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GENERALIZATION OF THE KANTOROVICH TYPE OPERATOR INEQUALITIES VIA GRAND FURUTA INEQUALITY

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(*communicated by N Elezović*)

Abstract. In this note we show the characterization of the δ-order by means of a generalized Kantorovich constant via Grand Furuta inequality, which is an extension result of that from M. Fujii, E. Kamei and Y. Seo, *Kantorovich type operator inequalities via grand Furuta inequality*, Sci. Math., **3**, (2000), 263–272. Among other, we show the following characterization of the δ-order: Let A, B be positive invertible operators on a Hilbert space H satisfying $MI \geqslant A \geqslant mI > 0$ and $NI \geqslant B \geqslant nI > 0$. Then the following statements are mutually equivalent for each $\delta \in [0,1]$:

(i) $A^{\delta} \geqslant B^{\delta}$,

(ii) $K(n^r, N^r, 1+\frac{p-\delta}{r}, 1+\frac{q-\delta}{r})A^q \geqslant B^p$ for all $p > \delta, q > \delta$ and $r > \delta$,

(iii) $\overline{K}(m^r, M^r, 1+\frac{q-\delta}{r}, 1+\frac{p-\delta}{r})A^q \geqslant B^p$ for all $p > \delta, q > \delta$ and $r > \delta$,

where the case $\delta = 0$ means the chaotic order $\log A \geqslant \log B$.

1. Introduction

Let $\mathscr{B}(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H and $\mathscr{B}_{++}(H)$ be the set of all positive invertible operators of $\mathscr{B}(H)$. We denote by $\mathrm{Sp}(A)$ the spectrum of the operator A.

The following Theorem F is an ingenious extension of the celebrated Löwner-Heinz theorem: $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for any $0 \leqslant p \leqslant 1$.

THEOREM F (FURUTA INEQUALITY)
If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$

(i) $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$

and

(ii) $\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

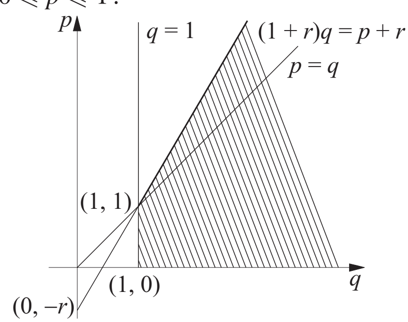

Figure: The range of Furuta inequality

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Furuta [5] established the following Theorem G as an extension of Theorem F.

THEOREM G (THE GRAND FURUTA INEQUALITY). *If* $A \geqslant B \geqslant 0$ *and* A *is invertible, then for each* $t \in [0,1]$,

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}A^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1}{q}} \geqslant \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for any $s \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ *and* $r \geqslant t$ *with* $(s-1)(p-1) \geqslant 0$ *and* $(1-t+r)q \geqslant (p-t)s+r$.

Related to the Löwner-Heinz theorem, the following proposition is also well known: $A \geqslant B \geqslant 0$ does not always assure $A^p \geqslant B^p$ for any $p > 1$. Associated with this result, Furuta [6] showed the following Kantorovich type operator inequality.

THEOREM A *If* $A \geqslant B \geqslant 0$ *with* $\mathrm{Sp}(A) \subseteq [m,M]$ *and* $\mathrm{Sp}(B) \subseteq [n,N]$ *for some scalars* $0 < m < M$ *and* $0 < n < N$, *then*

$$\left(\frac{N}{n}\right)^{p-1} A^p \geqslant K(n,N,p)A^p \geqslant B^p \qquad \textit{holds for all } p \geqslant 1$$

and

$$\left(\frac{M}{m}\right)^{p-1} A^p \geqslant K(m,M,p)A^p \geqslant B^p \qquad \textit{holds for all } p \geqslant 1,$$

where a generalized Kantorovich constant $K(m,M,p)$ *is defined as*

$$K(m,M,p) = \frac{mM^p - Mm^p}{(p-1)(M-m)}\left(\frac{p-1}{p}\frac{M^p-m^p}{mM^p-Mm^p}\right)^p \textit{ for all real number } p \in \mathbb{R}. \tag{1.1}$$

Especially $K(m,M,p)$ for $p > 1$ can be usually written by

$$K(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}} \text{ for all } p > 1.$$

The order between operators $A, B \in \mathscr{B}_{++}(H)$ defined by $\log A \geqslant \log B$ is said to be chaotic order $A \gg B$. We consider the class of orders $A^\delta \geqslant B^\delta$ for $\delta \in [0,1]$, where the case $\delta = 0$ means the chaotic order. The following lemma shows that the Furuta inequality interpolates the usual order and the chaotic one [3, Lemma 1].

LEMMA B. *Let* $A, B \in \mathscr{B}_{++}(H)$. *Then the following statements are mutually equivalent for each* $\delta \in [0,1]$:

(*i*) $A^\delta \geqslant B^\delta$, *where the case* $\delta = 0$ *means* $A \gg B$.

(*ii*) $A^{p+\delta} \geqslant \left(A^{\frac{p}{2}}B^{p+\delta}A^{\frac{p}{2}}\right)^{\frac{p+\delta}{2p+\delta}}$ *for all* $p \geqslant 0$.

(*iii*) $A^{u+\delta} \geqslant \left(A^{\frac{u}{2}}B^{p+\delta}A^{\frac{u}{2}}\right)^{\frac{u+\delta}{p+u+\delta}}$ *for all* $p \geqslant 0$ *and* $u \geqslant 0$.

As applications of Lemma B and the grand Furuta inequality, Fujii et al. gave in [3, Theorem 2] the following:

THEOREM C. *Let* $A, B \in \mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some scalars* $M > m > 0$. *Then the following statements are mutually equivalent for each* $\delta \in (0, 1]$:

(i) $A^{\delta} \geqslant B^{\delta}$.

(ii) *For each* $n \in \mathbb{N}$ *and* $\alpha \in [0, 1]$

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1)A^{(p-\delta+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-\delta}{2}} B^{p} A^{\frac{\alpha u-\delta}{2}}\right)^{s}$$

holds for $s \geqslant 1$, $p \geqslant \delta$ *and* $u \geqslant \delta$ *with* $(p-\delta+\alpha u)s \geqslant (n+\alpha)u$.

(iii) *For each* $n \in \mathbb{N}$

$$K(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n+1)^{\frac{1}{s}} A^{p} \geqslant B^{p}$$

holds for $s \geqslant 1$ *and* $p \geqslant \delta$ *with* $(p-\delta)s \geqslant n\delta$.

(iv) $(\frac{M}{m})^{p-\delta} A^{p} \geqslant B^{p}$ *holds for* $p \geqslant \delta$.

In this note we show the characterization of the δ-order by means of a generalized Kantorovich constant via Grand Furuta inequality, which is simultaneous extension results given in [3] and [10].

2. Results

The following theorem is our key theorem which is a two variable version of Theorem A.

