Mathematical
Inequalities
& Applications
Volume 9, Number 3 (2006), 511–519

ON THE RANGE KERNEL ORTHOGONALITY AND P–SYMMETRIC OPERATORS

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(*communicated by N. Elezović*)

Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on H. For given $A \in L(H)$, we define the derivation $\delta_A : L(H) \longrightarrow L(H)$ by $\delta_A(X) = AX - XA$. In this paper we establish the orthogonality of the range $R(\delta_A)$ and the kernel $\ker(\delta_A)$ of a derivation δ_A induced by a cyclic subnormal operator A, in the usual sense. We give a version of the Putnam - Fuglede theorem. We establish a short proof of the principal result of F. Wenying and J. Guoxing in [10]. Relatad results for P-symmetric operators are also given.

1. Introduction

Let H be an infinite dimensional complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators acting on H. If $A \in L(H)$, then the inner derivation induced by A is the operator δ_A defined by

$$\begin{aligned} \delta_A : & L(H) \longrightarrow L(H) \\ & X \longmapsto \delta_A(X) = AX - XA. \end{aligned}$$

Given subspaces M and N of a Banach space V with norm $\|.\|$, M is said to be orthogonal to N if $\|m+n\| \geqslant \|n\|$ for all $m \in M$ and $n \in N$. This definition generalizes the idea of orthogonality in Hilbert space.

Let A be a normal operator, Anderson [1] has shown that if S is in the commutant $\{A\}'$ of A (i.e. $[A,S] = AS - SA = 0$), then for all $X \in L(H)$ we have

$$\|\delta_A(X) + S\| \geqslant \|S\|$$

where $\|\cdot\|$ is the usual operator norm. The above inequality says that the range $R(\delta_A)$ of the derivation δ_A is orthogonal to the kernel $\ker(\delta_A)$ of δ_A.

The study of the range-kernel orthogonality of derivations has been considered in a number of papers (see [9], [15], [16], [17] and some of the references cited in these papers), and much attention has been given to its investigations with respect to different norms (see [9], [15], [16] and [18]).

Mathematics subject classification (2000): 47B47, 47B10, 47A30.

Key words and phrases: Range-kernel orthogonality, subnormal operator, cyclic vector, Putnam - Fuglede theorem, P-symmetric operator.

It has been shown in Theorem 4 [14] that if $A \in L(H)$ is a cyclic subnormal operator and if $S \in C_2(H) \cap \{A\}'$, where $C_2(H)$ is the Hilbert-Schmidt class associated with the norm $\|\cdot\|_2$, then for all $X \in L(H)$ we have

$$\|\delta_A(X) + S\|_2^2 = \|\delta_A(X)\|_2^2 + \|S\|_2^2.$$

In the same direction, it should be noted that F. Kittaneh remarked that the Theorem 2 in [15], can be modified to insure that if $A \in L(H)$ is a cyclic subnormal operator and $S \in \mathcal{J} \cap \{A\}'$, such that $\mathcal{J}$ is the norm ideal associated with the unitarily invariant norm $\|\cdot\|_{\mathcal{J}}$, then for all $X \in L(H)$ we have also

$$\|\delta_A(X) + S\|_{\mathcal{J}} \geqslant \|S\|_{\mathcal{J}}.$$

The purpose of the first section is to prove the orthogonality of the range and the kernel of an inner derivation induced by a cyclic subnormal operator in the usual operator norm (i.e. on the whole space $L(H)$). Moreover, we give an example showing that the cyclicity assumption on a subnormal operator A is sufficient for the range-kernel orthogonality to be hold. Finally, it is natural to ask if this range-kernel orthogonality result has a τ_A analogue, where τ_A is the elementary operator defined on $L(H)$ by $\tau_A(X) = AXA - X$ and A is a cyclic subnormal operator.

In the second section we give a version of the Putnam-Fuglede theorem. Given $A, B \in L(H)$ and let $\mathcal{F}$ be a two sided ideal of $L(H)$. The pair (A,B) is said to possess the Putnam-Fuglede commutativity theorem $(PF)_{\mathcal{F}}$ if $AT = TB$ and $T \in \mathcal{F}$ implies $A^*T = TB^*$. We show that the set

$$\sum(\mathcal{F}) = \{A \in L(H) : (A,A) \text{ has property } (PF)_{\mathcal{F}}\}$$

is not norm closed. This result allow us to give a characterization of operators A such that the pair (A,A) has the property $(PF)_{C_p}$, where C_p denote the Von Newmann schatten class for $p > 1$. Consequently, we obtain a short proof of the principal result of F. Wenying and G. Guoxing in [10]. We conclude this section with some notations.

