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ON (H_{pq}, L_{pq})–TYPE INEQUALITY OF MAXIMAL OPERATOR OF MARCINKIEWICZ–FEJÉR MEANS OF DOUBLE FOURIER SERIES WITH RESPECT TO THE KACZMARZ SYSTEM

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Abstract. The main aim of this paper is to prove that the maximal operator of Marcinkiewicz-Fejér means of double Fourier series with respect to the Kaczmarz system is bounded from the dyadic Hardy-Lorentz space H_{pq} into the Lorentz space L_{pq} for every $p > \frac{1}{2}$ and $0 < q \leqslant \infty$ provided that the supremum in the maximal operator is taken over special indices. As a consequence, we obtain the a.e. convergence of Marcinkiewicz-Fejér means of double Fourier series for special indices with respect to the Walsh-Kaczmarz system. That is, $\sigma_{2^n}(f, x^1, x^2) \to f(x^1, x^2)$ a.e. as $n \to \infty$.

1. Introduction

In 1939 Marcinkiewicz [6] proved for two-dimensional trigonometric system that the Marcinkiewicz means of a function converge to the function itself almost evrywhere for all $f \in L \log L([0, 2\pi]^2)$. Zhizhiashvili [15] improved this result for $f \in L([0, 2\pi]^2)$. Dyachenko [1] proved this result for dimensions greater than 2.

For the two-dimensional Walsh-Fourier series Weisz [13] proved that the maximal operator

$$\sigma^* f = \sup_{n \geqslant 1} \frac{1}{n} \left| \sum_{j=1}^{n} S_{j,j}(f) \right|$$

is bounded from the two-dimensional dyadic martingale Hardy-Lorentz space H_{pq} to the Lorentz space L_{pq} for $p > 2/3$ and $0 < q \leqslant \infty$ and is of weak type (1,1). Goginava [4] generalized the theorem of Weisz for d-dimensional Walsh-Fourier series. The a.e. convergence of the arithmetic means of quadratical partial sums of double Vilenkin-Fourier series was studied by Gát [3].

In 1948 Šneider [11] introduced the Walsh-Kaczmarz system and showed that the inequality

$$\limsup_{n \to \infty} \frac{D_n^{\kappa}(x)}{\log n} \geqslant C > 0$$

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holds a.e. In 1974 Schipp [8] and Young [12] proved that the Walsh-Kaczmarz system is a convergence system. Skvorcov in 1981 [10] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to f for any continuous functions f. Gát [2] proved, for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. Gát's Theorem was extended by Simon [9] to H_{pq} spaces, namely that the maximal operator of Fejér means of the one-dimensional Fourier series is bounded from Hardy-Lorentz spaces into Lorentz spaces for $p > 1/2$ and $0 < q \leqslant \infty$. He also showed (H_{pq}, L_{pq})-boundedness for every $0 < p \leqslant 1$ if the maximal operator of the Fejér means is considered only of order 2^n.

The main aim of this paper is to prove that the maximal operator of Marcinkiewicz-Fejér means of double Fourier series with respect to the Kaczmarz system is bounded from the dyadic Hardy-Lorentz space H_{pq} into the Lorentz space L_{pq} for every $p > \frac{1}{2}$ and $0 < q \leqslant \infty$ provided that the supremum in the maximal operator is taken over special indices. As a consequence, we obtain the a.e. convergence of Marcinkiewicz-Fejér means of double Fourier series for special indices with respect to the Walsh-Kaczmarz system. That is, $\sigma_{2^n}(f, x^1, x^2) \to f(x^1, x^2)$ a.e. as $n \to \infty$.