THEOREM 2.1. *Let* $A, B \in \mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A) \subseteq [m, M]$ *and* $\mathrm{Sp}(B) \subseteq [n, N]$ *for some scalars* $0 < m < M$ *and* $0 < n < N$. *If* $A \geqslant B \geqslant 0$, *then*

$$\frac{N^{p-1}}{n^{q-1}} A^{q} \geqslant K(n, N, p, q) A^{q} \geqslant B^{p} \qquad \text{for all } p > 1 \text{ and } q > 1 \tag{2.1}$$

and

$$\frac{M^{q-1}}{m^{p-1}} A^{p} \geqslant \overline{K}(m, M, p, q) A^{p} \geqslant B^{q} \qquad \text{for all } p > 1 \text{ and } q > 1, \tag{2.2}$$

where

$$K(n,N,p,q) = \begin{cases} \dfrac{nN^{p}-Nn^{p}}{(q-1)(N-n)}\left(\dfrac{q-1}{q}\dfrac{N^{p}-n^{p}}{nN^{p}-Nn^{p}}\right)^{q} & \text{if } qn^{p-1} \leqslant \dfrac{N^{p}-n^{p}}{N-n} \leqslant qN^{p-1}, \\ n^{p-q} & \text{if } \dfrac{N^{p}-n^{p}}{N-n} < qn^{p-1}, \\ N^{p-q} & \text{if } qN^{p-1} < \dfrac{N^{p}-n^{p}}{N-n}, \end{cases} \tag{2.3}$$

and

$$\overline{K}(m,M,p,q)=\begin{cases} \dfrac{(mM^p-Mm^p)M^{q-p}m^{q-p}}{(q-1)(M-m)}\left(\dfrac{q-1}{q}\dfrac{M^p-m^p}{mM^p-Mm^p}\right)^q & \\ \qquad\qquad\qquad\qquad \text{if } qm^{p-1}\leqslant \dfrac{M^p-m^p}{M-m}\leqslant qM^{p-1}, \\ M^{q-p} & \text{if } \dfrac{M^p-m^p}{M-m}<qm^{p-1}, \\ m^{q-p} & \text{if } qM^{p-1}<\dfrac{M^p-m^p}{M-m}. \end{cases} \tag{2.4}$$

REMARK 1. We have the following relations between $\overline{K}$ and K:

(1) For $0<m<M$

 (i) $\overline{K}(m,M,p,q)=(mM)^{q-p}K(m,M,p,q)$,

 (ii) $K(M^{-1},m^{-1},p,q)=(mM)^{q-p}K(m,M,p,q)$.

(2) If we put $p=q$ in (i), then we have $\overline{K}(m,M,p,p)=K(m,M,p,p)=K(m,M,p)$.

By virtue of Lemma B we show the following Kantorovich type characterization of the δ-order by means of a generalized Kantorovich constant via Grand Furuta inequality, which is simultaneous extension both of results due to Fujii-Kamei-Seo [3] and Mićić-Pečarić-Seo [10].

THEOREM 2.2. *Let $A,B\in\mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A)\subseteq[m,M]$ for some scalars $M>m>0$, and let $\overline{K}(m,M,p,q)$ be defined in (2.4). Then the following statements are mutually equivalent for each $\delta\in(0,1]$:*

(i) $A^\delta\geqslant B^\delta$.

$(ii-1)$ *For each $n>0$ and $\alpha\in[0,1]$*

$$\overline{K}\left(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}},M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}},n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1,n+1\right)A^{(q-\delta+\alpha u)s} \geqslant\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s\geqslant1$, $p\geqslant\delta$, $q\geqslant\delta$ and $u\geqslant\delta$ with $(p-\delta+\alpha u)s\geqslant(\alpha+n)u$.

$(ii-2)$ *For each $n>0$ and $\alpha\in[0,1]$*

$$\overline{K}\left(m^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}},M^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}},(n+1)\frac{(q-\delta+\alpha u)s+(1-\alpha)u}{(p-\delta+\alpha u)s+(1-\alpha)u},n+1\right)A^{(q-\delta+\alpha u)s} \geqslant\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s\geqslant1$, $p\geqslant\delta$, $q\geqslant\delta$ and $u\geqslant\delta$ with $(p-\delta+\alpha u)s\leqslant(\alpha+n)u$.

$(iii-1)$ *For each $n>0$*

$$\overline{K}\left(m^{\frac{(p-\delta)s}{n}},M^{\frac{(p-\delta)s}{n}},n\frac{q-\delta}{p-\delta}+1,n+1\right)^{\frac{1}{s}}A^q\geqslant B^p$$

holds for $s \geqslant 1$, $p \geqslant \delta$ *and* $q \geqslant \delta$ *with* $(p-\delta)s \geqslant n\delta$.

$(iii-2)$ *For each* $n > 0$

$$\overline{K}(m^{\frac{(p-\delta)s+\delta}{n+1}}, M^{\frac{(p-\delta)s+\delta}{n+1}}, (n+1)\frac{(q-\delta)s+\delta}{(p-\delta)s+\delta}, n+1)^{\frac{1}{s}} A^q \geqslant B^p$$

holds for $s \geqslant 1$, $p \geqslant \delta$ *and* $q \geqslant \delta$ *with* $(p-\delta)s \leqslant n\delta$.

$(iv-1)$ $\frac{M^{p-\delta}}{m^{q-\delta}} A^q \geqslant B^p$ *holds for* $p \geqslant \delta$ *and* $q \geqslant \delta$.

$(iv-2)$ $(\frac{M}{m})^{p-\delta} A^p \geqslant B^p$ *holds for* $p \geqslant \delta$.

Suppose that besides the conditions above (i.e. $A, B \in \mathscr{B}_{++}(H)$ *with* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some scalars* $M > m > 0$*) also a condition* $\mathrm{Sp}(B) \subseteq [m, M]$ *holds. If we replace* $\overline{K}(m, M, q, p)$ *by* $K(m, M, p, q)$ *in* $(ii-1)$, $(ii-2)$, $(iii-1)$ *and* $(iii-2)$, *then the statements* (i) – $(iv-2)$ *are mutually equivalent for each* $\delta \in (0, 1]$.

In particular, if we put $\delta = 1$ in Theorem 2.2, then we obtain the following Kantorovich type characterization of the operator order, which is a two variable generalization of [2, Theorem 3].

THEOREM 2.3. *Let* $A, B \in \mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some scalars* $M > m > 0$, *and let* $\overline{K}(m, M, p, q)$ *be defined in* (2.4). *Then the following statements are mutually equivalent:*

(i) $A \geqslant B$

$(ii-1)$ *For each* $n > 0$ *and* $\alpha \in [0, 1]$

$$\overline{K}(m^{\frac{(p-1+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-1+\alpha u)s-\alpha u}{n}}, n\frac{(q-1+\alpha u)s-\alpha u}{(p-1+\alpha u)s-\alpha u}+1, n+1) A^{(q-1+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-1}{2}} B^p A^{\frac{\alpha u-1}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 1$, $q \geqslant 1$ *and* $u \geqslant 1$ *with* $(p-1+\alpha u)s \geqslant (\alpha+n)u$.

