

$$U_n(A, b_0, b_1) \leqslant |b_0| U_n(A, 1, 0) + |b_1| U_n(A, 0, 1), \tag{5}$$

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for $n \geqslant 2$. As well, if $b_0 = 1$ and $b_1 = 0$, then $b_2 = -\alpha_2$, and hence, for $n \geqslant 3$,

$$U_n(A,1,0) = U_{n-1}(A,0,-\alpha_2) = \alpha_2 U_{n-1}(A,0,-1) \leqslant AU_{n-1}(A,0,-1). \tag{6}$$

Thus, for simplicity, we will restrict attention to the case $b_0 = 0$ and $b_1 = -1$.

Among our results is the following.

THEOREM 1. *Suppose that $b_0 = 0$ and $b_1 = -1$.*

(i) If $A < (1/3)^{1/3}$, then $\{U_i\}$ tends to zero at an exponential rate.

(ii) If $A > (1/3)^{1/3}$, then $\{U_i\}$ tends to infinity at an exponential rate.

(iii) If $A = (1/3)^{1/3}$, then $\{U_i\}_{i=76}^{\infty}$ is periodic with period five, with all values nonzero.

Note that Theorem 1 (i) implies that all solutions to (1) tend to zero at an exponential rate regardless of any erratic behavior in $\{(\alpha_i,\beta_i)\}$, so long as $0 \leqslant \alpha_n, \beta_n \leqslant (1/3)^{1/3} - \epsilon$, for some $\epsilon > 0$ and all $n \geqslant 2$.

In proving Theorem 1 we will rely on the following recent result (see [2]) which connects bounds for solutions to (1) with maximal products over integer partitions.

THEOREM 2. *Suppose $A > 0$ and for given x and y, consider $G(x,y)$, the maximal product over partitions of x into y parts, i.e.*

$$G(x,y) = \max_{\substack{e_1+e_2+\cdots+e_y=x \\ e_j \in \mathbf{N}^+}} e_1 e_2 \cdots e_y. \tag{7}$$

Then, for $n \geqslant 2$,

$$U_n = \max_{\left[\frac{n}{2}\right]+1 \leqslant k \leqslant n} G(k-1, 2k-n-1)A^{k-1}. \tag{8}$$

Proof. See [2]. □

It is worthwhile to note that the proof of Theorem 2 relies heavily on the following lemma, which serves to discretize the problem of determining $\{U_j\}$. While the Lemma is proven in [2], we include it here for completeness.

LEMMA 1. *Suppose that $\{b_i\}$, $\{\alpha_i\}$, $\{\beta_i\}$ and $A > 0$ satisfy (1) and (2) with $b_0 = 0$ and $b_1 = -1$. Let $\mathscr{P} = \{n \geqslant 0 : b_n \geqslant 0\}$ and $\mathscr{N} = \{n \geqslant 0 : b_n < 0\}$ partition the sign configuration of $\{b_i\}_{i=1}^{\infty}$, and define B_n (a polynomial in A) recursively in n from $\mathscr{N}$ and $\mathscr{P}$ via $B_0 = 0$, $B_1 = -1$ and*

$$B_n = \begin{cases} -A \cdot \chi_{\mathscr{N}}(n-1)B_{n-1} - A \cdot \chi_{\mathscr{N}}(n-2)B_{n-2}, & n \in \mathscr{P} \\ -A \cdot \chi_{\mathscr{P}}(n-1)B_{n-1} - A \cdot \chi_{\mathscr{P}}(n-2)B_{n-2}, & n \in \mathscr{N} \end{cases}, \tag{9}$$

for $n \geqslant 2$, where χ_V indicates the characteristic function for the set V.

Then, B_i and b_i have the same sign and

$$|b_i| \leqslant |B_i|, \tag{10}$$

for all $i \geqslant 1$.

Proof. Simple induction with (9) will show that B_n and b_n have the same sign for $n \geqslant 1$.

