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STABILITY OF GROUP AND RING HOMOMORPHISMS

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Abstract. In this paper, we give generalization of Hyers' theorem on the stability of approximately additive mapping and a generalization of Badora's theorem on approximate ring homomorphism. We also obtain a more general stability theorem, which gives stability theorems on Jordan and Lie homomorphisms. The proofs of the theorems given in this paper follow essentially the D. H. Hyers - Th. M. Rassias approach to stability of functional equations connected with S. M. Ulam's problem.

1. Introduction

In 1940, Ulam [1] raised the following question concerning the stability of homomorphisms:

Ulam's Question. Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leqslant \delta$ for all $x, y \in G_1$, then there is a homomorphism $g : G_1 \to G_2$ with $d(f(x), g(x)) \leqslant \varepsilon$ for all $x \in G_1$?

One of the first result in this direction is the result proved by Hyers (see [2]) which establishes the stability of a group homomorphism.

THEOREM (D. H. Hyers). *Let $\varepsilon \geqslant 0$ and let f be a function defined on an Abelian group $(G,+)$ with values in a Banach space $(Y, \|\cdot\|)$ satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in G$. Then there exists a unique additive mapping $h : G \to Y$, such that

$$\|f(x) - h(x)\| \leqslant \varepsilon, \quad \forall x \in G.$$

In 1978, Th. M. Rassias [3] provided the following drastic generalization of Hyers's result which allows the Cauchy difference to be unbounded.

THEOREM (Th. M. Rassias.) *Consider E_1, E_2 to be two Banach spaces and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geqslant 0$ and $p \in [0,1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p), \quad \forall x, y \in E_1.$$

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Then there exists a unique linear mapping $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{2-2^p}\|x\|^p, \quad \forall x \in E_1.$$

R. Badora in [4] proved the following result concerning the stability of a ring homomorphism.

THEOREM (R. Badora). *Let* R *be a ring and* $\mathscr{B}$ *be a Banach algebra and let* $\varepsilon, \delta \geqslant 0$. *Assume that* $f : R \to \mathscr{B}$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

and

$$\|f(x \cdot y) - f(x)f(y)\| \leqslant \delta,$$

for all $x, y \in R$. *Then there exists a unique ring homomorphism* $T : R \to \mathscr{B}$ *such that*

$$\|f(x) - T(x)\| \leqslant \varepsilon, \quad \forall x \in R.$$

During the last decades, Hyers' theorem was generalized in various directions, see [5-10]. In this note, we will generalize Hyers' theorem and Badora's theorem above. Moreover, we will give a stability theorem on Jordan homomorphism and a stability theorem on Lie homomorphism.

2. Stability of group homomorphisms

We first prove a theorem on stability of group homomorphisms, which generalizes Hyers' theorem.

THEOREM 2.1. *Let* E_1 *be an Abelian group and* E_2 *be a Banach space, if* $\varepsilon \geqslant 0$, $r \in \mathbb{N}$, $r \geqslant 2$ *and* $f : E_1 \to E_2$ *is such that*

$$\|f(\sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(x_k)\| \leqslant \varepsilon, \quad \forall x_1, x_2, \ldots, x_r \in E_1, \tag{2.1}$$

then there exists a unique additive mapping $T : E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leqslant \frac{1}{r-1}\varepsilon, \quad \forall x \in E_1. \tag{2.2}$$

Proof. First, we use induction to prove that for all $n \in \mathbb{N}$,

$$\left\|\frac{f(r^n x)}{r^n} - f(x)\right\| \leqslant \varepsilon \sum_{m=1}^{n} r^{-m}, \quad \forall x \in E_1. \tag{2.3}$$

Indeed, the case $n = 1$ is clear because by the hypothesis (2.1), we have

$$\left\|\frac{f(rx)}{r} - f(x)\right\| = \frac{1}{r}\|f(x + x + \cdots + x) - rf(x)\| \leqslant \frac{1}{r}\varepsilon, \ \forall x \in E_1.$$