$(ii-2)$ *For each* $n > 0$ *and* $\alpha \in [0, 1]$

$$\overline{K}(m^{\frac{(p-1+\alpha u)s+(1-\alpha)u}{n+1}}, M^{\frac{(p-1+\alpha u)s+(1-\alpha)u}{n+1}}, n\frac{(q-1+\alpha u)s-\alpha u}{(p-1+\alpha u)s-\alpha u}+1, n+1) A^{(q-1+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-1}{2}} B^p A^{\frac{\alpha u-1}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 1$, $q \geqslant 1$ *and* $u \geqslant 1$ *with* $(p-1+\alpha u)s \leqslant (\alpha+n)u$.

$(iii-1)$ *For each* $n > 0$

$$\overline{K}(m^{\frac{(p-1)s}{n}}, M^{\frac{(p-1)s}{n}}, n\frac{q-1}{p-1}+1, n+1)^{\frac{1}{s}} A^q \geqslant B^p$$

holds for $s \geqslant 1$, $p \geqslant 1$ *and* $q \geqslant 1$ *with* $(p-1)s \geqslant n$.

$(iii-2)$ *For each* $n > 0$

$$\overline{K}(m^{\frac{(p-1)s+1}{n+1}}, M^{\frac{(p-1)s+1}{n+1}}, (n+1)\frac{(q-1)s+1}{(p-1)s+1}+1, n+1)^{\frac{1}{s}} A^q \geqslant B^p$$

holds for $s \geqslant 1$, $p \geqslant 1$ *and* $q \geqslant 1$ *with* $(p-1)s \leqslant n$.

$(iv-1)$ $\frac{M^{p-1}}{m^{q-1}}A^q \geqslant B^p$ *holds for* $p \geqslant 1$ *and* $q \geqslant 1$.

$(iv-2)$ $(\frac{M}{m})^{p-1}A^p \geqslant B^p$ *holds for* $p \geqslant 1$.

Suppose that besides the conditions above (i.e. $A,B \in \mathscr{B}_{++}(H)$ *with* $\mathrm{Sp}(A) \subseteq [m,M]$ *for some scalars* $M > m > 0$*) also a condition* $\mathrm{Sp}(B) \subseteq [m,M]$ *holds. If we replace* $\overline{K}(m,M,q,p)$ *by* $K(m,M,p,q)$ *in* $(ii-1)$, $(ii-2)$, $(iii-1)$ *and* $(iii-2)$, *then the statements* (i) – $(iv-2)$ *are mutually equivalent.*

We show the following Kantorovich type characterization of the chaotic order which are parallel to the operator order versions of Theorem 2.3. Moreover, it is a two variable generalization of [9, Theorem 4], cf. [2, Theorem 4].

THEOREM 2.4. *$A,B \in \mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A) \subseteq [m,M]$ for some scalars $M > m > 0$, and let $\overline{K}(m,M,p,q)$ be defined in* (2.4). *Then the following statements are mutually equivalent:*

(i) $A \gg B$ *(i.e.* $\log A \geqslant \log B$*).*

$(ii-1)$ *For each* $n > 0$ *and* $\alpha \in [0,1]$

$$\overline{K}(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n\frac{(q+\alpha u)s-\alpha u}{(p+\alpha u)s-\alpha u}+1, n+1)A^{(q+\alpha u)s} \geqslant \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 0$, $q \geqslant 0$ *and* $u \geqslant 0$ *with* $(p+\alpha u)s \geqslant (\alpha+n)u$.

$(ii-2)$ *For each* $n > 0$ *and* $\alpha \in [0,1]$

$$\overline{K}(m^{\frac{(p+\alpha u)s+(1-\alpha)u}{n+1}}, M^{\frac{(p+\alpha u)s+(1-\alpha)u}{n+1}}, (n+1)\frac{(q+\alpha u)s+(1-\alpha)u}{(p+\alpha u)s+(1-\alpha)u}, n+1)A^{(q+\alpha u)s} \geqslant \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 0$, $q \geqslant 0$ *and* $u \geqslant 0$ *with* $(p+\alpha u)s \leqslant (\alpha+n)u$.

(iii) *For each* $n > 0$

$$\overline{K}(m^{\frac{ps}{n}}, M^{\frac{ps}{n}}, n\frac{q}{p}+1, n+1)^{\frac{1}{s}}A^q \geqslant B^p$$

holds for $s \geqslant 1$, $p \geqslant 0$ *and* $q \geqslant 0$.

(iv) $\overline{S}(h,q,p)A^q \geqslant B^p$ *holds for* $p \geqslant 0$ *and* $q \geqslant 0$,
where $h = \frac{M}{m} > 1$ *and*

$$\overline{S}(h,p,q) = \begin{cases} M^{q-p}\dfrac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h} & \text{if } q \leqslant \dfrac{h^p-1}{\log h} \leqslant qh^p, \\ M^{q-p} & \text{if } \dfrac{h^p-1}{\log h} \leqslant q, \\ m^{q-p} & \text{if } qh^p \leqslant \dfrac{h^p-1}{\log h}. \end{cases}$$

Suppose that besides the conditions above (i.e. $A,B \in \mathscr{B}_{++}(H)$ *with* $\mathrm{Sp}(A) \subseteq [m,M]$ *for some scalars* $M > m > 0$*) also a condition* $\mathrm{Sp}(B) \subseteq [m,M]$ *holds. If we replace* $\overline{K}(m,M,q,p)$ *by* $K(m,M,p,q)$ *in* (ii) *and* (iii), *and* $\overline{S}(h,q,p)$ *by* $S(h,p,q) = M^{q-p}m^{q-p}\overline{S}(h,p,q)$ *in* (iv), *then the statements* (i) – (iv) *are mutually equivalent.*

3. Applications

In this section, as applications of our results in Section 2, we show extensions of results given in [3, 10, 13].

By Theorem 2.2 we show a characterization of the δ-order by means of a generalized Kantorovich constant, which is a two variable generalization of [3, Corollary 4].

THEOREM 3.1. *Let $A, B \in \mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A) \subseteq [m, M]$ and $\mathrm{Sp}(B) \subseteq [n, N]$ for some scalars $0 < m < M$ and $0 < n < N$, and let $\overline{K}(m, M, p, q)$ and $K(m, M, p, q)$ be defined in (2.4) and (2.3), respectively. Then the following statements are mutually equivalent for each $\delta \in (0, 1]$:*

(i) $A^{\delta} \geqslant B^{\delta}$.

(ii) $K(n^r, N^r, 1 + \frac{p-\delta}{r}, 1 + \frac{q-\delta}{r})A^q \geqslant B^p$ *for all* $p > \delta$, $q > \delta$ *and* $r > \delta$.

(iii) $\overline{K}(m^r, M^r, 1 + \frac{q-\delta}{r}, 1 + \frac{p-\delta}{r})A^q \geqslant B^p$ *for all* $p > \delta$, $q > \delta$ *and* $r > \delta$.

In particular, if we put $\delta = 1$ in Theorem 3.1, then we have the following characterization of the operator order by means of a generalized Kantorovich constant.