Now, note that under the inherent assumptions, $b_1 = -1 = B_1$ and $b_2 = \alpha_2 \leqslant A = B_2$. We shall prove the lemma by induction. Suppose that $n > 1$ and that (10) is satisfied for all $i \leqslant n-1$. Now, assume that $n \in \mathscr{P}$. Then,

$$\begin{aligned} b_n &= -\alpha_n b_{n-1} - \beta_n b_{n-2} \\ &\leqslant -A \cdot \chi_{\mathscr{N}}(n-1) b_{n-1} - A \cdot \chi_{\mathscr{N}}(n-2) b_{n-2} \\ &= A \cdot \chi_{\mathscr{N}}(n-1)|b_{n-1}| + A \cdot \chi_{\mathscr{N}}(n-2)|b_{n-2}| \\ &\leqslant A \cdot \chi_{\mathscr{N}}(n-1)|B_{n-1}| + A \cdot \chi_{\mathscr{N}}(n-2)|B_{n-2}| \\ &= -A \cdot \chi_{\mathscr{N}}(n-1) B_{n-1} - A \cdot \chi_{\mathscr{N}}(n-2) B_{n-2} \\ &= B_n, \end{aligned} \tag{11}$$

where the first inequality follows from (2) and the second from an application of the induction hypothesis.

An analogous argument works when $n \in \mathscr{N}$. □

REMARK 1. Note that while Lemma 1 reduces the computations required in obtaining U_n to 2^{n-1} comparisons, Theorem 2 further reduces that number to roughly $n/2$. □

Rewriting (8), we have

COROLLARY 1. *Under the assumptions in Theorem 2,*

$$U_n = \max_{\left[\frac{n}{2}\right]+1 \leqslant k \leqslant n} W_{k,n} A^{k-1}, \tag{12}$$

where

$$W_{k,n} = \begin{cases} {r_{k,n}}^{i_{k,n}} (r_{k,n} - 1)^{j_{k,n}}, & 2k - n - 1 > 0 \\ 1, & \text{otherwise} \end{cases}, \tag{13}$$

with[1]

$$r_{k,n} = \left\lceil \frac{k-1}{2k-n-1} \right\rceil, \tag{14}$$

$$j_{k,n} = r_{k,n}(2k-n-1) - (k-1), \tag{15}$$

and

$$i_{k,n} = 2k - n - 1 - j_{k,n}. \tag{16}$$

[1] We use the notations $\lceil y \rceil$, $[y]$, and $\{y\}$ to denote the least integer greater than or equal to y, the greatest integer less than or equal to y, and the fractional part of y, respectively.

In [2], the following closed-form result for the case when $A = 1$ was obtained via Theorem 2.

THEOREM 3. *Suppose $A = 1$ and $\{U_j\}$ is as in Theorem 2. Then, we have $U_2 = U_3 = 1$, $U_4 = U_5 = 2$, $U_6 = 3$, $U_7 = U_8 = 4$ and for $n \geqslant 9$,*

$$U_n = \begin{cases} 3 \cdot 2^{\frac{n-6}{3}}, & n \equiv 0 \mod 3 \\ 2^{\frac{n-1}{3}}, & n \equiv 1 \mod 3 \\ 3^2 \cdot 2^{\frac{n-11}{3}}, & n \equiv 2 \mod 3 \end{cases} . \tag{17}$$

Proof. See [2]. □

A by-product of the proof of Theorem 1 is the following similar result for

$$\frac{2}{3} < A < \frac{3}{4}. \tag{18}$$

THEOREM 4. *Suppose A satisfies (18) and $\{U_j\}$ is as in Theorem 2. For given $n \geqslant 76$, express n in the form $n = 15x + a + 1$, for some $0 \leqslant a \leqslant 14$. Then,*

$$U_n = \begin{cases} C_2(A,a)(3A^3)^{3x}, & \text{if } a \equiv 2 \mod 5 \\ C_1(A,a)(3A^3)^{3x}, & \text{otherwise} \end{cases} , \tag{19}$$

where

$$C_1(A,a) = 3^{-a+3\left[\frac{2a}{5}\right]} 2^{2a-5\left[\frac{2a}{5}\right]} A^{a-\left[\frac{2a}{5}\right]} \tag{20}$$

and

$$C_2(A,a) = 4^{-2a+5\left[\frac{2a}{5}\right]+5} 3^{3a-7\left[\frac{2a}{5}\right]-7} A^{a-\left[\frac{2a}{5}\right]-1}. \tag{21}$$

Recurrences with varying or random coefficients have been studied by many previous authors. A partial survey of such literature can be found in [1].

