Mathematical
Inequalities
& Applications
Volume 9, Number 4 (2006), 567–576

SCHUR–CONVEXITY OF THE COMPLETE SYMMETRIC FUNCTION

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(*communicated by I. Olkin*)

Abstract. This paper investigates Schur-convexity of the complete symmetric function $c_r(x) = \sum_{i_1+\ldots+i_n=r} x_1^{i_1}\ldots x_n^{i_n}$ and the function $\phi_r(x) = \frac{c_r(x)}{c_{r-1}(x)}$, where $i_1,\ldots,i_n$ are non-negative integers and $r \geqslant 1$. Some inequalities, including Ky Fan type inequality, are established by use of the theory of majorization. It is also concerned with an open problem proposed by Menon [1].

1. Introduction and notation

Let $x=(x_1,x_2,\ldots,x_n)$ be a positive sequence and $R_+^n=\{x=(x_1,x_2,\ldots,x_n)|x_i>0,\ i=1,\ldots,n.\}$. The $r-$ th elementary symmetric function [11, p. 33; 5, p. 81] is defined as

$$e_r = e_r(x) = E_n^r(x) = \sum_{1\leqslant i_1<i_2<\ldots<i_r\leqslant n} \prod_{j=1}^{r} x_{i_j},\ r=1,2,\ldots,n. \tag{1.1}$$

Define $e_0(x)=1$, then $e_r(x)$ is the coefficient of t^r in the polynomial $\prod_{i=1}^n(1+x_it)$, that is, e_r satisfies the equation

$$\sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n}(1+x_it).$$

Using this method, Whiteley [10] defined the Whitely's symmetric function $T_n^{[r,s]}(x)$ by the following equation

$$\sum_{r=0}^{+\infty} T_n^{[r,s]}(x)t^r = \begin{cases} \prod_{i=1}^n(1+x_it)^s, & s>0,\\ \prod_{j=1}^n(1-x_it)^s, & s<0.\end{cases}$$

The author proved that for $s>0$,

$$\left(T_n^{[r,s]}(x+y)\right)^{\frac{1}{r}} \geqslant \left(T_n^{[r,s]}(x)\right)^{\frac{1}{r}} + \left(T_n^{[r,s]}(y)\right)^{\frac{1}{r}}, \tag{1.2}$$

and the inequality sign (1.2) is reversed for $s<0$.

Mathematics subject classification (2000): 05E05, 26E60, 26D20.

Key words and phrases: complete symmetric function, Schur-convexity, theory of majorization.

A Project Supported by Scientific Research Fund of Hunan Provincial Education Department (China) (Granted 03C427).

Paper MIA-09-52

The complete symmetric function [1; 5, p. 81; 8, p. 105] which is Whiteley's symmetric function as $s=-1$ reads as follows

$$c_r = c_r(x) = C_n^{[r]}(x) = \sum_{i_1+...+i_n=r} x_1^{i_1}...x_n^{i_n}, \tag{1.3}$$

where $i_1, i_2, ..., i_n$ are non-negative integers, and $c_0(x)=1$. It has been investigated by many mathematicians and some interesting results are obtained. See, for example, [1-3, 5, 8-10, 12] and the references cited therein. In particular, it follows, from (1.2), that (also see [12])

$$\left(C_n^{[r]}(a+b)\right)^{\frac{1}{r}} \leqslant \left(C_n^{[r]}(a)\right)^{\frac{1}{r}} + \left(C_n^{[r]}(b)\right)^{\frac{1}{r}}. \tag{1.4}$$

Which shows that the function $(c_r(x))^{\frac{1}{r}}$ is convex in R_+^n. It is obvious that $(c_r(x))^{\frac{1}{r}}$ is symmetric. Thus, by the proposition of C. 2 in [5, p. 67] (see also [9]), the function $(c_r(x))^{\frac{1}{r}}$ is Schur-convex. It is well known [5] that the function $e_r(x)$ and $\frac{e_r(x)}{e_{r-1}(x)}$ are Schur-concave in R_+^n. Thus, a problem arises naturally: whether the functions $c_r(x)$ and $\phi_r(x) = \frac{c_r(x)}{c_{r-1}(x)} (r \geqslant 1)$ are Schur-convex? This paper will be concerned with it in section 3.