THEOREM 3.2. *Let $A, B \in \mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A) \subseteq [m, M]$ and $\mathrm{Sp}(B) \subseteq [n, N]$ for some scalars $0 < m < M$ and $0 < n < N$, and let $\overline{K}(m, M, p, q)$ and $K(m, M, p, q)$ be defined in (2.4) and (2.3), respectively. Then the following statements are mutually equivalent:*

(i) $A \geqslant B$.

(ii) $K(n^r, N^r, 1 + \frac{p-1}{r}, 1 + \frac{q-1}{r})A^q \geqslant B^p$ *for all* $p > 1$, $q > 1$ *and* $r > 1$.

(iii) $\overline{K}(m^r, M^r, 1 + \frac{q-1}{r}, 1 + \frac{p-1}{r})A^q \geqslant B^p$ *for all* $p > 1$, $q > 1$ *and* $r > 1$.

By Theorem 2.4 we show a characterization of the chaotic order, which is two variable generalization of [13, Theorem 3].

THEOREM 3.3. *Let $A, B \in \mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A) \subseteq [m, M]$ and $\mathrm{Sp}(B) \subseteq [n, N]$ for some scalars $0 < m < M$ and $0 < n < N$, and let $\overline{K}(m, M, p, q)$ and $K(m, M, p, q)$ be defined in (2.4) and (2.3), respectively. Then the following statements are mutually equivalent:*

(i) $A \gg B$ (*i.e.* $\log A \geqslant \log B$).

(ii) $K(n^r, N^r, 1 + \frac{p}{r}, 1 + \frac{q}{r})A^q \geqslant B^p$ *for all* $p > 0$, $q > 0$ *and* $r > 0$.

(iii) $\overline{K}(m^r, M^r, 1 + \frac{q}{r}, 1 + \frac{p}{r})A^q \geqslant B^p$ *for all* $p > 0$, $q > 0$ *and* $r > 0$.

If we put $p = q$ in Theorem 2.2, then we obtain an extension of Theorem C.

THEOREM 3.4. *Let $A, B \in \mathscr{B}_{++}(H)$ be positive invertible operators on a Hilbert space H with $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars $M > m > 0$, and let $K(m, M, p)$ be defined in (1.1). Then the following statements are mutually equivalent for each $\delta \in (0, 1]$:*

(i) $A^{\delta} \geqslant B^{\delta}$.

$(ii-1)$ *For each* $n>0$ *and* $\alpha\in[0,1]$

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}},M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}},n+1)A^{(p-\delta+\alpha u)s}\geqslant\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s\geqslant 1$, $p\geqslant\delta$ *and* $u\geqslant\delta$ *with* $(p-\delta+\alpha u)s\geqslant(\alpha+n)u$.

$(ii-2)$ *For each* $n>0$ *and* $\alpha\in[0,1]$

$$K(m^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}},M^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}},n+1)A^{(p-\delta+\alpha u)s}\geqslant\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s\geqslant 1$, $p\geqslant\delta$ *and* $u\geqslant\delta$ *with* $(p-\delta+\alpha u)s\leqslant(\alpha+n)u$.

$(iii-1)$ *For each* $n>0$

$$K(m^{\frac{(p-\delta)s}{n}},M^{\frac{(p-\delta)s}{n}},n+1)^{\frac{1}{s}}A^p\geqslant B^p$$

holds for $s\geqslant 1$ *and* $p\geqslant\delta$ *with* $(p-\delta)s\geqslant n\delta$.

$(iii-2)$ *For each* $n>0$

$$K(m^{\frac{(p-\delta)s+\delta}{n+1}},M^{\frac{(p-\delta)s+\delta}{n+1}},n+1)^{\frac{1}{s}}A^p\geqslant B^p$$

holds for $s\geqslant 1$ *and* $p\geqslant\delta$ *with* $(p-\delta)s\leqslant n\delta$.

(iv) $\left(\frac{M}{m}\right)^{p-\delta}A^p\geqslant B^p$ *holds for* $p\geqslant\delta$.

Finally, if we put $n=1$ in $(ii-1)$ and $(iii-1)$ of Theorem 2.2, then we obtain the following Kantorovich type characterization of the δ-order by means of the Kantorovich constant.

COROLLARY 3.5. *Let* $A,B\in\mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A)\subseteq[m,M]$ *for some scalars* $M>m>0$. *Then the following statements are mutually equivalent for each* $\delta\in(0,1]$*:*

(i) $A^\delta\geqslant B^\delta$.

(ii) *For each* $\alpha\in[0,1]$

$$\frac{(M^{P(\delta)+Q(\delta)}-m^{P(\delta)+Q(\delta)})^2}{4M^{Q(\delta)}m^{Q(\delta)}(M^{P(\delta)}-m^{P(\delta)})(M^{Q(\delta)}-m^{Q(\delta)})}A^{(q-\delta+\alpha u)s}\geqslant\left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds for $s\geqslant 1$, $p\geqslant\delta$, $q\geqslant\delta$ *and* $u\geqslant\delta$ *with* $(p-\delta+\alpha u)s\geqslant(\alpha+1)u$, *where* $P(\delta)=(p-\delta+\alpha u)s-\alpha u$ *and* $Q(\delta)=(q-\delta+\alpha u)s-\alpha u$.

(iii) $\left(\frac{(M^{(q+p-2\delta)s}-m^{(q+p-2\delta)s})^2}{4M^{(q-\delta)s}m^{(q-\delta)s}(M^{(p-\delta)s}-m^{(p-\delta)s})(M^{(q-\delta)s}-m^{(q-\delta)s})}\right)^{\frac{1}{s}}A^q\geqslant B^p$
holds for $s\geqslant 1$, $p\geqslant\delta$ *and* $q\geqslant\delta$ *with* $(p-\delta)s\geqslant\delta$.

In particular, if we put $\delta=1$ in Corollary 3.5, then we have the following Kantorovich type characterizations of the operator order.

COROLLARY 3.6. *Let* $A,B\in\mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A)\subseteq[m,M]$ *for some scalars* $M>m>0$. *Then the following statements are mutually equivalent:*

(i) $A\geqslant B$.

(ii) *For each* $\alpha \in [0,1]$

$$\frac{(M^{P(1)+Q(1)} - m^{P(1)+Q(1)})^2}{4M^{Q(1)}m^{Q(1)}(M^{P(1)} - m^{P(1)})(M^{Q(1)} - m^{Q(1)})} A^{(q-1+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-1}{2}} B^p A^{\frac{\alpha u-1}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 1$, $q \geqslant 1$ *and* $u \geqslant 1$ *with* $(p-1+\alpha u)s \geqslant (\alpha+1)u$, *where* $P(1) = (p-1+\alpha u)s - \alpha u$ *and* $Q(1) = (q-1+\alpha u)s - \alpha u$.