REMARK 2. Suppose that the coefficients in (1) are constant, i.e.

$$b_n + \alpha b_{n-1} + \beta b_{n-2} = 0, \quad (n \geqslant 2). \tag{22}$$

Then, it is well known that if both roots of the auxiliary equation $x^2 + \alpha x + \beta = 0$ have modulus less than one, all solutions to (1) tend to zero (cf. Goldberg [3]). The region of (α, β) satisfying this root requirement is shaded in Figure 1(a). Theorem 1 guarantees that if all pairs (α_i, β_i) are in the smaller rectangular region shaded in Figure 1(b), then all solutions also tend to zero, *regardless* of any erratic behavior in the sequence $\{(\alpha_i, \beta_i)\}$.

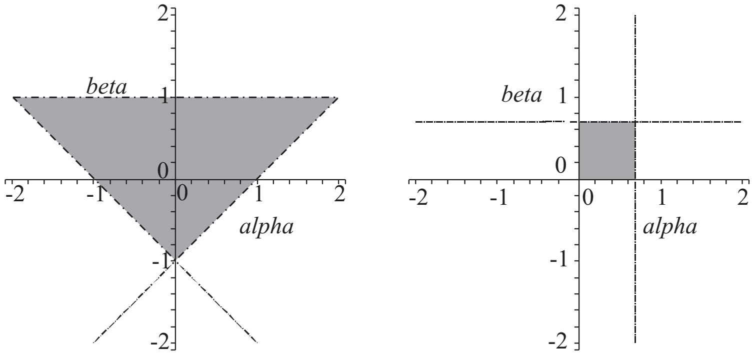

Figure $1(a)$ *Figure* $1(b)$

Figure 1. *Shaded regions (in the* (α,β) *-plane) for convergence to zero of all solutions, for constant and nonconstant coefficients (Figures* $1(a)$ *and* $1(b)$*, respectively).*

We now turn to a proof of Theorem 1.

2. Proof of Theorem 1

For fixed $n>1$, define the sequence $\{v_i\}$ by

$$v_i = G\left(\left[\frac{n}{2}\right]+i, 2\left[\frac{n}{2}\right]-n+1+2i\right), \tag{23}$$

for $0 \leqslant i \leqslant n-[n/2]-1$.

We will use the following lemma.

LEMMA 2. *The sequence* $\{v_i\}$ *is logarithmically concave and hence unimodal.*

Proof. We will show that for all $u,v \geqslant 0$,

$$G(u,v)G(u+2,v+4) \leqslant G(u+1,v+2)^2. \tag{24}$$

From the definition of G, we have for $r^* = \left\lceil \frac{2(u+1)}{2(v+2)} \right\rceil = \left\lceil \frac{u+1}{v+2} \right\rceil$,

$$\begin{aligned} G(u,v)G(u+2,v+4) &\leqslant G(u+(u+2), v+(v+4)) \\ &= r^{*\,2(v+2)-(r^*2(v+2)-2(u+1))}(r^*-1)^{r^*2(v+2)-2(u+1)} \\ &= \left(r^{*\,(v+2)-(r^*(v+2)-(u+1))}(r^*-1)^{r^*(v+2)-(u+1)}\right)^2 \\ &= G(u+1,v+2)^2. \end{aligned} \tag{25}$$

The first and last equalities in (25) follow by reasoning similar to that in the proof of Corollary 1. □

Proof of Theorem 1. It suffices to prove the theorem for $b_0=0$ and $b_1=-1$. Suppose $n=15x+a+1$, for some $0 \leqslant a \leqslant 14$ and $x \geqslant 5$, and that A satisfies (18).