Notations. Let $K(H)$ be the ideal of all compact operators. For $A \in L(H)$, let $[A]$ denote the coset of A in the Calkin algebra $\mathcal{C}(H) = L(H)/K(H)$. Let $C_1(H)$ be the ideal of trace class operators, the trace function is defined on $C_1(H)$ by $tr(T) = \sum_n (Te_n, e_n)$, where $(e_n)_n$ is any complete orthonormal sequence in H. For $1 < p < \infty$ we denote $C_P(H)$ the Von Neumann-Schatten class and $\|\cdot\|_p$ its associated norm. $\overline{R(\delta_A/C_P)}$ is the norm closure of the range of δ_A/C_P. The annhilateur of $R(\delta_A/C_P)$ is denoted by

$$R(\delta_A/C_P)^\circ = \{f \in (C_p(H))' : f(AX - XA) = 0 \text{ for all } X \in C_P(H)\}.$$

In addition to the notation already introduced, we shall use the following notation. Given $X \in L(H)$, we shall denote the kernel, the orthogonal complement of the kernel and the range of X by $\ker(X)$, $(\ker(X))^\perp$ and $R(X)$ respectively. The spectrum, the essential spectrum, the point spectrum and the spectral radius of X will be denoted by $\sigma(X)$, $\sigma_e(X)$, $\sigma_p(X)$ and $r(X)$ respectively. Any other notation will be explained as and when required.

2. The range-kernel orthogonality

DEFINITION 2.1. A vector $e_\circ \in H$ is cyclic for $A \in L(H)$ if H is the smallest invariant subspace for A that contains $e_\circ$. The operator A is said to be cyclic if it has a cyclic vector.

DEFINITION 2.2. Let $A \in L(H)$. The operator A is said to be subnormal if there exists a normal operator B on a Hilbert space K such that H is a subspace of K, the subspace H is invariant under the operator B, and the restriction of B to H coincides with A.

The basic tools in the main result of this section is to use other technics that this used, stated below as a proposition and a remark.

PROPOSITION 2.1. *Let a be a normal element of a C^*-algebra $\mathcal{A}$. Then for every element $c \in \mathcal{A}$ satisfying $ac = ca$, we have*

$$\|ax - xa + c\| \geqslant \|c\|$$

for all $x \in \mathcal{A}$.

Proof. It is well known that there exists an $*$-isometric isomorphism ψ and a Hilbert space H such that $\psi : \mathcal{A} \longrightarrow L(H)$ preserving the order [13]. It follows that $\psi(a)$ is a normal operator and commutes with $\psi(c)$. Then combining the Anderson's Result for normal operators [1] and the isometric isomorphism, we get the related inequality

$$\|ax - xa + c\| \geqslant \|c\|$$

for all $x \in \mathcal{A}$.

REMARK 2.1. ([11, p.187]) A coset $[A]$ has norm equal to its spectral radius in each of the following cases:

(i) $[A]$ is hyponormal.

(ii) $[A]$ is a Toeplitz operator.

(iii) A has norm equal to its spectral radius and A has no isolated eigenvalues of finite multiplicity.

THEOREM 2.1. *Let $A \in L(H)$ be a cyclic subnormal operator. For every bounded linear operator T such that $AT = TA$, we have*

$$\|AX - XA + T\| \geqslant \|T\|$$

for all $X \in L(H)$.

Proof. Let T be in $L(H)$ such that $AT = TA$. Since A is a cyclic subnormal operator, then it follows from Yoshino's result [20] that T is subnormal. This implies that $r(T) = \|T\|$. Hence it is enough to prove that

$$\|AX - XA + T\| \geqslant |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma(T)$. Furthermore, since T is a subnormal operator, then it is well known that $\sigma(T) = \sigma_p(T) \cup \sigma_e(T)$ (see [11]).

Let $\lambda \in \sigma(T)$. We consider the following cases for the location of λ:

Case 1. Assume that $\lambda \in \sigma_p(T)$. We shall divide this case into two different steps.