(iii) $\left(\frac{(M^{(q+p-2)s} - m^{(q+p-2)s})^2}{4M^{(q-1)s}m^{(q-1)s}(M^{(p-1)s} - m^{(p-1)s})(M^{(q-1)s} - m^{(q-1)s})}\right)^{\frac{1}{s}} A^q \geqslant B^p$ *holds for* $s \geqslant 1$, $p \geqslant 1$ *and* $q \geqslant 1$ *with* $(p-1)s \geqslant 1$.

The following corollary is a two variable generalization of [8, Theorem 4].

COROLLARY 3.7. *Let* $A, B \in \mathscr{B}_{++}(H)$ *be positive invertible operators on a Hilbert space* H *with* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some scalars* $M > m > 0$. *Then the following statements are mutually equivalent:*

(i) $A \gg B$ $($*i.e.* $\log A \geqslant \log B$ $)$.

(ii) *For each* $\alpha \in [0,1]$

$$\frac{(M^{P(0)+Q(0)} - m^{P(0)+Q(0)})^2}{4M^{Q(0)}m^{Q(0)}(M^{P(0)} - m^{P(0)})(M^{Q(0)} - m^{Q(0)})} A^{(q+\alpha u)s} \geqslant \left(A^{\frac{\alpha u}{2}} B^p A^{\frac{\alpha u}{2}}\right)^s$$

holds for $s \geqslant 1$, $p \geqslant 0$, $q \geqslant 0$ *and* $u \geqslant 0$ *with* $(p+\alpha u)s \geqslant (\alpha+1)u$, *where* $P(0) = (p+\alpha u)s - \alpha u$ *and* $Q(0) = (q+\alpha u)s - \alpha u$.

(iii) $\left(\frac{(M^{(q+p)s} - m^{(q+p)s})^2}{4M^{qs}m^{qs}(M^{ps} - m^{ps})(M^{qs} - m^{(qs)})}\right)^{\frac{1}{s}} A^q \geqslant B^p$ *holds for* $s \geqslant 1$, $p \geqslant 0$ *and* $q \geqslant 0$.

4. Proofs of the results in Sections 2 and 3

Proof of Theorem 2.1. The first inequalities (2.1) are showed by Mićić et al. in [10, Theorem 3.1]. We shall prove (2.2). As $0 < A^{-1} \leqslant B^{-1}$ and $M^{-1} \leqslant A^{-1} \leqslant m^{-1}$ holds, then by applying the right hand inequality of (2.1) we obtain

$$A^{-p} \leqslant \frac{(q-1)^{q-1}}{q^q} \frac{(m^{-p} - M^{-p})^q}{(m^{-1} - M^{-1})(M^{-1}m^{-p} - m^{-1}M^{-p})^{q-1}} B^{-q}$$

if $qM^{-(p-1)} \leqslant \frac{m^{-p} - M^{-p}}{m^{-1} - M^{-1}} \leqslant qm^{-(p-1)}$,

$$A^{-p} \leqslant M^{-(p-q)} B^{-q} \qquad \text{if } \frac{m^{-p} - M^{-p}}{m^{-1} - M^{-1}} < qM^{-(p-1)}$$

and

$$A^{-p} \leqslant m^{-(p-q)} B^{-q} \qquad \text{if } qm^{-(p-1)} < \frac{m^{-p} - M^{-p}}{m^{-1} - M^{-1}}.$$

Then a simple calculation implies

$$A^{-p} \leqslant \frac{(q-1)^{q-1}}{q^q} \frac{(M^p - m^p)^q M^{q-p} m^{q-p}}{(M-m)(mM^p - Mm^p)^{q-1}} B^{-q} = M^{q-p} m^{q-p} K(m, M, p, q) B^{-q}$$

if $qm^{p-1} \leqslant \frac{M^p-m^p}{M-m} \leqslant qM^{p-1}$,

$$A^{-p} \leqslant M^{q-p}B^{-q} = M^{q-p}m^{q-p}K(m,M,p,q)B^{-q} \quad \text{if} \quad \tfrac{M^p-m^p}{M-m} < qm^{p-1}$$

and

$$A^{-p} \leqslant m^{q-p}B^{-q} = M^{q-p}m^{q-p}K(m,M,p,q)B^{-q} \quad \text{if} \quad qM^{p-1} < \tfrac{M^p-m^p}{M-m}.$$

We obtain the right hand inequality of (2.2) by taking inverses in both sides of inequalities above. We have from the left hand inequality of (2.1) that $K(m,M,p,q) \leqslant \frac{M^{p-1}}{m^{q-1}}$ and we obtain

$$B^q \leqslant M^{q-p}m^{q-p}K(m,M,p,q)A^p \leqslant \frac{M^{q-1}}{m^{p-1}}A^p \qquad \text{for all } p>1 \text{ and } q>1,$$

so the proof of theorem is complete. □

Proof of Theorem 2.2. We use same idea as in the proof of [3, Theorem 2].

$(i) \implies (ii-1)$: For given $p \geqslant \delta$ and $u \geqslant \delta$, put $A_1 = A^u$ and $B_1 = \left(A^{\frac{u-\delta}{2}}B^pA^{\frac{u-\delta}{2}}\right)^{\frac{u}{p+u-\delta}}$ in (iii) of Lemma B. Then we have $A_1 \geqslant B_1 \geqslant 0$. By the grand Furuta inequality, it follows that for each $t \in [0,1]$,

$$A_1^{\frac{(p_1-t)s+r}{q_1}} \geqslant \{A_1^{\frac{r}{2}}(A_1^{-\frac{t}{2}}B_1^{p_1}A_1^{-\frac{t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{1}{q_1}} \tag{4.1}$$

holds for any $s \geqslant 1$, $p_1 \geqslant 1$, $q_1 \geqslant 1$ satisfying the following two conditions

$$r \geqslant t, \tag{4.2}$$

$$(1-t+r)q_1 \geqslant (p_1-t)s+r. \tag{4.3}$$

For given $n>0$, $\alpha \in [0,1]$ and $s \geqslant 1$, we put $p_1 = \frac{p+u-\delta}{u}$, $q_1 = n+1 \geqslant 1$, $\alpha = 1-t$ and $r = \frac{(p-\delta+\alpha u)s}{nu} - \frac{n+1}{n}\alpha$. Then (4.2) is equivalent to the assumption in $(ii-1)$:

$$(p-\delta+\alpha u)s \geqslant (n+\alpha)u \tag{4.4}$$

and (4.3) is satisfied as the equality holds.