2. Definitions and notation

Let $K := [0,1)$ denote the unique interval in $\mathbb{R}$. By a dyadic interval in K we mean one of the form $[l/2^k, (l+1)/2^k)$ for some $k, l \in \mathbb{N}$ ($\mathbb{N} := \{0, 1, ...\}$). For a $K \ni x = \sum_{i=0}^{\infty} x_i / 2^{i+1}$ the sets $I_n(x_0, ..., x_{n-1}) := \{y \in K : y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$ are the dyadic intervals of length 2^{-n}. Let $I_n := I_n(0, ..., 0)$. The σ-algebra generated by the dyadic 2-dimensional cubes I_k^2 of length $2^{-k} \times 2^{-k}$ will be denoted by F_k $(k \in \mathbb{N})$. Let $+$ denote the dyadic or so called logical addition [7]. Let $L_p(K)$ denote the usual Lebesgue spaces on K with the corresponding norm $\|.\|_p$ (and the elements of L_p are bounded 1-periodic functions).

The Lorentz space $L_{pq}(K^2)$, $0 < p, q \leqslant \infty$ with norms or quasi-norms $\|\cdot\|_{pq}$ is defined in the usual way (For details see e.g. Weisz [14]).

Denote by $f = (f_n, n \in \mathbb{N})$ a one-parameter martingale with respect to $(F_n, n \in \mathbb{N})$. The maximal function of the martingale f is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f_n|.$$

For $0 < p, q \leqslant \infty$ the Hardy-Lorentz martingale space $H_{p,q}(K^2)$ consists of all martingales for which

$$\|f\|_{H_{p,q}} = \|f^*\|_{p,q} < \infty.$$

A bounded measurable function a is a p-atom, if there exists a dyadic 2-dimensional cube I^2, such that

$a)$ $\int_{I^2} a d\mu = 0$;

$b)$ $\|a\|_\infty \leqslant \mu(I^2)^{-1/p}$;

$c)$ supp $a \subset I^2$.

An operator T which maps the set of martingale into the collection of measurable functions will be called p-quasi-local if there exists a constant $C_p > 0$ such that for evey p-atom a

$$\int_{K^2 \setminus I^2} |Ta|^p \leqslant C_p < \infty,$$

where I^2 is the support of the atom.

The Rademacher functions are defined by

$$r_n(x) := r_0(2^n x), \quad n \geqslant 1 \text{ and } x \in K, \text{ where } r_0(x) := \begin{cases} 1 & \text{if } x \in [0,1/2), \\ -1 & \text{if } x \in [1/2,1), \end{cases}$$

and $r_0(x+1) := r_0(x)$. Each natural number n can be uniquely expressed as $n = \sum_{i=0}^{\infty} n_i 2^i$, $n_i \in \{0,1\}$ $(i \in \mathbb{N})$, where only a finite number of n_i's are different from zero. Let the order of $1 \leqslant n$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$.

The Walsh-Paley functions are defined by

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k}.$$

The Walsh-Kaczmarz functions are defined by $\kappa_0 := 1$ and for $n \geqslant 1$

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k}.$$

Each $x \in K = [0,1)$ can be expressed as $x = \sum_{j=0}^{\infty} x_j 2^{-n-1}$, where $x_j \in \{0,1\}$ $(j \in \mathbb{N})$. This expression is unique if x is not a dyadic rational. In other words, if x is not of the form $j/2^n$, where j,n are nonnegative integers. If x is a dyadic rational, then we choose the expansion which terminates in zeros. In this way we have the unicity of this expression for all x. Later we need the notation $e_s := \frac{1}{2^s}$, $x_s e_s = \frac{x_s}{2^s}$.

For $A \in \mathbb{N}$ define the transformation $\tau_A : K \to K$ by

$$\tau_A(x) := \frac{x_{A-1}}{2^1} + \frac{x_{A-2}}{2^2} + \cdots + \frac{x_0}{2^{A-1}} + \sum_{j=A}^{\infty} \frac{x_j}{2^{j+1}}.$$

In other words, if the coordinates of x are $x_0, x_1, \ldots, x_{A-1}, x_A, \ldots$, then the coordinates of $\tau_A(x)$ are $x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, \ldots$. By the definition of τ_A (see [10]), we have

$$\kappa_n(x) = r_{|n|}(x) w_n(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in [0,1)).$$