(i) if $\lambda \in \sigma_p(T)$ such that $\dim\ker(T-\lambda) < \infty$. Let us denote E_λ the subspace $\ker(T-\lambda)$. Since $AT = TA$ and T is subnormal , then the subspace E_λ is invariant by T and A. Moreover A/E_λ is normal, then E_λ reduces A [18, p. 514]. Hence for A and T we get the following decomposition

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad T = \begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}.$$

For an operator $X = \begin{pmatrix} Y & Z \\ R & S \end{pmatrix}$, we have

$$AX - XA + T = \begin{pmatrix} BY - YB + \lambda & * \\ * & * \end{pmatrix}.$$

Recall that the norm of an operator matrix is always greater than or equal to the norm of the operator matrix consisting of its diagonal entries only [8, p. 82], applying this twice, from the above equality we have

$$\|AX - XA + T\| \geqslant \|BY - YB + \lambda\|$$

A is a finite operator because A is subnormal [19], hence B thus. Then we obtain

$$\|BY - YB + \lambda\| \geqslant |\lambda|.$$

Consequently we have

$$\|AX - XA + T\| \geqslant |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma_p(T)$ such that $\dim\ker(T-\lambda) < \infty$.

(ii) If $\lambda \in \sigma_p(T)$ such that $\dim\ker(T-\lambda) = \infty$. Since T is a subnormal operator then $\dim\ker(T-\lambda)^* = \infty$. It follows that $T-\lambda$ is not a Fredholm operator which is equivalent to $\lambda \in \sigma_e(T)$ (see the Case 2.).

Case 2. If $\lambda \in \sigma_e(T)$. For this case we may distinguish two steps.

(i) T has no isolated eigenvalues of finite multiplicity.

The condition $AT = TA$ implies that $[A][T] = [T][A]$. Since A is a cyclic subnormal operator then $[A]$ is a normal operator according to Shaw and Berger's result [4]. Using the Proposition 2.1 we obtain that $R(\delta_{[A]})$ is orthogonal to $\ker(\delta_{[A]})$. From this it follows that

$$\|AX - XA + T\| \geqslant \|[A][X] - [X][A] + [T]\| \geqslant \|[T]\|$$

for all $X \in L(H)$. Since T is subnormal and has no isolated eigenvalues of finite multiplicity, then by Remark 2.1 we have $\|[T]\| = r([T])$. Hence by a standart argument we have

$$\|[A][X] - [X][A] + [T]\| \geqslant |\lambda|$$

for all $X \in L(H)$. It follows

$$\|AX - XA + T\| \geqslant |\lambda|$$

for all $X \in L(H)$.

(ii) If T has isolated eigenvalues of finite multiplicity. We consider the subspace $E = \bigvee_{\mu\in\beta(T)} \ker(T-\mu)$, where $\beta(T)$ is the set of all isolated eigenvalues of T with finite multiplicity. The condition $AT = TA$ implies that T is subnormal. Since T/E is a normal operator then E reduces T. With respect to the decomposition $H = E \oplus E^{\perp}$, we have

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Applying Proposition 2.1 to the Calkin algebra, it is easily seen that

$$\|AX - XA + T\| \geqslant \|[A][X] - [X][A] + [T]\| \geqslant \|[T]\|.$$

On the other hand it is clear to check that T is a Fredholm operator if and only if T_2 is a Fredholm operator (see [7] Exercise 3 p. 352). It follows that $\lambda \in \sigma_e(T)$ if and only if $\lambda \in \sigma_e(T_2)$. Consequently we get $\sigma_e(T) = \sigma_e(T_2)$. By hypothesis we have $\lambda \in \sigma_e(T) = \sigma_e(T_2)$ and $T = T_1 \oplus T_2$. Using [8, p. 82] one obtains

$$\inf\left\{\left\| \begin{pmatrix} K_1 + T_1 & K_2 \\ K_3 & T_2 + K_4 \end{pmatrix} \right\|, K_1, K_2, K_3, K_4 \text{ compacts}\right\} \geqslant \inf\left\{\|T_2 + K_4\|, K_4 \text{ compact}\right\}.$$

Then it follows immediately

$$\|[T]\| \geqslant \|[T_2]\|.$$

Since T_2 has no isolated eigenvalues of finite multiplicity, then by the Remark 1 we have $\|AX - XA + T\| \geqslant |\lambda|$. This implies that

$$\|AX - XA + T\| \geqslant |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma_e(T)$.

Finally, we conclude that

$$\|AX - XA + T\| \geqslant |\lambda|$$

for all $X \in L(H)$ and all $\lambda \in \sigma(T)$. Then

$$\|AX - XA + T\| \geqslant \|T\|$$

for all $X \in L(H)$ and all $T \in \{A\}'$.