Therefore (4.1) implies that

$$A^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}} \geqslant \left\{A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}})^sA^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}\right\}^{\frac{1}{n+1}} \tag{4.5}$$

holds for $n>0$, $p \geqslant \delta$, $\alpha \in [0,1]$ and $s \geqslant 1$ with the condition (4.4). By raising the left hand side to power $n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1$ for some $q \geqslant \delta$ and the right hand side to power $n+1$, it follows from (2.2) that

$$\begin{aligned}\overline{K}(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1, n+1)\times \\ \times A^{(q-\delta+\alpha u)s-\alpha u+\frac{(p-\delta+\alpha u)s-\alpha u}{n}} \geqslant A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}})^sA^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}.\end{aligned} \tag{4.6}$$

By rearranging (4.6), we have the desired inequality $(ii-1)$:

$$\overline{K}(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1, n+1)A^{(q-\delta+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s.$$

$(i) \implies (ii-2)$ can be proved in the same way as $(i) \implies (ii-1)$. For given $n>0$, $\alpha\in[0,1]$ and $s\geqslant 1$ we put $p_1=\frac{p+u-\delta}{u}$, $q_1=n+1\geqslant 1$ and $r=t=1-\alpha$ in (4.1). Then (4.2) is satisfied as the equality and (4.3) is equivalent to the assumption in $(ii-2)$:

$$(p-\delta+\alpha u)s\leqslant (n+\alpha)u. \tag{4.7}$$

Therefore (4.1) implies that

$$A^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}} \geqslant \left\{A^{\frac{u(1-\alpha)}{2}}(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}})^sA^{\frac{u(1-\alpha)}{2}}\right\}^{\frac{1}{n+1}} \tag{4.8}$$

holds for $n>0$, $p\geqslant\delta$, $\alpha\in[0,1]$ and $s\geqslant 1$ with the condition (4.7). By raising the left hand side to power $(n+1)\frac{(q-\delta+\alpha u)s+(1-\alpha)u}{(p-\delta+\alpha u)s+(1-\alpha)u}$ for some $q\geqslant\delta$ and the right hand side to power $n+1$, it follows from (2.2) that

$$\overline{K}(m^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}}, M^{\frac{(p-\delta+\alpha u)s+(1-\alpha)u}{n+1}}, (n+1)\frac{(q-\delta+\alpha u)s+(1-\alpha)u}{(p-\delta+\alpha u)s+(1-\alpha)u}, n+1)A^{(q-\delta+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}}\right)^s$$

holds. So $(i)\implies(ii-2)$ is proved.

$(ii-1)\implies(iii-1)$ and $(ii-2)\implies(iii-2)$: We have only to put $\alpha=0$, $u=\delta$ in $(ii-1)$ and $(ii-2)$.

$(iii-1)\implies(iv-1)$: If we put $x=\frac{M}{m}$ in (iii-1), then we have

$$\begin{aligned}
&\overline{K}(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1)^{\frac{1}{s}} \\
&\quad=\left(\frac{n^n}{(n+1)^{n+1}}M^{(p-q)s}\frac{(x^{(q-\delta)s+\frac{(p-\delta)s}{n}}-1)^{n+1}}{(x^{\frac{(p-\delta)s}{n}}-1)(x^{(q-\delta)s+\frac{(p-\delta)s}{n}}-x^{\frac{(p-\delta)s}{n}})^n}\right)^{\frac{1}{s}} \\
&\quad\to 1\cdot M^{p-q}\frac{x^{(n+1)(q-\delta+\frac{p-\delta}{n})}}{x^{\frac{p-\delta}{n}}x^{p-\delta}x^{n(q-\delta)}}=\frac{M^{p-\delta}}{m^{q-\delta}} \quad \text{as } s\to\infty
\end{aligned}$$

if $(n+1)m^{(q-\delta)s}\leqslant\frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}}\leqslant(n+1)M^{(q-\delta)s}$.

But, if $\frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}}<(n+1)m^{(q-\delta)s}$, then

$$\overline{K}(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1)^{\frac{1}{s}}=\left((M^{\frac{(p-\delta)s}{n}})^{n-n\frac{q-\delta}{p-\delta}}\right)^{\frac{1}{s}}=M^{p-q}\leqslant\frac{M^{p-\delta}}{m^{q-\delta}}.$$

Similarly, if $(n+1)M^{(q-\delta)s} < \frac{M^{(q-\delta)s+(p-\delta)s/n}-m^{(q-\delta)s+(p-\delta)s/n}}{M^{(p-\delta)s/n}-m^{(p-\delta)s/n}}$, then

$$\overline{K}(m^{\frac{(p-\delta)s}{n}}, M^{\frac{(p-\delta)s}{n}}, n\frac{q-\delta}{p-\delta}+1, n+1)^{\frac{1}{s}} = m^{p-q} \leqslant \frac{M^{p-\delta}}{m^{q-\delta}}.$$

Hence it follows from $(iii-1)$ that $(iv-1)$ holds.

$(iii-2) \implies (iv-2)$: Putting $q=p$, $s=1$ and $n=\frac{p}{\delta}-1$ in $(iii-2)$ we have

$$K(m^{\delta}, M^{\delta}, \frac{p}{\delta})A^p \geqslant B^p \qquad \text{for} \quad p \geqslant \delta. \tag{4.9}$$

Since $K(m,M,p) \leqslant \left(\frac{M}{m}\right)^{p-1}$ for $p \geqslant 1$ by Theorem A, it follows

$$K(m^{\delta}, M^{\delta}, \frac{p}{\delta}) \leqslant \left(\frac{M^{\delta}}{m^{\delta}}\right)^{\frac{p}{\delta}-1} = \left(\frac{M}{m}\right)^{p-\delta} \qquad \text{for} \quad \frac{p}{\delta} \geqslant 1,$$

which give the desired inequality $(iv-2)$.

$(iv-1) \implies (i)$ and $(iv-2) \implies (i)$: We have only to put $p=q=\delta$ in $(iv-1)$ and $p=\delta$ in $(iv-2)$.

When $A,B \in \mathscr{B}_{++}(H)$ satisfy $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m,M]$, the proof is similar to above. Therefore we shall prove only $(i) \implies (ii-1)$. By raising the left hand side of (4.5) to power $n+1$, the right hand side to power $n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1$ for some $q \geqslant \delta$ and using that

$$\begin{aligned} m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}} &\leqslant \left\{A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}})^sA^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}\right\}^{\frac{1}{n+1}} \\ &\leqslant M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, \end{aligned}$$

it follows from (2.1) that

$$\begin{aligned} K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{n}}, n+1, n\frac{(q-\delta+\alpha u)s-\alpha u}{(p-\delta+\alpha u)s-\alpha u}+1)\times \\ \times A^{(q-\delta+\alpha u)s-\alpha u+\frac{(p-\delta+\alpha u)s-\alpha u}{n}} \\ \geqslant A^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}}(A^{\frac{\alpha u-\delta}{2}}B^pA^{\frac{\alpha u-\delta}{2}})^sA^{\frac{(p-\delta+\alpha u)s-(n+1)\alpha u}{2n}} \end{aligned} \tag{4.10}$$

holds. By rearranging (4.10), we have the desired inequality $(ii-1)$. □

Proof of Theorem 2.4. $(i) \implies (ii-1)$, $(i) \implies (ii-2)$ and $(ii-1) \implies (iii)$ can be proved in the same way as in Theorem 2.2 if we put $\delta = 0$.

$(ii-2) \implies (iii)$: We have only to put $\alpha = 0$ and $u = \frac{ps}{u}$ in $(ii-2)$.