On the other hand, K. V. Menon [1] defined the generalized $r-$ th order symmetric mean as follows

$$D_r(x) = D_n^{[r]}(x) = \binom{r+n-1}{n-1}^{-1} C_n^{[r]}(x), \tag{1.5}$$

where $\binom{r+n-1}{n-1} = \frac{(n+r-1)!}{(n-1)!r!}$. The author proved that for n variables and for $r=1,2,3$,

$$D_{r-1}(a)D_{r+1}(a) - D_r^2(a) \geqslant 0. \tag{1.6}$$

1979, D. W. Detemple and J. M. Robertson [2] proved that when $n=2$ the inequality (1.6) is true for all $r \in N = \{1,2,...\}$. The question whether inequality (1.6) is valid for $r \geqslant 4$ and for $n \geqslant 3$ was published in [3] and [8] and solved by Guan in [4].

By Theorem 2 of [2], one can easily prove that when $n=2$,

$$D_{r-2}(x)D_{r+2}(x) - D_{r-1}(x)D_{r+1}(x) \geqslant 0, \quad \forall \ r \geqslant 2. \tag{1.7}$$

Thus, the problem raises naturally: whether or not the inequality (1.7) holds for $n>2$. This paper will also deal with it in section 4.

REMARK 1. The inequality of Theorem 11 in [1] is written as

$$D_{r-1}(x)D_{r+1}(x) - D_{r-2}(x)D_{r+2}(x) \geqslant 0, \quad \forall \ r \geqslant 2.$$

There may be misprint. As matter of fact, let $r=2$, $x_1=1$ and $x_2=2$, simply calculation shows that $D_1(x)=\frac{3}{2}$, $D_2(x)=\frac{7}{3}$, $D_3(x)=\frac{15}{4}$, and $D_4(x)=\frac{31}{5}$. Thus,

$$D_1(x)D_3(x) - D_0(x)D_4(x) = \frac{45}{8} - \frac{31}{5} < 0.$$

The main purpose of this paper is to prove that the functions $c_r(x)$ and $\frac{c_r(x)}{c_{r-1}(x)}$ are Schur-convex in R_+^n. Some inequalities, including Ky Fan type inequality, are established by use of the theory of majorization. We also show that inequality (1.7) holds for $n > 2$.

The Schur-convex function was introduced by I. Schur in 1923 [5], and has many important applications in analytic inequalities. Hardy, Littlewood and Pólya were also interested in some inequalities that are related to Schur-convex functions [6]. The following definitions can be found in many references such as [3, 5, 8].

For fixed $n \geqslant 2$, let

$$x = (x_1, x_2, ..., x_n), \qquad y = (y_1, y_2, ..., y_n)$$

be two n-tuples of real numbers. And let

$$x_{[1]} \geqslant x_{[2]} \geqslant ... \geqslant x_{[n]}, \quad y_{[1]} \geqslant y_{[2]} \geqslant ... \geqslant y_{[n]},$$

be their ordered components.

DEFINITION 1.1. The n-tuple x is said to be majorized by y (in symbols $x \prec y$), if

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]}, \ m = 1, 2, ..., n-1; \tag{1.8}$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \tag{1.9}$$

DEFINITION 1.2. A real-valued function ϕ defined on a set $\Omega \subset R^n$ is said to be Schur-convex function on Ω if

$$x \prec y \quad \text{on} \quad \Omega \Longrightarrow \phi(x) \leqslant \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on Ω. ϕ is Schur-concave function on Ω if and only if $-\phi$ is Schur-convex function; ϕ is a strictly Schur-concave function on Ω if and only if $-\phi$ is strictly Schur-convex function on Ω.

