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ON THE RATE OF CONVERGENCE FOR CERTAIN SUMMATION–INTEGRATION TYPE OPERATORS

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Abstract. In the present paper, we study the certain summation integral type operators which includes the well known Baskakov-Durrmeyer and Szasz-Durrmeyer operators as special cases. We obtain the rate of convergence for functions of bounded variation, for these generalized sequences of linear positive operators together with the exact bounds for Baskakov basis functions and Szasz basis functions.

1. Introduction

To approximate Lebesgue integrable functions on the interval $[0,1]$ Durrmeyer [3], defined the integral modification of Bernstein polynomials as

$$B_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}.$$

Derriennic [2] studied some approximation properties for these Bernstein Durrmeyer operators. Guo [4] estimated the rate of convergence for bounded variation functions for the operators B_n. Motivated by the integral modification of Bernstein polynomials and subsequent work on Bernstein Durrmeyer operators, Mazhar and Totik [9] and Sahai and Prasad [10] defined the Durrmeyer variants of Szasz-Mirakyan and Lupas operators respectively. Gupta [5] defined some other type of Baskakov Durrmeyer operators and studied asymptotic formulas and error estimates for the operators, it was observed in [5] that the operators with different weight give better results over the Lupas Durrmeyer operators defined in [10]. To approximate Lebesgue integrable functions on the interval $[0,\infty)$, we now consider the following operators

$$M_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x,c)\int_0^{\infty} b_{n,k}(t,c)f(t)dt \tag{2}$$

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where

$$p_{n,k}(x,c) = (-1)^k \frac{x^k}{k!} \phi_{n,c}^{(k)}(x), \quad b_{n,k}(t,c) = (-1)^{k+1} \frac{t^k}{k!} \phi_{n,c}^{(k+1)}(t)$$

and

(i) for $c > 0$, $\phi_{n,c}(x) = (1+cx)^{-n/c}$ and $x \in [0,\infty)$
(ii) for $c = 0$, $\phi_{n,c}(x) = e^{-nx}$ and $x \in [0,\infty)$.

Here we observe that for case (i) i.e. $c > 0$ the operators M_n reduce to Baskakov Durrmeyer operators and for case (ii) i.e. $c = 0$ our operators M_n become Szasz Durrmeyer operators. Some approximation properties of these operators were studied in [8]. Operators with the rate of convergence for the particular value $c = 1$ were studied in [6].

Let

$$K_n(x,t,c) = \sum_{k=0}^{\infty} p_{n,k}(x,c) b_{n,k}(t,c)$$

and

$$\beta_{n,c}(x,y) = \int_0^y K_n(x,t,c)dt$$

In particular

$$\beta_{n,c}(x,\infty) = \int_0^\infty K_n(x,t,c)dt = 1$$

In the present paper, we study the rate of approximation for the functions of bounded variation of these generalized operators $M_n(f,x)$, defined by (2).

2. Auxiliary results

In this section we give certain results, which are necessary to prove the main result.

LEMMA 1. *For $m \in N \cup \{0\}$, if we define the m-th order moment by*

$$\mu_{n,m}(x,c) = \sum_{k=0}^{\infty} p_{n,k}(x,c) \int_0^\infty b_{n,k}(t,c)(t-x)^m dt$$

then

$$\mu_{n,0}(x,c) = 1, \mu_{n,1}(x,c) = \frac{1+cx}{n-c}$$

and

$$\mu_{n,2}(x,c) = \frac{2cx^2(n+c) + 2x(n+2c) + 2}{[n-c][n-2c]}.$$

Also the following recurrence relation holds

$$[n - c(m+1)]\mu_{n,m+1}(x,c) = x(1+cx)[\mu_{n,m}^{(1)}(x,c) + 2m\mu_{n,m-1}(x,c)] + [(1+2cx)(m+1) - cx]\mu_{n,m}(x,c).$$