$(iii) \implies (iv)$: Putting $s=1$ and $r=\frac{p}{n}$ in (iii), we obtain that

$$\overline{K}(m^r, M^r, \frac{q}{r}+1, \frac{p}{r}+1)A^q \geqslant B^p$$

holds for $r>0$, $p \geqslant 0$ and $q \geqslant 0$. Letting $r \to +0$ we have (iv) since $\overline{K}(m,M,p,q) = M^{q-p}m^{q-p}K(m,M,p,q)$ by Remark 1 and

$$K(m^r, M^r, \frac{q}{r}+1, \frac{p}{r}+1) \to S(h,q,p) \quad \text{as} \quad r \to +0$$

holds for each $p \geqslant 0$ and $q \geqslant 0$ by the proof of [10, Theorem 4], where

$$S(h,p,q) = \begin{cases} m^{p-q}\dfrac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h} & \text{if } q \leqslant \dfrac{h^p-1}{\log h} \leqslant qh^p, \\ m^{p-q} & \text{if } \dfrac{h^p-1}{\log h} \leqslant q, \\ M^{p-q} & \text{if } qh^p \leqslant \dfrac{h^p-1}{\log h}. \end{cases}$$

$(iv) \implies (i)$: If we put $p = q$ in (iv) we have that $S(h,p,p)A^p \geqslant B^p$ holds for each $p > 0$, since $p \leqslant \frac{h^p-1}{\log h} \leqslant ph^p$ holds for all $p > 0$. Therefore the constant $S(h,p,p)$ coincides with the generalized Specht's ratio $S(h,p)$ [13] defined as

$$S(h,p) = \frac{h^{\frac{p}{h^p-1}}}{e\log\left(h^{\frac{p}{h^p-1}}\right)}. \tag{4.11}$$

Then $S(h,p)A^p \geqslant B^p$ for all $p > 0$ implies $\log A \geqslant \log B$ by [13, Theorem 5]. We remark that the statements (iv) for $p = 0$ and (i) are identical since $S(h,0) = \lim_{p\to+0} S(h,p) = 1$.

When $A, B \in \mathscr{B}_{++}(H)$ satisfy $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m,M]$, the proof is similar to the proof of Theorem 2.2. □

To prove Theorem 3.1 we need the following lemma [1, Lemma 4], which follows from Lemma B.

LEMMA B-2. *Let $A, B \in \mathscr{B}_{++}(H)$. Then the following statements are mutually equivalent for each $\delta \in [0,1]$:*

(i) $A^\delta \geqslant B^\delta$, *where the case $\delta = 0$ means $A \gg B$.*

(ii) $\left(B^{\frac{p}{2}}A^{p+\delta}B^{\frac{p}{2}}\right)^{\frac{p+\delta}{2p+\delta}} \geqslant B^{p+\delta}$ *for all $p \geqslant 0$.*

(iii) $\left(B^{\frac{u}{2}}A^{p+\delta}B^{\frac{u}{2}}\right)^{\frac{u+\delta}{p+u+\delta}} \geqslant B^{u+\delta}$ *for all $p \geqslant 0$ and $u \geqslant 0$.*

Proof of Theorem 3.1. $(i) \implies (ii)$: It follows from Lemma B-2 that $A^\delta \geqslant B^\delta$ ensures

$$\left(B^{\frac{r}{2}}A^{q+\delta}B^{\frac{r}{2}}\right)^{\frac{r+\delta}{q+r+\delta}} \geqslant B^{r+\delta} \qquad \text{for all } q > 0 \text{ and } r > 0.$$

If we put $A_1 = \left(B^{\frac{r}{2}}A^{q+\delta}B^{\frac{r}{2}}\right)^{\frac{r+\delta}{q+r+\delta}}$ and $B_1 = B^{r+\delta}$, then we have $A_1 \geqslant B_1 > 0$ and $M^{r+\delta}I \geqslant B_1 \geqslant m^{r+\delta}I > 0$. Applying (2.1) of Theorem 2.1 to A_1 and B_1, we obtain

$$K(m^{r+\delta}, M^{r+\delta}, p_1, q_1)A_1^{q_1} \geqslant B_1^{p_1} \qquad \text{for all } p_1 > 1 \text{ and } q_1 > 1.$$

If we put $p_1 = \frac{p+r+\delta}{r+\delta} > 1$ and $q_1 = \frac{q+r+\delta}{r+\delta} > 1$, then we obtain

$$K(m^{r+\delta}, M^{r+\delta}, \frac{p+r+\delta}{r+\delta}, \frac{q+r+\delta}{r+\delta})A^{q+\delta} \geqslant B^{q+\delta} \qquad \text{for all } p > 0 \text{ and } q > 0.$$

Replacing $p+\delta$ by p, $q+\delta$ by q and $r+\delta$ by r, we have the desired inequality (ii).

$(i) \implies (iii)$: We have only to put $n=\frac{p-\delta}{r}$ and $s=1$ in $(iii-1)$ of Theorem 2.2.

$(ii) \implies (i)$: It follows from (2.1) of Theorem 2.1 that

$$\frac{(N^r)^{\frac{p-\delta}{r}}}{(n^r)^{\frac{q-\delta}{r}}}A^q \geqslant K(n^r,N^r,1+\frac{p-\delta}{r},1+\frac{q-\delta}{r})A^q \geqslant B^p \qquad \text{for all } p>\delta \text{ and } q>\delta.$$

Therefore we have $\frac{N^{p-\delta}}{m^{q-\delta}}A^q \geqslant B^p$. Putting $p=q$ and letting $p\to\delta$ we obtain (i).

$(iii) \implies (i)$ can be proved in the same way as $(ii) \implies (i)$ by applying (2.2) of Theorem 2.1. □

Proof of Theorem 3.3. $(i) \implies (ii)$ can be proved in the same way as $(i) \implies (ii)$ in Theorem 3.1 if we put $\delta=0$.

$(i) \implies (iii)$: We have only to put $s=1$ and $r=\frac{p}{n}$ in (iii) of Theorem 2.4.

$(ii) \implies (i)$: Putting $q=p$ in (ii) we have

$$K(n^r,N^r,1+\frac{p}{r})A^p \geqslant B^p \qquad \text{for all } p>0 \text{ and } r>0. \tag{4.12}$$

By [13, Lemma 11] we have

$$K(n^r,N^r,1+\frac{p}{r}) \to S(h,p) \quad \text{as} \quad r\to+0,$$

where $S(h,p)$ is the generalized Specht's ratio defined as (4.11). Letting $r\to+0$ in (4.12), we have $S(h,p)A^p \geqslant B^p$ for all $p>0$ and it implies $\log A \geqslant \log B$ by [13, Theorem 5].

$(iii) \implies (i)$ can be proved in the same way as $(ii) \implies (i)$ by replacing N by M and n by m. □

Proof of Corollary 3.7. $(i) \implies (ii)$ and $(ii) \implies (iii)$: We have only to put $n=1$ in (i), $(ii-1)$ and (iii) of Theorem 2.4.