The Dirichlet kernels are defined by

$$D_n^{\alpha}(x) := \sum_{k=0}^{n-1} \alpha_k(x),$$

where $\alpha_k = w_k$ or κ_k. Recall that

$$D_{2^n}(x) := D_{2^n}^{w}(x) = D_{2^n}^{\kappa}(x) = \begin{cases} 2^n, & \text{if } x \in [0,1/2^n), \\ 0, & \text{if } x \in [1/2^n,1). \end{cases}$$

The Fourier coefficients (if f is an integrable function), the partial sums of Fourier series, the Fejér means and the Fejér kernels are defined as follows:

$$\hat{f^{\alpha}}(n) := \int_K f\,\alpha_n, \quad S_n^{\alpha}(f,x) := \sum_{k=0}^{n-1} \hat{f^{\alpha}}(k)\alpha_k(x)$$

$$t_n^{\alpha}(f,x) := \frac{1}{n}\sum_{k=0}^{n} S_k^{\alpha}(f,x), \quad K_n^{\alpha}(x) := \frac{1}{n}\sum_{k=0}^{n} D_k^{\alpha}(x),$$

where $\alpha_n = w_n$ or κ_n. The 2-dimensional Dirichlet kernels and Marcinkiewicz-Fejér kernels are defined by

$$D_{k,l}^{\alpha}(x^1,x^2) := D_k^{\alpha}(x^1)D_l^{\alpha}(x^2), \quad K_n^{\alpha}(x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n} D_{k,k}^{\alpha}(x^1,x^2).$$

The Marcinkiewicz means of the two dimensional function f is

$$\sigma_n^{\alpha}(f,x^1,x^2) := \frac{1}{n}\sum_{k=0}^{n} S_{k,k}^{\alpha}(f,x^1,x^2).$$

If f is a martingale, that is, $f = (f_0, f_1, \dots)$, then the Fourier coefficients must be defined in a little bit different way:

$$\hat{f^{\alpha}}(n,m) := \lim_{k\to\infty} \int_K f_k \alpha_n \alpha_m,$$

For f we consider the maximal operator

$$\sigma^{\#}f(x^1,x^2) = \sup_A |\sigma_{2^A}^k(f,x^1,x^2)|.$$

3. Formulation of main results

THEOREM 1. *Let* $f \in H_{p,q}(K^2)$, $p > \frac{1}{2}$, $0 < q \leqslant \infty$. *Then*

$$\left\|\sigma^{\#}f\right\|_{p,q} \leqslant C(p,q)\,\|f\|_{H_{p,q}}.$$

COROLLARY 1. *Let* $f \in L_1(K^2)$. *Then*

$$\left\|\sigma^{\#}f\right\|_{weak\_L_1} \leqslant C\,\|f\|_{L_1}.$$

COROLLARY 2. *Let* $f \in L_1(K^2)$. *Then*

$$\sigma_{2^n}(f,x^1,x^2) \to f(x^1,x^2) \text{ a.e. as } \quad n \to \infty.$$

4. Auxiliary propositions

We shall need the following lemmas (see [13, 2, 5]).

LEMMA 1. *(Weisz) Suppose that the operator* T *is sublinear and* p*-quasi-local for each* $0 < p_0 < p \leqslant 1$*. If* T *is bounded from* $L_\infty(K^2)$ *to* $L_\infty(K^2)$*, then*

$$\|Tf\|_{pq} \leqslant C(p,q)\|f\|_{pq} \quad (f \in H_{pq}(K^2))$$

for every $0 < p_0 < p < \infty$ *and* $0 < q \leqslant \infty$*. In particular, for* $f \in L_1(K^2)$*, it holds*

$$\|Tf\|_{1,\infty} = \|Tf\|_{weak\_L_1(K^d)} \leqslant C\|f\|_1 .$$

LEMMA 2. *(Gát) Let* $A, s \in \mathbb{N}$*,* $A > s$*. Suppose that* $x \in I_s \backslash I_{s+1}$*. Then for the one dimensional Fejér kernel*

$$K_{2^A}^w(x) = \begin{cases} 0, & \text{if} \quad x - e_s x_s \notin I_A \\ 2^{s-1} & \text{if} \quad x - e_s x_s \in I_A. \end{cases}$$