DEFINITION 1.3. $f : R^n \to R$ is called monotonic increasing function if

$$x \leqslant y \Longrightarrow f(x) \leqslant f(y),$$

where $x \leqslant y$ implies that $x_i \leqslant y_i, i = 1, 2, ..., n$.

2. Lemmas

In order to verify our main results, the following lemmas are necessary.

LEMMA 2.1. ([5, p. 57]) *Assume that $f(x) = f(x_1, x_2, ..., x_n)$ is symmetric, and has continuous partial derivatives on I^n, where I is an open interval. Then $f : I^n \to R$ is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0 \tag{2.1}$$

on I^n. It is strictly Schur-convex if (2.1) is a strict inequality for $x_i \neq x_j$, $1 \leqslant i, j \leqslant n$.

The inequality (2.1) is commonly referred to as Schur's condition. $f : I^n \to R$ is Schur-concave if and only if the inequality sign of (2.1) is reversed. In Schur's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n, Lemma 2.1 remains true if we replace I^n by a set $A \subseteq R^n$ with the following properties ([5, p. 57]):

(i) A is convex and has a nonempty interior;

(ii) A is symmetric in the sense that $x \in A$ implies $Px \in A$ for any $n \times n$ permutation matrix P.

LEMMA 2.2. ([1]) *If* $0 < r < s$. *Then*
(i) $c_r(a)c_{s-1}(a) > c_{r-1}(a)c_s(a)$,
(ii) $(c_r(a))^{\frac{1}{r}} \geqslant (c_s(a))^{\frac{1}{s}}$.

LEMMA 2.3. ([7]) *Assume that* $x_i > 0, i = 1, 2, ..., n, \sum_{i=1}^n x_i = s$, *and* $c \geqslant s$. *Then*

$$\frac{c-x}{\frac{nc}{s}-1} = \left(\frac{c-x_1}{\frac{nc}{s}-1}, ..., \frac{c-x_n}{\frac{nc}{s}-1}\right) \prec (x_1, x_2, ..., x_n) = x.$$

LEMMA 2.4. ([7]) *Suppose that* $x_i > 0, i = 1, 2, ..., n, \sum_{i=1}^n x_i = s$, *and* $c \geqslant 0$. *Then*

$$\frac{c+x}{s+nc} = \left(\frac{c+x_1}{s+nc}, \frac{c+x_2}{s+nc}, ..., \frac{c+x_n}{s+nc}\right) \prec \left(\frac{x_1}{s}, \frac{x_2}{s}, ..., \frac{x_n}{s}\right) = \frac{x}{s}.$$

LEMMA 2.5. ([4]) *Assume that* $a = (a_1, a_2, ..., a_n)$, $a_i > 0, i = 1, 2, ..., n$, $n \geqslant 2$ *and that* $r \geqslant 1$ *is an integer. Then*

$$D_r^2(a) \leqslant D_{r-1}(a)D_{r+1}(a).$$

LEMMA 2.6. *Suppose that* $x_i > 0, i = 1, 2, ..., n$ *and let*

$$\bar{x_i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Then

$$c_r(x) = x_i c_{r-1}(x) + c_r(\bar{x_i}).$$

Proof. It is easy to see that

$$c_r(x) = \sum_{i_1+i_2+...+i_n=r} x_i^{i_1} ... x_n^{i_n} = x_i^r + x_i^{r-1}c_1(\bar{x_i}) + ... + c_r(\bar{x_i}),$$

and

$$c_{r-1}(x) = x_i^{r-1} + x_i^{r-2}c_1(\bar{x_i}) + ... + c_{r-1}(\bar{x_i}).$$

There follows that

$$c_r(x) = x_i c_{r-1}(x) + c_r(\bar{x_i}).$$

3. Schur-convexity of the functions $c_r(x)$ and $c_r(x)/c_{r-1}(x)$

In this section, we investigated Schur-convexity of the complete symmetric function $c_r(x)$ and the function $c_r(x)/c_{r-1}(x)$. As its applications, some analytic inequalities, including Ky Fan type inequality, are established.