$(iii) \implies (i)$: Putting $p=q$ and $s=1$ in (iii), we have (i) since $\frac{(M^p+m^p)^2}{4M^pm^p}A^p \geqslant B^p$ for all $p\geqslant 0$ implies $A \gg B$ by [13, Theorem 2]. □

5. Remark

We can not obtain a more precise estimation then the constant is given in Theorem C if we replace $n+1$ with $T+R$ for some $T,R\in\mathbb{R}$. In fact we obtain that $T>0$ and $R\geqslant 1$ and

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}},M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}},T+R) \geqslant K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}},M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}},T+1)$$

for each $\delta\in(0,1]$, $\alpha\in[0,1]$, $s\geqslant 1$, $p\geqslant\delta$ and $u\geqslant\delta$ with $R(p-\delta+\alpha u)s \geqslant (\alpha+T)u$.

PROPOSITION 5.1. *Let* $A,B \in \mathscr{B}_{++}(H)$ *with* $\mathrm{Sp}(A) \subseteq [m,M]$ *for some scalars* $M > m > 0$. *If* $A^{\delta} \geqslant B^{\delta}$ *for* $\delta \in (0,1]$, *then*

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R)A^{(p-\delta+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-\delta}{2}} B^{p} A^{\frac{\alpha u-\delta}{2}}\right)^{s}$$

holds for each $\alpha \in [0,1]$, $T > 0$, $R \geqslant 1$, $s \geqslant 1$, $p \geqslant \delta$ *and* $u \geqslant \delta$ *with* $R(p-\delta+\alpha u)s \geqslant (\alpha+T)u$ *and*

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R) \geqslant K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+1)$$

holds.

Proof. We have that the inequality (4.1) holds for $s \geqslant 1$, $p_1 \geqslant 1$ and $q_1 \geqslant 1$ with conditions (4.2) and (4.3). For given $T,R \in \mathbb{R}$, $\alpha \in [0,1]$ and $s \geqslant 1$, we put $p_1 = \frac{p+u-\delta}{u}$, $q_1 = T+R \geqslant 1$ and $\alpha = 1-t$. As we desire that the power of M and m in $K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R)$ be $\frac{(p-\delta+\alpha u)s-\alpha u}{T}$, we have $\frac{(p_1-t)s+r}{u(R+T)} = \frac{(p-\delta+\alpha u)s-\alpha u}{T}$. It follows that $r = \frac{R(p-\delta+\alpha u)s}{Tu} - \frac{R+T}{T}\alpha$. The condition (4.2) is equivalent to the assumption in Proposition 5.1:

$$R(p-\delta+\alpha u)s \geqslant (\alpha+T)u \tag{5.1}$$

and (4.3) is equivalent to

$$(R-1)\frac{(R+T)}{u}\frac{(p-\delta+\alpha u)s-\alpha u}{T} \geqslant 0. \tag{5.2}$$

Because $(R-1)\frac{(R+T)}{u}\frac{(p-\delta+\alpha u)s-\alpha u}{T} = (R-1)[(p_1-t)s+r]$ and $(p_1-t)s+r \geqslant 0$ for $p_1 \geqslant 1 \geqslant t \geqslant 0$, $s \geqslant 1$ and $r \geqslant t$, it follows $R \geqslant 1$. Next, because $(p-\delta+\alpha u)s-\alpha u = (p-\delta)+\alpha u(s-1) \geqslant 0$ for $p \geqslant \delta$, $u \geqslant \delta$, $\alpha \in [0,1]$ and $s \geqslant 1$, it follows $T > 0$.

Therefore (4.1) implies that

$$A^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}} \geqslant \left\{A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}(A^{\frac{\alpha u-\delta}{2}}B^{p}A^{\frac{\alpha u-\delta}{2}})^{s}A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}\right\}^{\frac{1}{T+R}}$$

holds. By raising both sides to power $T+R$, it follows from Theorem A that

$$\begin{aligned} K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R)&A^{(p-\delta+\alpha u)s-\alpha u+R\frac{(p-\delta+\alpha u)s-\alpha u}{T}} \\ &\geqslant A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}(A^{\frac{\alpha u-\delta}{2}}B^{p}A^{\frac{\alpha u-\delta}{2}})^{s}A^{\frac{R(p-\delta+\alpha u)s-(R+T)\alpha u}{2T}}. \end{aligned} \tag{5.3}$$

By rearranging (5.3), we have the desired inequality

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R)A^{(p-\delta+\alpha u)s} \geqslant \left(A^{\frac{\alpha u-\delta}{2}}B^{p}A^{\frac{\alpha u-\delta}{2}}\right)^{s}.$$

By [13, Proposition 4], we have that $F(p,r,m,M) = K(m^{r}, M^{r}, \frac{p}{r}+1)$ is an increasing function of p, r and M for $p > 0$, $r > 0$ and $M > m > 0$. It follows that

$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, \ T+R)$ is an increasing function of R for $R \geqslant 1$. Then we have

$$K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+R) \geqslant K(m^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, M^{\frac{(p-\delta+\alpha u)s-\alpha u}{T}}, T+1).$$

□

REFERENCES

[1] J. I. FUJII, Y. SEO, *Characterizations of chaotic order associated with the Mond-Shisha difference*, Math. Inequal. Appl., **5**, (4) (2002), 725–734.

[2] M. FUJII, M. HASHIMOTO, Y. SEO AND M. YANAGIDA, *Characterizations of usual and chaotic order via Furuta and Kantorovich inequalities*, Sci. Math., **3**, (2000), 405–418.

[3] M. FUJII, E. KAMEI AND Y. SEO, *Kantorovich type operator inequalities via grand Furuta inequality*, Sci. Math., **3**, (2000), 263–272.

[4] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$, for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with* $(1+2r)q \geqslant p+2r$, Proc. Amer. Math. Soc., **101**, (1987), 85–88.

[5] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **219**, (1995), 139–155.

[6] T. FURUTA, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl., **2**, (1998), 137–148.

[7] T. FURUTA, M. GIGA, *A complementary result of Kantorovich type order preserving inequalities by Mićić-Pečarić-Seo*, Linear Algebra Appl., **369**, (2003), 27–40.

[8] T. FURUTA, Y. SEO, *An application of generalized Furuta inequality to Kantorovich type inequalities*, Sci. Math., **2**, (1999), 393–399.

[9] M. HASHIMOTO, M. YANAGIDA, *Further extensions of characterizations of chaotic order associated with Kantorovich type inequalities*, Sci. Math., **3**, (2000), 127–136.

[10] J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, *Function order of positive operators based on the Mond-Pečarić method*, Linear Alg. Appl., **360**, (2003), 15–34.

[11] J. MIĆIĆ, J. PEČARIĆ, *Characterizations of δ–order associated with Kantorovich type operator inequalities*, Sci. Math. Japon., **63**, (2006), 149–155.

[12] Y. SEO, *A characterization of operator order via grand Furuta inequality*, J. Inequal. Appl., **6**, (2001), 473–481.

[13] T. YAMAZAKI, M. YANAGIDA, *Characterizations of chaotic order associated with Kantorovich inequality*, Sci. Math., **2**, (1999), 37–50.

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