Suppose that (2.3) holds for some $n \in \mathbb{N}$. Then for each $x \in E_1$, by (2.3) we have

$$\left\|\frac{f(r^n \cdot rx)}{r^n} - f(rx)\right\| \leqslant \varepsilon \sum_{m=1}^{n} r^{-m},$$

therefore

$$\left\|\frac{f(r^{n+1}x)}{r^{n+1}} - \frac{1}{r} f(rx)\right\| \leqslant \varepsilon \sum_{m=2}^{n+1} r^{-m}.$$

The triangle inequality yields that

$$\begin{aligned} \left\|\frac{f(r^{n+1}x)}{r^{n+1}} - f(x)\right\| &\leqslant \left\|\frac{1}{r^{n+1}} f(r^{n+1}x) - \frac{1}{r} f(rx)\right\| + \left\|\frac{1}{r} f(rx) - f(x)\right\| \\ &\leqslant \frac{\varepsilon}{r} + \varepsilon \sum_{m=2}^{n+1} r^{-m} \\ &= \varepsilon \sum_{m=1}^{n+1} r^{-m}. \end{aligned}$$

Thus, (2.3) is valid for all $n \in \mathbb{N}$. Since $\sum_{m=1}^{n} r^{-m}$ is increasingly convergent to $\frac{1}{r-1}$, we get from (2.3) that

$$\left\|\frac{f(r^n x)}{r^n} - f(x)\right\| \leqslant \frac{1}{r-1}\varepsilon, \quad \forall x \in E_1. \tag{2.4}$$

Fixed an $x \in E_1$, for all $m, n \in \mathbb{N}$ with $m > n$, we have from (2.4) that

$$\begin{aligned} \left\|\frac{1}{r^m} f(r^m x) - \frac{1}{r^n} f(r^n x)\right\| &= \frac{1}{r^n} \left\|\frac{1}{r^{m-n}} f(r^m x) - f(r^n x)\right\| \\ &\leqslant \frac{1}{r^n} \cdot \frac{1}{r-1} \varepsilon. \end{aligned}$$

Therefore

$$\lim_{n,m \to \infty} \left\|\frac{1}{r^m} f(r^m x) - \frac{1}{r^n} f(r^n x)\right\| = 0.$$

Since E_2 is a Banach space, the sequence $\{\frac{f(r^n x)}{r^n}\}$ converges. Set

$$T(x) = \lim_{n \to \infty} \frac{1}{r^n} f(r^n x), \quad \forall x \in E_1, \tag{2.5}$$

then we obtain a mapping $T : E_1 \longrightarrow E_2$. From (2.1), for all $x_1, x_2, \ldots, x_r \in E_1$ and all $n \in \mathbb{N}$, we compute that

$$\|f(r^n(x_1 + x_2 + \cdots + x_r)) - f(r^n x_1) - f(r^n x_2) - \cdots - f(r^n x_r)\| \leqslant \varepsilon,$$

and so

$$\frac{1}{r^n} \left\|f(r^n \sum_{k=1}^{r} x_k) - \sum_{k=1}^{r} f(r^n x_k)\right\| \leqslant \frac{1}{r^n} \varepsilon.$$

Consequently,

$$\lim_{n\to\infty}\left\|\frac{1}{r^n}f\,(r^n\sum_{k=1}^{r}x_k)-\sum_{k=1}^{r}\frac{1}{r^n}f\,(r^nx_k)\right\|=0.$$

It follows from (2.5) that

$$\|T(x_1+x_2+\cdots+x_r)-T(x_1)-T(x_2)-\cdots-T(x_r)\|=0.$$

Hence

$$T(x_1+x_2+\cdots+x_r)=T(x_1)+T(x_2)+\cdots+T(x_r)$$

for all $x_1,x_2,\ldots,x_r\in E_1$. Clearly, $T(0)=0$ and so T is an additive mapping. From (2.4) and (2.5) we obtain

$$\|T(x)-f\,(x)\|\leqslant\frac{1}{r-1}\varepsilon,\ \ \forall x\in E_1.$$

Now we prove the uniqueness of T. Assume that $T_1:E_1\longrightarrow E_2$ is an additive mapping such