LEMMA 3. *(Nagy) Let* $A, s, l \in \mathbb{N}, s \leqslant l < A, \left(x^1, x^2\right) \in (I_s \backslash I_{s+1}) \times (I_l \backslash I_{l+1})$*. Then*

$$K_{2^A}^w\left(x^1, x^2\right) = \begin{cases} 0 & \text{if} \quad \exists i \in B_1, x_i^1 \neq x_i^2, \\ 0 & \text{if} \quad \forall i \in B_1, x_i^1 = x_i^2, \exists m \in B_2, x^1 - e_s - e_m \notin I_{l+1}, x_m^1 = 1, \\ 2^{s+m-2} & \text{if} \quad \forall i \in B_1, x_i^1 = x_i^2, \exists m \in B_2, x^1 - e_s - e_m \in I_{l+1}, x_m^1 = 1, \\ 2^{2s-1} & \text{if} \quad x^1 - e_s \in I_{l+1} \left(\forall i \in B_1, x_i^1 = x_i^2\right), \end{cases}$$

where $B_1 = \{l+1, ..., A-1\}$*,* $B_2 = \{s+1, ..., l\}$.

LEMMA 4. *(Nagy) Let* $A, s, l \in \mathbb{N}, s \leqslant l < A, \left(x^1, x^2\right) \in I_A \times (I_t \backslash I_{t+1})$ *and* $t < t + l < A$*. Then*

$$K_{2^A}^w\left(x^1, x^2\right) = \begin{cases} 0 & \text{if} \quad \exists l, t < t+l < A, x^2 - x_t^2 e_t - e_{t+l} \notin I_A, x_{t+l}^2 \neq 0, \\ 2^{2t+l-2} & \text{if} \quad \exists l, t < t+l < A, x^2 - x_t^2 e_t - e_{t+l} \in I_A, x_{t+l}^2 \notin 0, \\ 2^{t-2} n(A,t) & \text{if} \quad x^2 - x_t^2 e_t \in I_A, \end{cases}$$

where $n(A,t) = \left[-2^{t-A}\left(2^A - 2^{t-1} + 1/2\right) - \left(2^A - 2\right)\right]$.

LEMMA 5. *(Nagy) Let* $A \in \mathbb{N}$*,* $\left(x^1, x^2\right) \in G \times G$*. Then*

$$2^A K_{2^A}^\kappa\left(x^1, x^2\right) = 1 + \sum_{j=0}^{A-1} 2^j D_{2^j, 2^j}\left(x^1, x^2\right) + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^1\right) r_j\left(x^2\right) K_{2^j}^w\left(\tau_j\left(x^2\right)\right)$$
$$+ \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^2\right) r_j\left(x^1\right) K_{2^j}^w\left(\tau_j\left(x^1\right)\right) + \sum_{j=0}^{A-1} 2^j r_j\left(x^1 + x^2\right) K_{2^j}^w\left(\tau_j\left(x^1\right), \tau_j\left(x^2\right)\right).$$

COROLLARY 3.
$$\sup_A \int_{K^2} K_{2^A}^{\kappa}\left(x^1,x^2\right) dx^1 dx^2 < \infty.$$

Proof of Corollary 3. Since $\sup_A \int_K K_{2^A}^{w}(x)\,dx < \infty$ and $\sup_A \int_{K^2} K_{2^A}^{w}\left(x^1,x^2\right) dx^1 dx^2 < \infty$ we obtain the proof of Corollary 3 from Lemma 5.

5. Proofs of the main results

Proof of Theorem 1. By Lemma 1, the proof of Theorem 1 will be complete if we show that the operator $\sigma^{\#}$ is p-quasi-local for each $1/2 < p \leqslant 1$ and bounded from $L_\infty(K^2)$ to $L_\infty(K^2)$.

The boundedness follows from Corollary 3.