REMARK 1. In particular, by Lemma 1, for $c \geqslant 0$ given any number $\lambda > 2$ and $x > 0$, we have for n sufficiently large

$$\mu_{n,2}(x,c) \leqslant \frac{\lambda x(1+cx)}{n}. \tag{3}$$

LEMMA 2. *For all* $x \in (0,\infty)$, $c \geqslant 0$ *and* $k \in N$, *we have*
(i) $p_{n,k}(x,c) \leqslant \frac{\sqrt{1+cx}}{\sqrt{2enx}}$, *where* $\phi_{n,c}(x) = (1+cx)^{-n/c}, c > 0$
(ii) $p_{n,k}(x,c) \leqslant \frac{1}{\sqrt{2enx}}$, *where* $\phi_{n,c}(x) = e^{-nx}, c = 0$
where the constant $1/\sqrt{2e}$ *and the estimation order* $n^{-1/2}$ *(for* $n \to \infty$*) are the best possible.*

Proof. By [12, Th. 2], it follows that

$$\binom{n+k-1}{k} t^k (1-t)^n < \frac{1}{\sqrt{2ent}}, t \in (0,1]$$

Replacing the variable t by $\frac{cx}{1+cx}$ and n by n/c in the above inequality, we get

$$\prod_{l=0}^{k-1} \frac{l+n/c}{k!} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} \equiv p_{n,k}(x,c) < \frac{\sqrt{1+cx}}{\sqrt{2enx}}, \text{for} x \in (0,\infty.)$$

This completes the proof of (i).

Next, we prove (ii). Following [13], we have

$$p_{n,k}(x,c) \leqslant \frac{H(j)}{\sqrt{nx}}, \text{ where } \phi_{n,c}(x) = e^{-nx}, c = 0, k \geqslant j$$

where $H(j) = \frac{(j+1/2)^{j+1/2} e^{-(j+1/2)}}{j!}$.

Since $\max_{j \geqslant 0} H(j) = H(0) = 1/\sqrt{2e}$, it follows that:

$$p_{n,k}(x,c) \leqslant \frac{1}{\sqrt{2enx}}, \text{ for each integer} k \geqslant 0.$$

REMARK 2. For particular value $c = 1$ Wang and Guo [11] gave the following bound for Baskakov basis functions:

$$p_{n,k}(x,1) \leqslant \frac{33}{\sqrt{n}} \left(\frac{1+x}{x}\right)^{3/2}, x \in (0,\infty), k \in N.$$

It could be observed that our Lemma 2 (i) for $c = 1$ gives the sharp bound over the bound of Wang and Guo [11].

For $c = 0$, also Lemma 2 (ii) could be utilized to give better estimate over the main result of [7].

LEMMA 3. *Suppose $0 < x < \infty$, $\lambda > 2$ and $c \geqslant 0$ for n sufficiently large, there hold*

$$\int_0^y K_n(x,t,c)dt \leqslant \frac{\lambda x(1+cx)}{n(x-y)^2}, 0 \leqslant y < x, \tag{4}$$

$$\int_z^\infty K_n(x,t,c)dt \leqslant \frac{\lambda x(1+cx)}{n(z-x)^2}, x < z < \infty. \tag{5}$$

Proof. We first prove (4) as follows:

$$\int_0^y K_n(x,t,c)dt \leqslant \int_0^y \frac{(x-t)^2}{(x-y)^2} K_n(x,t,c)dt$$
$$\leqslant \frac{1}{(x-y)^2} M_n((t-x)^2, x) \leqslant \frac{\mu_{n,2}(x,c)}{(x-y)^2} \leqslant \frac{\lambda x(1+cx)}{n(x-y)^2}$$

by using (3). The proof of (5) follows along the similar lines.

3. Rate of convergence

In this section we prove the following general theorem, for $c > 0$ and $c = 0$, we get the particular results for Baskakov Durrmeyer type operators and Szasz Durrmeyer operators respectively.

THEOREM 1. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and if $f(x) = O((1+x)^r), x \to \infty$. Then for $\lambda > 2$, $c \geqslant 0, x \in (0,\infty)$ and n sufficiently large, we have*

$$\left| M_n(f,x) - \frac{1}{2}\{f(x+)+f(x-)\} \right| \leqslant \frac{\sqrt{1+cx}}{\sqrt{8enx}}. |f(x+)-f(x-)|$$
$$+ \frac{x+3\lambda.(1+cx)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + C_1 \left(\frac{1+x}{nx^2}\right)^r + C_2 \frac{(1+cx)(1+x)^r}{nx},$$

where the constants C_1 and C_2 are independent of n and x and $g_x(t)$ is the auxiliary function defined by

$$g_x(t) = \begin{cases} f(t) - f(x-), 0 \leqslant t < x \\ 0, t = x \\ f(t) - f(x+), x < t < \infty \end{cases}$$