$$\|f\,(x)-T_1(x)\|\leqslant\frac{1}{r-1}\varepsilon,\ \ \forall x\in E_1.$$

Since both T and T_1 are additive, we deduce that for each $x\in E_1$ and all $n\in\mathbb{N}$,

$$\begin{aligned}n\|T(x)-T_1(x)\| &= \|T(nx)-T_1(nx)\|\\ &\leqslant \|T(nx)-f\,(nx)\|+\|f\,(nx)-T_1(nx)\|\\ &\leqslant \frac{2\varepsilon}{r-1},\end{aligned}$$

so that

$$\|T(x)-T_1(x)\|\leqslant\frac{2\varepsilon}{n(r-1)}$$

for all $x\in E_1$ and hence $T(x)=T_1(x)$ for all $x\in E_1$. This completes the proof.

REMARK 2.1. From Theorem 2.1, Hyers' theorem can be easily proved. However, Theorem 2.1 is not a consequence of Hyers' theorem, because the condition (2.1) implies only

$$\|f\,(x+y)-f\,(x)-f\,(y)\|\leqslant\varepsilon+(r-2)\|f\,(0)\|,\ \ \forall x,y\in E_1.$$

Hence Hyers' theorem says only that there exists a unique mapping $T:E_1\to E_2$ such that

$$\|f\,(x)-T(x)\|\leqslant\varepsilon+(r-2)\|f\,(0)\|,\ \ \forall x\in E_1$$

rather than (2.2) since $\varepsilon+(r-2)\|f\,(0)\|>\frac{\varepsilon}{r-1}$.

REMARK 2.2. The condition (2.1) does not imply $f\,(0)=0$. For example, let $f\,(x)=1,\forall x\in\mathbb{R}$, then (2.1) holds for $\varepsilon=r-1$ but $f\,(0)=1$. Also, in this case the mapping T is identically zero.

3. Stability of a ring homomorphism

In this section, we prove some results concerning stability of a ring homomorphism and generalize R. Badora's theorem in two directions.

THEOREM 3.1. *Let R be a ring, $\mathscr{B}$ be a Banach algebra and $r \in \mathbb{N}, r \geqslant 2$ and $\varepsilon, \delta \geqslant 0$. If $f : R \to \mathscr{B}$ satisfies (2.1) and*

$$\|f(x_1x_2\cdots x_r) - f(x_1)f(x_2)\cdots f(x_r)\| \leqslant \delta, \quad \forall x_1, x_2, \dots, x_r \in R, \tag{3.1}$$

then there exists a unique additive mapping $T : R \to \mathscr{B}$ such that

$$T(x_1)T(x_2)\cdots T(x_r) = T(x_1x_2\cdots x_r), \quad \forall x_1, x_2, \dots, x_r \in R \tag{3.2}$$

and

$$\|f(x) - T(x)\| \leqslant \frac{\varepsilon}{r-1}, \quad \forall x \in R. \tag{3.3}$$

Proof. Theorem 2.1 shows that there exists a unique additive mapping $T : R \to \mathscr{B}$ satisfies (3.3). By the proof of Theorem 2.1, we see that the mapping T is given by

$$T(x) = \lim_{n\to\infty} \frac{1}{r^n} f(r^n x), \ \ \forall x \in R. \tag{3.4}$$

For all $x_1, x_2, \dots, x_r \in R$, let

$$g(x_1, x_2, \dots, x_r) = f(x_1x_2\cdots x_r) - f(x_1)f(x_2)\cdots f(x_r),$$

then using inequality (3.1), we get $\lim_{n\to\infty} \frac{1}{r^n} g(r^n x_1, x_2, \dots, x_r) = 0$. Therefore