Note that by Lemma 2, the sequence $\{a_i\}$, with

$$a_i = v_i A^{\left[\frac{n}{2}\right]+i} \tag{26}$$

is unimodal, and by Theorem 2,

$$U_n = \max_{0 \leqslant i \leqslant n - \left[\frac{n}{2}\right]-1} a_i. \tag{27}$$

Let $j^* = 9x + a - [2a/5] - [n/2]$. We will show that $U_n \in \{a_{j^*-1}, a_{j^*}\}$. First, it may be readily verified that for $x \geqslant 5$ and $0 \leqslant a \leqslant 14$,

$$3 < \left(\frac{9x + a - \left[\frac{2a}{5}\right] - 2}{3x + a - 2\left[\frac{2a}{5}\right] - 4}\right), \left(\frac{9x + a - \left[\frac{2a}{5}\right] - 1}{3x + a - 2\left[\frac{2a}{5}\right] - 2}\right) \leqslant 4 \tag{28}$$

and

$$2 < \left(\frac{9x + a - \left[\frac{2a}{5}\right]}{3x + a - 2\left[\frac{2a}{5}\right]}\right), \left(\frac{9x + a - \left[\frac{2a}{5}\right] + 1}{3x + a - 2\left[\frac{2a}{5}\right] + 2}\right) \leqslant 3. \tag{29}$$

Thus, from (23), (26) and Corollary 1, we have

$$\begin{aligned} a_{j^*-2} &= G\left(9x + a - \left[\frac{2a}{5}\right] - 2, 3x + a - 2\left[\frac{2a}{5}\right] - 4\right) A^{9x+a-\left[\frac{2a}{5}\right]-2} \\ &= 4^{-2a+5\left[\frac{2a}{5}\right]+10} 3^{3x+3a-7\left[\frac{2a}{5}\right]-14} A^{9x+a-\left[\frac{2a}{5}\right]-2}, \end{aligned} \tag{30}$$

$$\begin{aligned} a_{j^*-1} &= G\left(9x + a - \left[\frac{2a}{5}\right] - 1, 3x + a - 2\left[\frac{2a}{5}\right] - 2\right) A^{9x+a-\left[\frac{2a}{5}\right]-1} \\ &= 4^{-2a+5\left[\frac{2a}{5}\right]+5} 3^{3x+3a-7\left[\frac{2a}{5}\right]-7} A^{9x+a-\left[\frac{2a}{5}\right]-1}, \end{aligned} \tag{31}$$

$$\begin{aligned} a_{j^*} &= G\left(9x + a - \left[\frac{2a}{5}\right], 3x + a - 2\left[\frac{2a}{5}\right]\right) A^{9x+a-\left[\frac{2a}{5}\right]} \\ &= 3^{3x-a+3\left[\frac{2a}{5}\right]} 2^{2a-5\left[\frac{2a}{5}\right]} A^{9x+a-\left[\frac{2a}{5}\right]} \end{aligned} \tag{32}$$

and

$$\begin{aligned} a_{j^*+1} &= G\left(9x + a - \left[\frac{2a}{5}\right] + 1, 3x + a - 2\left[\frac{2a}{5}\right] + 2\right) A^{9x+a-\left[\frac{2a}{5}\right]+1} \\ &= 3^{3x-a+3\left[\frac{2a}{5}\right]-3} 2^{2a-5\left[\frac{2a}{5}\right]+5} A^{9x+a-\left[\frac{2a}{5}\right]+1}. \end{aligned} \tag{33}$$

By (18) and (30) – (33), we have

$$\begin{aligned} \frac{a_{j^*-2}}{a_{j^*-1}} &= \frac{4^5}{3^7 A} \\ &\in [0.6243, 0.7024] \end{aligned} \tag{34}$$

and

$$\frac{a_{j^*}}{a_{j^*+1}} = \frac{3^3}{2^5 A} \in [1.125, 1.266], \tag{35}$$

and by the unimodality of $\{a_i\}$, $U_n \in \{a_{j^*-1}, a_{j^*}\}$ as claimed.