$V_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

Proof. Clearly

$$\left| M_n(f,x) - \frac{1}{2}\{f(x+)+f(x-)\} \right|$$
$$\leqslant |M_n(g_x,x)| + \frac{1}{2}|f(x+)-f(x-)|.|M_n(sign(t-x),x)| \tag{6}$$

In order to prove the result we need the estimates for $M_n(g_x,x)$ and $M_n(sign(t-x),x)$. First we estimate $M_n(sign(t-x),x)$ as follows:

$$\begin{aligned} M_n(sign(t-x),x) &= \sum_{k=0}^{\infty} p_{n,k}(x,c)\left(\int_x^{\infty} b_{n,k}(t,c)dt - \int_0^x b_{n,k}(t,c)dt\right) \\ &= \sum_{k=0}^{\infty} p_{n,k}(x,c)\left(\int_0^{\infty} b_{n,k}(t,c)dt - 2\int_0^x b_{n,k}(t,c)dt\right) \\ &= 1-2\sum_{k=0}^{\infty} p_{n,k}(x,c)\int_0^x b_{n,k}(t,c)dt \end{aligned}$$

Using the fact that $\int_x^{\infty} b_{n,k}(t,c)dt = \sum_{j=0}^{k} p_{n,j}(x,c)$ and applying Lemma 2, we have

$$|M_n(sign(t-x),x)| \leqslant \begin{cases} \dfrac{\sqrt{1+cx}}{\sqrt{2enx}}, c>0 \\ \dfrac{1}{\sqrt{2enx}}, c=0 \end{cases} \tag{7}$$

Next we estimate $M_n(g_x,x)$. By Lebesgue-Stieltjes integral representation, we have

$$\begin{aligned} M_n(g_x,x) &= \int_0^{\infty} g_x(t)K_n(x,t,c)dt \\ &= \left(\int_{I_1}+\int_{I_2}+\int_{I_3}+\int_{I_4}\right)K_n(x,t,c)g_x(t)dt = R_1+R_2+R_3+R_4, \end{aligned} \tag{8}$$

say, where $I_1=[0,x-x/\sqrt{n}]$, $I_2=[x-x/\sqrt{n},x+x/\sqrt{n}]$, $I_3=[x+x/\sqrt{n},2x]$ and $I_4=[2x,\infty)$. First we estimate R_1. Writing $y=x-x/\sqrt{n}$ and using integration by parts, we have

$$R_1 = \int_0^y g_x(t)d_t(\beta_{n,c}(x,t)) = g_x(y+)\beta_n(x,y) - \int_0^y \beta_{n,c}(x,t)d_t(g_x(t))$$

Since $|g_x(y+)| \leqslant V_{y+}^x(g_x)$ by (4), we obtain

$$\begin{aligned} |R_1| &\leqslant V_{y+}^x(g_x)\beta_{n,c}(x,y) + \int_0^y \beta_{n,c}(x,t)d_t(-V_t^x(g_x)) \\ &\leqslant V_{y+}^x(g_x)\frac{\lambda x(1+cx)}{n(x-y)^2} + \frac{\lambda(1+cx)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)). \end{aligned}$$

Integrating by parts the last term after simple computation we have

$$|R_1| \leqslant \frac{\lambda x(1+cx)}{n}\left[\frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{t}$, we obtain

$$|R_1| \leqslant \frac{2\lambda(1+cx)}{nx}\sum_{k=1}^{n} V^x_{x-x/\sqrt{k}}(g_x). \tag{9}$$

Now, we estimate R_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V^{x+x/\sqrt{n}}_{x-x/\sqrt{n}}(g_x).$$

Also by the fact that $\int_a^b d_t(\beta_{n,c}(x,t)) \leqslant 1$ for $(a,b) \subset [0,\infty)$ it follows

$$|R_2| \leqslant V^{x+x/\sqrt{n}}_{x-x/\sqrt{n}}(g_x) \leqslant \frac{1}{n}\sum_{k=1}^{n} V^{x+x/\sqrt{k}}_{x-x/\sqrt{k}}(g_x). \tag{10}$$