$$\begin{aligned} T(x_1x_2\cdots x_r) &= \lim_{n\to\infty} \frac{1}{r^n} f[r^n(x_1x_2\cdots x_r)] \\ &= \lim_{n\to\infty} \frac{1}{r^n} f[(r^n x_1)x_2\cdots x_r] \\ &= \lim_{n\to\infty} \frac{1}{r^n} [g(r^n x_1, x_2, \dots, x_r) + f(r^n x_1)f(x_2)\cdots f(x_r)] \\ &= T(x_1)f(x_2)\cdots f(x_r) \end{aligned}$$

for all $x_1, x_2, \dots, x_r \in R$. From the last equation and the additivity of T we see that for all $n \in \mathbb{N}$,

$$\begin{aligned} T(x_1)f(r^n x_2)f(x_3)\cdots f(x_r) &= T(x_1 \cdot r^n x_2 \cdot x_3 \cdots , x_r) \\ &= T(r^n x_1 \cdot x_2 \cdots x_r) \\ &= r^n T(x_1)f(x_2)\cdots f(x_r), \end{aligned}$$

and so

$$T(x_1)\frac{f(r^n x_2)}{r^n} f(x_3)\cdots f(x_r) = T(x_1)f(x_2)f(x_3)\cdots f(x_r).$$

Sending n to infinity, we see that

$$T(x_1)T(x_2)f(x_3)\cdots f(x_r) = T(x_1x_2x_3\cdots x_r), \ \ \forall x_1, x_2, \dots, x_r \in R. \tag{3.5}$$

Suppose that

$$T(x_1)T(x_2)\cdots T(x_{r-1})f(x_r) = T(x_1x_2\cdots x_r),\ \forall x_1,x_2,\ldots,x_r \in R. \tag{3.6}$$

Then from (3.6) we get that for all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{r^n}T(x_1)T(x_2)\cdots T(x_{r-1})f(r^n x_r) &= \frac{1}{r^n}T(x_1x_2\cdots x_{r-1}\cdot r^n x_r)\\ &= \frac{1}{r^n}T(r^n(x_1x_2\cdots x_r))\\ &= T(x_1x_2\cdots x_r).\end{aligned}$$

By letting $n \to \infty$ we see that

$$T(x_1)T(x_2)\cdots T(x_r) = T(x_1x_2\cdots x_r),\ \ \forall x_1,x_2,\ldots,x_r \in R,$$

which is the desired identity (3.2).

COROLLARY 3.1. *Let R be a ring with a unit 1 and $\mathscr{B}$ be a Banach algebra with a unit e, and $r \in \mathbb{N}, r \geqslant 2, \varepsilon, \delta \geqslant 0$. If a mapping $f : R \to \mathscr{B}$ satisfies (2.1) and (3.1) and $f(1) = e$, then there exists a unique ring homomorphism $T : R \to \mathscr{B}$ such that*

$$\|f(x) - T(x)\| \leqslant \frac{\varepsilon}{r-1},\ \ \forall x \in R. \tag{3.7}$$

Proof. From Theorem 3.1, there exists a unique additive mapping $T : R \to \mathscr{B}$ satisfying (3.2) and (3.7). Using (3.5) we see that

$$T(x_1)T(x_2) = T(x_1x_2),\ \ \forall x_1,x_2 \in R.$$

Thus, the mapping $T : R \to \mathscr{B}$ is also a ring homomorphism.

REMARK 3.1. Under the assumptions of Theorem 3.1, the condition $f(1) = e$ does not implies $T(1) = e$, where T is given by (3.4). For example, let $f(1) = 1, f(x) = 0, \forall x \in \mathbb{R} \setminus \{1\}$, then we get a bounded function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x+y) - f(x) - f(y)| \leqslant \varepsilon = 2,\ \ \forall x,y \in \mathbb{R},$$

and

$$|f(xy) - f(x)f(y)| \leqslant \delta = 1,\ \ \forall x,y \in \mathbb{R}.$$

But the function T given by (3.4) is identically zero. However, by (3.6) the condition $T(1) = e$ does imply $f(1) = e$.

REMARK 3.2. By Theorem 3.1, R. Badora's theorem can be easily proved. However Theorem 3.1 is not a simple consequence of Badora's theorem, see Remark 2.1 for the reason.