Let a be an arbitrary atom with support $R = I \times J$ and $\mu(I) = \mu(J) = 2^{-N}$. We may assume that $I = J = I_N$. It is easy to see that $\sigma_{2^A}(a) = 0$ if $A \leqslant N$. Therefore, we can suppose that $A > N$.

Using Lemma 5 and the fact that
$$D_{2^n}^{w}(x) = \begin{cases} 2^n, x \in I_n, \\ 0, x \notin I_n, \end{cases} \tag{1}$$
for $\left(x^1,x^2\right) \in K^2 \backslash (I_N \times I_N)$ we write
$$\begin{aligned}
\sigma_{2^A} a\left(x^1,x^2\right) &= \frac{1}{2^A}\int_{I_N\times I_N} a\left(t^1,t^2\right)\Bigg(1+\sum_{j=0}^{A-1} 2^j D_{2^j,2^j}\left(x^1+t^1,x^2+t^2\right) \\
&\quad + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^1+t^1\right) r_j\left(x^2+t^2\right) K_{2^j}^{w}\left(\tau_j\left(x^2+t^2\right)\right) \\
&\quad + \sum_{j=0}^{A-1} 2^j D_{2^j}\left(x^2+t^2\right) r_j\left(x^1+t^1\right) K_{2^j}^{w}\left(\tau_j\left(x^1+t^1\right)\right) \\
&\quad + \sum_{j=0}^{A-1} 2^j r_j\left(x^1+t^1+x^2+t^2\right) K_{2^j}^{w}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\Bigg) dt^1 dt^2 \\
&= \frac{1}{2^A}\int_{I_N\times I_N} a\left(t^1,t^2\right)\sum_{j=N+1}^{A-1} 2^j D_{2^j}\left(x^1+t^1\right) r_j\left(x^2+t^2\right) K_{2^j}^{w}\left(\tau_j\left(x^2+t^2\right)\right) dt^1 dt^2 \\
&\quad + \frac{1}{2^A}\int_{I_N\times I_N} a\left(t^1,t^2\right)\sum_{j=N+1}^{A-1} 2^j D_{2^j}\left(x^2+t^2\right) r_j\left(x^1+t^1\right) K_{2^j}^{w}\left(\tau_j\left(x^1+t^1\right)\right) dt^1 dt^2 \times \\
&\quad \times \frac{1}{2^A}\int_{I_N\times I_N} a\left(t^1,t^2\right)\sum_{j=N+1}^{A-1} 2^j r_j\left(x^1+t^1+x^2+t^2\right) K_{2^j}^{w}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right) dt^1 dt^2 \\
&= \sigma_{2^A}^{(1)} a\left(x^1,x^2\right) + \sigma_{2^A}^{(2)} a\left(x^1,x^2\right) + \sigma_{2^A}^{(3)} a\left(x^1,x^2\right).
\end{aligned} \tag{2}$$

Step 1. Integrating over $(K\backslash I_N)\times(K\backslash I_N)$. Using (1) and the fact that $|a|\leqslant c2^{2N/p}$ we have

$$\sigma_{2^A}^{(1)}a\left(x^1,x^2\right)=0, \tag{3}$$

$$\sigma_{2^A}^{(2)}a\left(x^1,x^2\right)=0, \tag{4}$$

$$\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|\leqslant c\frac{2^{2N/p}}{2^A}\sum_{j=N+1}^{A-1}2^j\int\limits_{I_N\times I_N}\left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|dt^1dt^2. \tag{5}$$

By Lemma 3 we see that $K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\neq 0$ implies that one of the four cases below must hold.