THEOREM 3.1. *The complete symmetric function* $c_r(x) = \sum_{i_1+\ldots+i_n=r} x_1^{i_1}\ldots x_n^{i_n}$ *is a Schur-convex function in* R_+^n, *and is increasing in* $x_i, i = 1,2,\ldots,n$.

Proof. We firstly prove that $c_r(x)$ is an increasing function with respect to x_i. As matter of fact, it follows from Lemma 2.6 that

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i}.$$

Using inductive method yields that

$$\frac{\partial c_r(x)}{\partial x_i} \geqslant 0, i = 1,2,\ldots,n.$$

Which shows that $c_r(x)$ is an increasing function in $x_i, i = 1,2,\ldots,n$.

Now we prove that $c_r(x)$ is a Schur-convex function in R_+^n. It is clear that $c_r(x)$ is symmetric and have continuous partial derivatives in R_+^n. By Lemma 2.1, we only need to prove that

$$(x_i - x_j)\left(\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_r(x)}{\partial x_j}\right) \geqslant 0, i \neq j. \tag{3.1}$$

This can be obtained by induction.

(i) When $r = 2$, differentiating $c_r(x)$ with respect to x_i, we obtain

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i} = \sum_{k=1}^{n} x_k + x_i.$$

And so

$$(x_i - x_j)\left(\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_r(x)}{\partial x_j}\right) = (x_i - x_j)^2 \geqslant 0.$$

(ii) Assume that (3.1) is true for $r-1$, that is,

$$(x_i - x_j)\left(\frac{\partial c_{r-1}(x)}{\partial x_i} - \frac{\partial c_{r-1}(x)}{\partial x_j}\right) \geqslant 0, i \neq j. \tag{3.2}$$

Then, still by Lemma 2.6, it follows that

$$\frac{\partial c_r(x)}{\partial x_i} = c_{r-1}(x) + x_i \frac{\partial c_{r-1}(x)}{\partial x_i} \quad \text{and} \quad \frac{\partial c_r(x)}{\partial x_j} = c_{r-1}(x) + x_j \frac{\partial c_{r-1}(x)}{\partial x_j}.$$

Simply calculation arrives at

$$\begin{aligned}
\frac{\partial c_r(x)}{\partial x_i} - \frac{\partial c_r(x)}{\partial x_j} &= x_i \frac{\partial c_{r-1}(x)}{\partial x_i} - x_j \frac{\partial c_{r-1}(x)}{\partial x_j} \\
&= x_i \frac{\partial c_{r-1}(x)}{\partial x_i} - x_j \frac{\partial c_{r-1}(x)}{\partial x_i} + x_j \frac{\partial c_{r-1}(x)}{\partial x_i} - x_j \frac{\partial c_{r-1}(x)}{\partial x_j} \\
&= (x_i - x_j) \frac{\partial c_{r-1}(x)}{\partial x_i} + x_j \left(\frac{\partial c_{r-1}(x)}{\partial x_i} - \frac{\partial c_{r-1}(x)}{\partial x_j}\right).
\end{aligned}$$

From this and (3.2), it follows that

$$(x_i-x_j)\left(\frac{\partial c_r(x)}{\partial x_i}-\frac{\partial c_r(x)}{\partial x_j}\right) = (x_i-x_j)^2\frac{\partial c_{r-1}(x)}{\partial x_i}+x_j(x_i-x_j)\left(\frac{\partial c_{r-1}(x)}{\partial x_i}-\frac{\partial c_{r-1}(x)}{\partial x_j}\right) \geqslant 0.$$

Thus, by mathematical induction method, inequality (3.1) is true. And therefore, the proof is complete.

REMARK 2. R. F. Muirhead established the following theorem.

MUIRHEAD'S THEOREM. ([5, p. 87; 15, p. 44]) *If* $y_k > 0, k = 1,2,...,n,\ a,b \in R^n$ *and* $a \prec b$, *then*

$$\sum_{\pi} y_{\pi(1)}^{a_1}...y_{\pi(n)}^{a_n} \leqslant \sum_{\pi} y_{\pi(1)}^{b_1}...y_{\pi(n)}^{b_n},$$

where $\sum_{\pi}$ *denotes summation over the* $n!$ *permutations of* $1,2,...,n$, *and* $\pi(1)...\pi(n)$ *is a permutation of* $1,2,...,n$.