EXAMPLE. Let U be the unilateral shift operator of multiplicity one. On $H \oplus H$, we consider the operators $A = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$, where $P = 1 - UU^*$ and $Q = PU^*$. Then A is a noncyclic subnormal operator and $T \in \{A\}'$. It is easy to see that $\delta_A(X) + T = 0$ but $\|T\| = 1$; and so $R(\delta_A)$ is not orthogonal to $\ker(\delta_A)$.

According to the preceding theorem, this example indicates that the cyclicity assumption on A is sufficient for the range-kernel orthogonality of δ_A to hold. It has been used earlier in [15].

REMARK 2.2. There exist subnormal operator A and operator X such that $AX = XA$ and $A^*X \neq XA^*$. Hence the Putnam-Fuglede commutativity theorem cannot be extended to subnormal operators.

PROPOSITION 2.2. *Let A be cyclic subnormal operator, then $\|T\| \leqslant dist(T, R(\tau_A))$ for all paranormal or hyponormal operator T in* $\ker(\tau_A)$.

Proof. Suppose that T is a hyponormal operator in $\ker(\tau_A)$. The condition $ATA = T$ implies that $AT^2 = T^2A$. Applying the Theorem 2.1 we get

$$\|T^2\| \leqslant \|AY - YA + T^2\|$$

for all $Y \in L(H)$. Replacing Y by XAT one obtains

$$\|T\| \leqslant \|AXA - X + T\|$$

for all $X \in L(H)$. Which completes the proof of the proposition.

REMARK 2.3. If A is a cyclic subnormal operator, then we deduce from the Theorem 2.1 that $R(\delta_A)$ is orthogonal to $\ker(\delta_A)$, hence $\overline{R(\delta_A)} \cap \{A\}' = \{0\}$. Anderson proved that $\overline{R(\delta_A)} \cap \{A\}' = \{0\}$ if A is normal or isometric (see [1]).

Open problem. Let τ_A denotes the elementary operator $\tau_A : L(H) \longrightarrow L(H)$ defined by $\tau_A(X) = AXA - X$. If A is a cyclic subnormal operator we ask if we have the range-kernel orthogonality for τ_A.

3. P-symmetric operators

DEFINITION 3.1. Let $A, B \in L(H)$ and $\mathcal{F}$ be a two sided ideal of $L(H)$. The pair (A, B) is said to possess the Fuglede-Putnam property (shortened to $(PF)_{\mathcal{F}}$) if $AT = TB$ and $T \in \mathcal{F}$ implies $A^*T = TB^*$.

REMARK 3.1. It is shown in Proposition 1 [4], that the pair (A, A) of operators has the property $(PF)_{\mathcal{F}}$, where $\mathcal{F}$ is a two sided ideal of $L(H)$, under one of the following hypothesis:

(i) A is a normal operator.
(ii) A is an isometry.
(iii) A is a cyclic subnormal operator.
(iv) A is invertible such that $\|A^{-1}\| \|A\| = 1$.

PROPOSITION 3.1. *Let $\mathcal{F}$ be a two sided ideal of $L(H)$. Then the set of operators*

$$\sum(\mathcal{F}) = \{A \in L(H) \ : \ (A, A) \textit{ has the property } (PF)_{\mathcal{F}}\}$$

is not norm closed in $L(H)$.

Proof. To see this, we define a sequence of operators $(S_n)_n$ and S as follows. Let $(e_k)_{k \geqslant 0}$ be an orthonormal basis for H, we consider the operators

$$S_n e_k = \begin{cases} \frac{1}{n} & \text{if } k = 0, \\ e_{k+1} & \text{otherwise.} \end{cases} \quad \text{and} \quad S e_k = \begin{cases} 0 & \text{if } k = 0, \\ e_{k+1} & \text{otherwise.} \end{cases}$$

It is clear that $\|S_n - S\| \longrightarrow 0$. On the other hand , for all $n \geqslant 1$, S_n is a cyclic subnormal operator. Then from the preceding remark, it follows that $S_n \in \sum(\mathcal{F})$ for all non-negative integer n.

Let us consider $T = e_\circ \otimes e_1$, the rank one operator defined by $Tx = (x, e_1)e_\circ$ for all $x \in H$. Evidently $T \in \mathcal{F}$ and $ST = TS$. However, a simple calculation show that $S^*T \neq TS^*$, which implies that $S \notin \sum(\mathcal{F})$. This completes the proof.

REMARK 3.2. It is elementary to show that the weighted shift S defined above is subnormal. Since $S \notin \sum(\mathcal{F})$ for all two sided ideal $\mathcal{F}$ of $L(H)$, it follows from the Corollary 5 [4], that the range $R(\delta_S/C_2)$ is not orthogonal to the kernel $\ker(\delta_S/C_2)$.