1)
$$x^1\in I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right),$$
$$x^2\in I_N\left(x_0^1,...,x_{s-1}^1,0,...,0,x_m^2=1,0,...,x_l^2=1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{j-1}^2,t_j^2,...\right);$$

2)
$$x^1\in I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right),$$
$$x^2\in I_N\left(x_0^1,...,x_{s-1}^1,0,...,0,x_m^2=1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{l-1}^2,1-x_l^2,x_{l+1}^2,...,x_{j-1}^2,t_j^2,...\right);$$

3)
$$x^1\in I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right),$$
$$x^2\in I_N\left(x_0^1,...,x_{s-1}^1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{m-1}^2,1-x_m^2,x_{m+1}^2,...,x_{l-1}^2,1-x_l^2,x_{l+1}^2,...,x_{j-1}^2,t_j^2,...\right);$$

4)
$$x^1\in I_N\left(x_0^1,...,x_{N-1}^1\right),$$
$$t^1=\left(0,...,0,t_N^1,...,t_{s-1}^1,1-x_s^1,x_{s+1},...,x_{j-1}^1,t_j^1,...\right),$$
$$x^2\in I_N\left(x_0^1,...,x_{N-1}^1\right),$$
$$t^2=\left(0,...,0,x_N^1+x_N^2+t_N^1,...,x_{s-1}^1+x_{s-1}^2+t_{s-1}^1,x_s^2,...,x_{m-1}^2,\right.$$
$$\left.1-x_m^2,x_{m+1}^2,...,x_{l-1}^2,1-x_l^2,x_{l+1}^2,...,x_{j-1}^2,t_j^2,...\right).$$

First, we consider the case 1). From Lemma 3 it is clear that

$$\int\limits_{I_N\times I_N}\left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right|dt^1dt^2$$
$$\leqslant c\frac{2^{j-l+j-m}}{2^{2j}}\mathbf{1}_{I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right)\times I_N\left(x_0^1,...,x_{s-1}^1,0,...,0,x_m^2=1,0,...,x_l^2=1,0,...,0\right)}\left(x^1,x^2\right) \tag{6}$$
$$\leqslant c2^{-l-m}\mathbf{1}_{I_N\left(x_0^1,...,x_{s-1}^1,x_s^1=1,0,...,0\right)\times I_N\left(x_0^1,...,x_{s-1}^1,0,...,0,x_m^2=1,0,...,x_l^2=1,0,...,0\right)}\left(x^1,x^2\right),$$

Next, we consider the case 2). We have

$$\int\limits_{I_N\times I_N} \left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2$$

$$\leqslant \frac{c}{2^{2j}}\sum_{l=N}^{j}2^{j-l+j-m}\mathbf{1}_{I_N\left(x^1_0,\ldots,x^1_{s-1},x^1_s=1,0,\ldots,0\right)\times I_N\left(x^1_0,\ldots,x^1_{s-1},0,\ldots,0,x^2_m=1,0,\ldots,0\right)}\left(x^1,x^2\right) \tag{7}$$

$$\leqslant c2^{-N-m}\mathbf{1}_{I_N\left(x^1_0,\ldots,x^1_{s-1},x^1_s=1,0,\ldots,0\right)\times I_N\left(x^1_0,\ldots,x^1_{s-1},0,\ldots,0,x^2_m=1,0,\ldots,0\right)}\left(x^1,x^2\right).$$

Now, we consider the case 3). We have

$$\int\limits_{I_N\times I_N} \left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2$$

$$\leqslant \frac{c}{2^{2j}}\sum_{m=N}^{j}\sum_{l=m}^{j}2^{j-l+j-m}\mathbf{1}_{I_N\left(x^1_0,\ldots,x^1_{s-1},x^1_s=1,0,\ldots,0\right)\times I_N\left(x^1_0,\ldots,x^1_{s-1},0,\ldots,0\right)}\left(x^1,x^2\right) \tag{8}$$

$$\leqslant c2^{-2N}\mathbf{1}_{I_N\left(x^1_0,\ldots,x^1_{s-1},x^1_s=1,0,\ldots,0\right)\times I_N\left(x^1_0,\ldots,x^1_{s-1},0,\ldots,0\right)}\left(x^1,x^2\right).$$