Next, we estimate R_3. By setting $z = x + x/\sqrt{n}$, we have

$$\begin{aligned} R_3 &= \int_z^{2x} K_n(x,t,c)g_x(t)dt = -\int_z^{2x} g_x(t)d_t(1-\beta_{n,c}(x,t)) \\ &= -g_x(2x)(1-\beta_{n,c}(x,2x)) + g_x(z)(1-\beta_{n,c}(x,z)) + \int_z^{2x}(1-\beta_{n,c}(x,t))d_t g_x(t). \end{aligned}$$

Since $|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_x^t(g_x)$ it follows, by Lemma 3

$$|R_3| \leqslant \frac{\lambda x(1+cx)}{n}\{x^{-2}V_x^{2x}(g_x) + (z-x)^{-2}V_x^z(g_x) + \int_z^{2x}(t-x)^{-2}d_t V_x^t(g_x)\}.$$

Again integrating by parts, we derive

$$|R_3| \leqslant \frac{\lambda x(1+cx)}{n}\{2x^{-2}V_x^{2x}(g_x) + 2\int_z^{2x} V_x^t(g_x)(t-x)^{-3}dt\}.$$

Thus arguing similarly as in the estimate of R_1, we obtain

$$|R_3| \leqslant \frac{3\lambda(1+cx)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{11}$$

Finally we estimate R_4. For $n > \gamma$, we have

$$|R_4| = \left| \int_{2x}^{\infty} K_n(x,t,c) g_x(t) dt \right| \leqslant M \int_{2x}^{\infty} K_n(x,t,c)[(1+t)^\gamma + (1+x)^\gamma] dt.$$

Using the identity

$$(1+t)^\gamma - (1+x)^\gamma \leqslant (2^\gamma - 1)\frac{(1+x)^\gamma}{x^\gamma}(t-x)^\gamma, \text{ for } t \geqslant 2x,$$

and Lemma 2, we have

$$\begin{aligned} |R_4| &\leqslant M(2^\gamma - 1)\frac{(1+x)^\gamma}{x^\gamma} \int_{2x}^{\infty} K_n(x,t,c)(t-x)^\gamma dt + 2M(1+x)^\gamma \int_{2x}^{\infty} K_n(x,t,c) dt \\ &\leqslant M(2^\gamma - 1)\frac{(1+x)^\gamma}{x^\gamma} \int_{2x}^{\infty} K_n(x,t,c)\frac{(t-x)^{2\gamma}}{x^\gamma} dt + 2M\frac{(1+x)^\gamma}{x^2} \int_{2x}^{\infty} K_n(x,t,c)(t-x)^2 dt \\ &\leqslant M(2^\gamma - 1)\frac{(1+x)^\gamma}{x^{2\gamma}} O\left(n^{-\gamma}\right) + 2M\frac{(1+x)^\gamma}{x^2}\frac{\lambda x(1+cx)}{n}. \end{aligned}$$

Hence

$$|R_4| \leqslant C_1 \left(\frac{1+x}{nx^2}\right)^r + C_2 \frac{(1+x)^\gamma (1+cx)}{nx}, \tag{12}$$

where the constants C_1 and C_2 are independent of n and x.

Hence if n is sufficiently large then by combining the estimates of (8), (12), we get the required result.

REMARK 3. Recently Bastien and Rogalski [1] answered the question raised by the first author in a private communication and obtained the better bound for this Baskakov basis functions (for the special case $c = 1$) as follows:

$$\binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \leqslant \frac{M}{\sqrt{nx(1+x)}} \tag{13}$$

where the constant M is 1 if $n = 1$. For $n \geqslant 2, k = 0$, the value of M is $\frac{2\sqrt{2}}{3\sqrt{3}}$. If $n \geqslant 2, k \geqslant 1$ the value of M depends on n and is given by $\left(\frac{3}{2}\right)^{3/2} n^{3/2} \frac{(n-1)^{n-1}}{(n+\frac{1}{2})^{n+\frac{1}{2}}}$.

By using the bound (13), our theorem gives an improved estimate for this particular case $c = 1$ also.

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