This theorem implies that for fixed $y_k (1 \leqslant k \leqslant n)$, $\phi(a) = \sum_{\pi} y_{\pi(1)}^{a_1}...y_{\pi(n)}^{a_n}$ is Schur-convex in $a \in R^n$. Thus, one can easily find that Theorem 3.1 is different from it.

THEOREM 3.2. *The function* $\phi_r(x) = \frac{c_r(x)}{c_{r-1}(x)}$ *is Schur-convex in* R_+^n, *and is increasing in* x_i, $i = 1,2,...,n$, *where* $r \geqslant 1$ *is a positive integer.*

Proof. It is obvious that $\phi_r(x)$ is symmetric and has continuous partial derivatives in R_+^n. Differentiating $\phi_r(x)$ with respect to x_i, we have

$$\frac{\partial \phi_r(x)}{\partial x_i} = \frac{1}{(c_{r-1}(x))^2}\left[c_{r-1}(x)\frac{\partial c_r(x)}{\partial x_i} - c_r(x)\frac{\partial c_{r-1}(x)}{\partial x_i}\right].$$

Using Lemma 2.6 and computing, we obtain

$$\frac{\partial \phi_r(x)}{\partial x_i} - \frac{\partial \phi_r(x)}{\partial x_j} = \frac{1}{(c_{r-1}(x))^2}\left[c_r(\bar{x}_j)\frac{\partial c_{r-1}(x)}{\partial x_i} - c_r(\bar{x}_i)\frac{\partial c_{r-1}(x)}{\partial x_i}\right].$$

Simply calculation shows that

$$\begin{aligned}\frac{\partial c_r(x)}{\partial x_i} &= c_{r-1}(x) + x_i\frac{\partial c_{r-1}(x)}{\partial x_i} = c_{r-1}(x) + x_i\left[c_{r-2}(x) + x_i\frac{\partial c_{r-2}(x)}{\partial x_i}\right] \\ &= c_{r-1}(x) + x_i c_{r-2}(x) + x_i^2\frac{\partial c_{r-2}(x)}{\partial x_i} = \cdots \\ &= c_{r-1}(x) + x_i c_{r-2}(x) + x_i^2 c_{r-3}(x) + ... + x_i^{r-2}c_1(x) + x_i^{r-1}.\end{aligned} \tag{3.3}$$

From Lemma 2.6 and (3.3), it follows that

$$\begin{aligned}\frac{\partial \phi_r(x)}{\partial x_i} &= (c_{r-1}(x)c_{r-1}(x) - c_r(x)c_{r-2}(x)) \\ &\quad + x_i((c_{r-1}(x)c_{r-2}(x) - c_r(x)c_{r-3}(x)) \\ &\quad + \cdots + x_i^{r-2}(c_{r-1}(x)c_1(x) - c_r(x)c_0(x)) + c_{r-1}(x)x_i^{r-1}.\end{aligned} \tag{3.4}$$

And so

$$
\begin{aligned}
\frac{\partial \phi_r(x)}{\partial x_i} - \frac{\partial \phi_r(x)}{\partial x_j} &= \frac{1}{(c_{r-1}(x))^2}\{[c_{r-}(x) - x_j c_{r-1}(x)][c_{r-2}(x) + x_j c_{r-3}(x) \\
&\quad + x_j^2 c_{r-4}(x) + \cdots + x_j^{r-3} c_1(x) + x_i^{r-2}] - [c_r(x) - x_i c_{r-1}(x)] \times \\
&\quad \times [c_{r-2}(x) + x_i c_{r-3}(x) + x_i^2 c_{r-4}(x) + \cdots + x_i^{r-3} c_1(x) + x_i^{r-2}]\} \\
&= \frac{1}{(c_{r-1}(x))^2}\{[c_{r-1}(x) c_{r-2}(x) - c_r(x) c_{r-3}(x)](x_i - x_j) \\
&\quad + [c_{r-1}(x) c_{r-3}(x) - c_r(x) c_{r-4}(x)](x_i^2 - x_j^2) + \cdots \\
&\quad + [c_{r-1}(x) c_1(x) - c_r(x) c_0(x)](x_i^{r-2} - x_j^{r-2}) \\
&\quad + c_{r-1}(x)(x_i^{r-1} - x_j^{r-1})\}.
\end{aligned} \tag{3.5}
$$