Next we give the following more general stability result.

THEOREM 3.2. *Let $\mathscr{A}$ be an Abelian group, $\mathscr{B}$ be a Banach space, $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ be such that*

$$2^n\phi(x,y) = \phi(2^n x,y) = \phi(x,2^n y),\ \ \forall n \in \mathbb{N}, x,y \in \mathscr{A},$$

and $\psi : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ *be a continuous mapping such that*

$$2^n\psi(x,y) = \psi(2^n x, y) = \psi(x, 2^n y), \quad \forall n \in \mathbb{N}, x, y \in \mathscr{B},$$

and $\varepsilon, \delta \geqslant 0$. *If* $f : \mathscr{A} \to \mathscr{B}$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad \forall x, y \in \mathscr{A} \tag{3.8}$$

and

$$\|f(\phi(x,y)) - \psi(f(x), f(y))\| \leqslant \delta, \quad \forall x, y \in \mathscr{A}, \tag{3.9}$$

then there exists a unique additive mapping $T : \mathscr{A} \to \mathscr{B}$ *such that*

$$T(\phi(x,y)) = \psi(T(x), T(y)), \quad \forall x, y \in \mathscr{A} \tag{3.10}$$

and

$$\|f(x) - T(x)\| \leqslant \varepsilon, \quad \forall x \in \mathscr{A}, \tag{3.11}$$

Proof. From Theorem 2.1, we know that the mapping T given by (3.4) is the unique additive mapping satisfying (3.8) and (3.11). To show that the mapping T satisfies (3.10), let us define

$$g(x,y) = f(\phi(x,y)) - \psi(f(x), f(y)), \quad \forall x, y \in \mathscr{A}.$$

Then from condition (3.9) we see that $\lim_{n\to\infty} \frac{1}{2^n} g(2^n x, y) = 0, \ \forall x, y \in \mathscr{A}$. Thus, by (3.4) we have for all $x, y \in \mathscr{A}$,

$$\begin{aligned} T(\phi(x,y)) &= \lim_{n\to\infty} \frac{1}{2^n} f(\phi(2^n x, y)) \\ &= \lim_{n\to\infty} \left(\psi(\frac{1}{2^n} f(2^n x), f(y)) + \frac{1}{2^n} g(2^n x, y) \right) \\ &= \psi(T(x), f(y)). \end{aligned}$$

From the last equation and the additivity of T, we obtain that

$$T(\phi(x,y)) = \frac{1}{2^n} T(\phi(x, 2^n y)) = \psi(T(x), \frac{1}{2^n} f(2^n y)), \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ yields (3.10). This completes the proof.

For example, in the case where $\mathscr{A}$ is an algebra, we can take $\phi(x,y) = \alpha xy + \beta yx$ and take ψ similarly. Especially, we obtain the following stability theorems on Jordan and Lie homomorphisms.

COROLLARY 3.2. *Let* $\mathscr{A}$ *be an algebra,* $\mathscr{B}$ *be a Banach algebra and* $\varepsilon, \delta \geqslant 0$. *If* $f : \mathscr{A} \to \mathscr{B}$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon, \quad \forall x, y \in \mathscr{A} \tag{3.12}$$

and $\forall x, y \in \mathscr{A}$,

$$\|f([x,y]) - [f(x), f(y)]\| (\textit{resp.} \|f(x \circ y) - f(x) \circ f(y)\|) \leqslant \delta, \tag{3.13}$$

then there exists a unique additive mapping $T:\mathscr{A}\to\mathscr{B}$ *such that*

$$T([x,y]) = [T(x),T(y)](resp. T(x\circ y) = T(x)\circ T(y)), \quad \forall x,y\in\mathscr{A} \tag{3.14}$$

and

$$\|f(x) - T(x)\| \leqslant \varepsilon, \quad \forall x\in\mathscr{A}, \tag{3.15}$$

where $[X,Y] = XY - YX$ *is the Lie product of* X,Y *and* $a\circ b = \frac{1}{2}(ab+ba)$ *is the Jordan product of* a,b.

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