Finally, we consider the case 4). We write

$$\int\limits_{I_N\times I_N} \left|K^w_{2^j}\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2$$

$$\leqslant \frac{c}{2^{2j}}\sum_{s=N}^{j}\sum_{l=s}^{j}\sum_{m=s}^{l}2^{j-l+j-m}2^{s-N}\mathbb{1}_{I_N\left(x^1_0,\ldots,x^1_{N-1}\right)\times I_N\left(x^1_0,\ldots,x^1_{N-1}\right)}\left(x^1,x^2\right) \tag{9}$$

$$\leqslant c2^{-2N}\mathbb{1}_{I_N\left(x^1_0,\ldots,x^1_{N-1}\right)\times I_N\left(x^1_0,\ldots,x^1_{N-1}\right)}\left(x^1,x^2\right).$$

From (5)-(9) we obtain

$$\int\limits_{(K\backslash I_N)\times(K\backslash I_N)} \sup_{A\geqslant N}\left|\sigma^{(3)}_{2^A}a\left(x^1,x^2\right)\right|^p dx^1dx^2$$

$$\leqslant c2^{2N}\sum_{s=1}^{N}\sum_{l=s}^{N}\sum_{m=s}^{l}\frac{1}{2^{(m+l)p}}\frac{1}{2^{2N}}2^s\leqslant c\sum_{s=1}^{N}\frac{1}{2^{(2p-1)s}}<\infty \text{ for } 1/2<p\leqslant 1. \tag{10}$$

Combining (3), (4) and (10) we obtain that $(1/2<p\leqslant 1)$

$$\int\limits_{(K\backslash I_N)\times(K\backslash I_N)} \left(\sigma^{\#}a\left(x^1,x^2\right)\right)^p dx^1dx^2\leqslant c_p<\infty. \tag{11}$$

Step 2. Integrating over $I_N\times(K\backslash I_N)$. Then, we can write

$$\sigma_{2^A}a\left(x^1,x^2\right)=\sigma^{(1)}_{2^A}a\left(x^1,x^2\right)+\sigma^{(3)}_{2^A}a\left(x^1,x^2\right) \tag{12}$$

From (1) we have

$$\left|\sigma_{2^A}^{(1)} a\left(x^1, x^2\right)\right| \leqslant c \frac{2^{2N/p}}{2^A} \sum_{j=N+1}^{A-1} 2^j \int_{I_N \times I_N} D_{2^j}\left(x^1+t^1\right) K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) dt^1 dt^2$$
$$\leqslant c \frac{2^{2N/p}}{2^A} \sum_{j=N+1}^{A-1} 2^j \int_{I_N} K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) dt^2. \tag{13}$$

Using Lemma 2 we conclude that $K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) \neq 0$ implies that $x^2 \in I_N\left(0,...,0,x_s^2=1,0,...,0\right)$ for some $s=1,...,N$ and $t=\left(0,...,0,x_N^2,...,x_{j-1}^2,t_j^2,...\right)$.

Hence,

$$\int_{I_N} K_{2^j}^w\left(\tau_j\left(x^2+t^2\right)\right) dt^2 \leqslant c \frac{2^{j-s}}{2^j} \mathbf{1}_{I_N\left(0,...,0,x_s^2=1,0,...,0\right)}\left(x^2\right) \leqslant \frac{c}{2^s} \mathbf{1}_{I_N\left(0,...,0,x_s^2=1,0,...,0\right)}\left(x^2\right),$$

and consequently,

$$\int_{I_N \times (I \backslash I_N)} \sup_{A>N}\left|\sigma_{2^A}^{(1)} a\left(x^1, x^2\right)\right|^p dx^1 dx^2 \leqslant c_p 2^{2N} \sum_{s=1}^{N} \frac{1}{2^{sp}} \frac{1}{2^{2N}} \leqslant c_p < \infty. \tag{14}$$

Using Lemmas 3–4 it is clear that $K_{2^j}^w\left(\tau_j\left(x^1+t^1\right), \tau_j\left(x^2+t^2\right)\right) \neq 0$ implies that:

5)
$$x^1 \in I_N(0),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{m-1}^1,1-x_m^1,x_{m+1}^1,...,x_{s-1}^1,1-x_s^1,x_{s+1}^1,...,x_{j-1}^1,t_j^1,...,\right),$$
$$x^2 \in I_N\left(0,...,0,x_l^2=1,0...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{j-1}^2,t_j^2,...\right);$$