Consequently, the cyclicity assumption on the subnormal operator S is fundamental for the orthogonality of $R(\delta_S/C_2)$ and $\ker(\delta_S/C_2)$ to be hold. This gives an affirmative answer to a question raised by F. Kittaneh in [14], and treated by the authors F. Wenying and J. Guoxing in [10].

PROPOSITION 3.2. *Let* $A \in L(H)$. *For* $1 < p < \infty$ *and if* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following statements are equivalent:*

(i) (A, A) *has the property* $(FP)_{C_p}$.

(ii) $\overline{R(\delta_A/C_q)} = \overline{R(\delta_{A^*}/C_q)}$.

(iii) *If* $T \in \ker(\delta_S/C_p)$, *then* $\overline{R(T)}$ *and* $(\ker(T))^\perp$ *reduces* A, *and the restriction* $A/\overline{R(T)}$ *and* $A/(\ker(T))^\perp$ *are normal operators.*

Proof. $(i) \Longleftrightarrow (ii)$ A simple calculation shows that

$$\overline{R(\delta_A/C_q)} = \overline{R(\delta_{A^*}/C_q)}$$

if and only if, whenever $f \in R(\delta_A/C_q)^\circ$ implies $f^* \in R(\delta_{A^*}/C_q)^\circ$, where we have $f^*(X) = \overline{f(X^*)}$ for all $X \in C_q$. Therefore, it suffices to show that

$$R(\delta_A/C_q)^\circ \cong \{A\}' \cap C_p.$$

It is convenient to note that

$$(C_q)' = \{f_T : T \in C_p\} \cong C_p$$

for all p and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

Consequently, if $f_T \in R(\delta_A/C_q)^\circ$ for some operator $T \in C_p$ we get

$$f_T(A(x \otimes y)) = f_T((x \otimes y)A)$$

for all x and y in H. From where

$$tr(TAx \otimes y) = tr(Tx \otimes A^*y).$$

But since $tr(u \otimes v) = (u, v)$, we obtain $(TAx, y) = (Tx, A^*y)$, hence $AT = TA$.

Conversely, suppose that $T \in \{A\}' \cap C_p$. From the above computation, it results easily that

$$f_T(A(x \otimes y)) = f_T((x \otimes y)A)$$

for all x and y in H. Since the class of all finite rank operators is dense in C_q for all $q \geqslant 1$, then the desired result follows immediately .

$(iii) \Longleftrightarrow (i)$ Is an obvious consequence of Lemma 2.3 [3].

Application. Let $\Omega_p(A)$, $\Lambda_p(A)$ and $\Delta_p(A)$ the Banach subalgebras of C_p associated with A defined as follows

$$\Omega_p(A) = \{C \in C_p \ : \ CC_p + C_pC \subset \overline{R(\delta_A/C_p)}\}$$

$$\Lambda_p(A) = \{Z \in C_p \ : \ ZR(\delta_A/C_p) + R(\delta_A/C_p)Z \subset \overline{R(\delta_A/C_p)}\}$$

$$\Delta_p(A) = \{B \in C_p \ : \ R(\delta_B/C_p) \subset \overline{R(\delta_A/C_p)}\}.$$

In the finite dimensional case, $\Omega_p(A)$, $\Lambda_p(A)$ and $\Delta_p(A)$ coincides with the subalgebras introduced in [2]. Consequently we get $\Omega_p(A) = \{0\}$, $\Lambda_p(A) = \{A\}'$ the commutant of A and $\Delta_p(A) = \{A\}''$ the bicommutant of A. By considering the Fuglede-Putnam theorem it follows that $\Omega_p(A)$, $\Lambda_p(A)$ and $\Delta_p(A)$ are C^* - subalgebras if and only if A is normal.

In the infinite dimensional case, by using the Proposition 3.1 ones obtain that $\Omega_p(A)$, $\Lambda_p(A)$ and $\Delta_p(A)$ are C^* - subalgebras if A satisfies one of the conditions of the previous Remark 3.1.

REMARK 3.3. The class of operators $A \in L(H)$ such that the pair (A,A) has the property $(FP)_{C_1}$ is called a class of P-symmetric operators. For a good accounts see ([5];[6]).

Acknowledgement. The authors would like to thank the Professor B. P. Duggal for his careful reading of the manuscript. His valuable suggestions and comments made numerous improvements throughout.

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(Received April 25, 2005)

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