Now, note that

$$\begin{aligned}\frac{a_{j^*-1}}{a_{j^*}} &= \frac{4^{-2a+5\left[\frac{2a}{5}\right]+5}3^{4a-10\left[\frac{2a}{5}\right]-7}}{2^{2a-5\left[\frac{2a}{5}\right]}A} \\ &= \frac{4^5}{3^7 A}\frac{3^{10\left\{\frac{2a}{5}\right\}}}{4^{5\left\{\frac{2a}{5}\right\}}2^{5\left\{\frac{2a}{5}\right\}}} \\ &= \frac{4^5}{3^7 A}\left(\frac{9}{8}\right)^{5\left\{\frac{2a}{5}\right\}}.\end{aligned} \tag{36}$$

Hence,

$$\frac{a_{j^*-1}}{a_{j^*}} \in \begin{cases} (0.624295, 0.702332), & \text{if } a \equiv 0 \mod 5 \\ (0.790123, 0.888889), & \text{if } a \equiv 1 \mod 5 \\ (1.000, 1.125), & \text{if } a \equiv 2 \mod 5 \\ (0.702331, 0.790124), & \text{if } a \equiv 3 \mod 5 \\ (0.88888, 1.000), & \text{if } a \equiv 4 \mod 5 \end{cases}, \tag{37}$$

and

$$U_n = \begin{cases} a_{j^*-1}, & \text{if } a \equiv 2 \mod 5 \\ a_{j^*}, & \text{otherwise} \end{cases}. \tag{38}$$

From (32) and (31), we have

$$\begin{aligned} a_{j^*} &= 3^{3x-a+3\left[\frac{2a}{5}\right]}2^{2a-5\left[\frac{2a}{5}\right]}A^{9x+a-\left[\frac{2a}{5}\right]} \\ &= C_1(A,a)(3A^3)^{3x}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} a_{j^*-1} &= 4^{-2a+5\left[\frac{2a}{5}\right]+5}3^{3x+3a-7\left[\frac{2a}{5}\right]-7}A^{9x+a-\left[\frac{2a}{5}\right]-1} \\ &= C_2(A,a)(3A^3)^{3x}, \end{aligned} \tag{40}$$

where $C_1(A,a)$ and $C_2(A,a)$ are as in (20) and (21), respectively.

Parts (i) and (ii) of the theorem now follow directly from (39) and (40).

Regarding Part (iii), note that for $\kappa = (1/3)^{1/3}$,

$$\begin{aligned} \max(C_1(\kappa,a), C_2(\kappa,a))(3\kappa^3)^{3x} &= \max(C_1(\kappa,a), C_2(\kappa,a)) \\ &= \begin{cases} 1, & \text{if } a \equiv 0 \mod 5 \\ 0.924482, & \text{if } a \equiv 1 \mod 5 \\ 0.924482, & \text{if } a \equiv 2 \mod 5 \\ 0.961500, & \text{if } a \equiv 3 \mod 5 \\ 0.888889, & \text{if } a \equiv 4 \mod 5 \end{cases}. \end{aligned} \tag{41}$$

□

REMARK 3. Note that Theorem 4 follows from (39) and (40).

3. Conclusion and future directions

To this point, we have not found corresponding theorems for general k^{th} order linear recurrences, and hence we restate the following question from [2].

Open question. What is (a workable general form for) the maximum possible value (as a function of n and A) of the n^{th} term of a k^{th} order linear recurrence satisfying

$$b_n + \sum_{i=n-k}^{n-1} \alpha_{n,i} b_i = 0; \quad n \geqslant 2, \tag{42}$$

where $b_0 = 0$, $b_1 = 1$, and for some $A > 0$,

$$\alpha_{n,i} \in [0, A]; \quad n-k \leqslant i \leqslant n-1, \quad n \geqslant 2. \tag{43}$$

In addition, it would be interesting to know constants satisfying Theorem 1, Parts (i) and (ii), for higher order difference equations.

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Kenneth S. Berenhaut
Wake Forest University
Department of Mathematics
Winston-Salem
NC 27109
USA
Telephone: (336)758-5922
Fax: (336)758-7190
e-mail: berenhks@wfu.edu

Eva G. Goedhart
Wake Forest University
Department of Mathematics
Winston-Salem
NC 27109
USA
Telephone: (336)758-5922
Fax: (336)758-7190
e-mail: goedeg3@wfu.edu