6)
$$x^1 \in I_N(0),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right)$$
$$x^2 \in I_N\left(0,...,0,x_l^2=1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{m-1}^1,1-x_m^1,x_{m+1}^1,...,x_{j-1}^2,t_j^2,...\right);$$

7)
$$x^1 \in I_N(0),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right)$$
$$x^2 \in I_N\left(0,...,0,x_l^2=1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,,x_{j-1}^2,t_j^2,...\right);$$

8)
$$x^1 \in I_N(0),$$
$$t^1=\left(0,...,0,x_N^1,...,x_{j-1}^1,t_j^1,...\right)$$
$$x^2 \in I_N\left(0,...,0,x_l^2=1,0,...,0,x_s=1,0,...,0\right),$$
$$t^2=\left(0,...,0,x_N^2,...,x_{j-1}^2,t_j^2,...\right);$$

Consider the case 5). As above we get that

$$\int\limits_{I_N\times I_N} \left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2 \leqslant c2^{-2N}\mathbf{1}_{I_N(0)\times I_N\left(0,\dots,0,x_l^2=1,0\dots,0\right)}\left(x^1,x^2\right). \tag{15}$$

Using Lemma 4 for case 6) we obtain

$$\begin{aligned}\int\limits_{I_N\times I_N} &\left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2\\ &\leqslant \frac{c}{2^{2j}}\sum_{m=N}^{j}2^{j-m+j-l}\mathbf{1}_{I_N(0)\times I_N\left(0,\dots,0,x_l^2=1,0\dots,0\right)}\left(x^1,x^2\right)\\ &\leqslant c2^{-N-l}\mathbf{1}_{I_N(0)\times I_N\left(0,\dots,0,x_l^2=1,0\dots,0\right)}\left(x^1,x^2\right).\end{aligned} \tag{16}$$

The estimation of cases 7) and 8) is analogous to the estimation of cases 5) and 6) and we have

$$\int\limits_{I_N\times I_N} \left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2 \leqslant c2^{-l}\mathbf{1}_{I_N(0)\times I_N\left(0,\dots,0,x_l^2=1,0\dots,0\right)}\left(x^1,x^2\right), \tag{17}$$

$$\int\limits_{I_N\times I_N} \left|K_{2^j}^w\left(\tau_j\left(x^1+t^1\right),\tau_j\left(x^2+t^2\right)\right)\right| dt^1dt^2 \leqslant c2^{-l-s}\mathbf{1}_{I_N(0)\times I_N\left(0,\dots,0,x_l^2=1,0,\dots,0,x_s=1,0,\dots,0\right)}\left(x^1,x^2\right). \tag{18}$$

By $(11)-(18)$ we have

$$\begin{aligned}\int\limits_{I_N\times(K\backslash I_N)} &\sup_{A>N}\left|\sigma_{2^A}^{(3)}a\left(x^1,x^2\right)\right|^p dx^1dx^2\\ &\leqslant c_p2^{2N}\left\{\frac{1}{2^{2Np}}\sum_{l=1}^{N}\frac{1}{2^{2N}}+\sum_{l=1}^{N}\frac{1}{2^{pl}}\frac{1}{2^{Np}}\frac{1}{2^{2N}}+\sum_{l=1}^{N}\frac{1}{2^{pl}}\frac{1}{2^{2N}}+\sum_{l=1}^{N}\sum_{s=1}^{N}\frac{1}{2^{p(l+s)}}\frac{1}{2^{2N}}\right\}\\ &\leqslant c_p<\infty.\end{aligned} \tag{19}$$

Combining (12), (14) and (19) we obtain that

$$\int\limits_{I_N\times(K\backslash I_N)}\left(\sigma^{\#}a\left(x^1,x^2\right)\right)^p dx^1dx^2\leqslant c_p<\infty.$$

Step 3. Integrating over $(K\backslash I_N)\times I_N$.
The case is analogous to step 2. □

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