It follows Lemma 2.2 that

$$
\frac{c_{r-1}(x)}{c_r(x)} > \frac{c_{r-3}(x)}{c_{r-2}(x)}, \frac{c_{r-1}(x)}{c_r(x)} > \frac{c_{r-4}(x)}{c_{r-3}(x)}, \cdots, \frac{c_{r-1}(x)}{c_r(x)} > \frac{c_0(x)}{c_1(x)}. \tag{3.6}
$$

By (3.4) and (3.6), we have

$$
\frac{\partial \phi_r(x)}{\partial x_i} \geqslant 0, \quad i = 1, 2, \cdots, n.
$$

Which shows that $\phi_r(x)$ is increasing with respect to x_i $(i = 1, 2, \cdots, n)$.
It is obvious that

$$
(x_i - x_j)(x_i^k - x_j^k) \geqslant 0 \ \ (1 \leqslant k \leqslant r - 1). \tag{3.7}
$$

From (3.5), (3.6) and (3.7), it follows that

$$
(x_i - x_j)\left(\frac{\partial \phi_r(x)}{\partial x_i} - \frac{\partial \phi_r(x)}{\partial x_j}\right) \geqslant 0.
$$

Thus, by Lemma 2.1, $\phi_r(x)$ is Schur-convex in R_+^n, and therefore the proof is complete.

THEOREM 3.3. *Assume that* $x_i > 0, i = 1, 2, \cdots, n,$ *and* $\sum_{i=1}^n x_i = s, c > 0,$ *then*

(*i*)

$$
\frac{c_r(c - x)}{c_r(x)} \leqslant \left(\frac{nc}{s} - 1\right)\frac{c_{r-1}(c - x)}{c_{r-1}(x)} \ (c \geqslant s), \tag{3.8}
$$

$$(ii) \qquad \frac{c_r(c+x)}{c_r(x)} \leqslant \left(\frac{nc}{s}+1\right)\frac{c_{r-1}(c+x)}{c_{r-1}(x)}. \tag{3.9}$$

Proof. (i) By Theorem 3.2 and Lemma 2.3, we have $\phi_r\left(\frac{c-x}{\frac{nc}{s}-1}\right) \leqslant \phi_r(x)$, which implies (3.8).

(ii) From Theorem 3.2 and Lemma 2.4, it follows that $\phi_r(\frac{c+x}{s+nc}) \leqslant \phi_r(\frac{x}{s})$. Simplifying get (3.9).

The unweighted arithmetic and geometric means of x, denoted by $A_n(x)$, $G_n(x)$, respectively, are defined as follows

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}.$$

For $0 < x_i \leqslant \frac{1}{2}$, $i = 1,2,\cdots,n$, the following inequality

$$\frac{G_n(x)}{G_n(1-x)} \leqslant \frac{A_n(x)}{A_n(1-x)}, \tag{3.10}$$

commonly referred to as the Ky Fan inequality [11, p. 5] has stimulated an interest of many researchers. New proofs, improvements and generalizations of the inequality (3.10). See, for example, [13, 14] and the references cited therein. Now we shall investigate it further.

Using Theorem 3.3, one can easily establish the following results.

COROLLARY 3.1. *Suppose that* $x_i > 0$, $\sum_{i=1}^{n} x_i = s$, *and* $c \geqslant s$. *Then*

$$\begin{aligned} \frac{c_r(c-x)}{c_r(x)} &\leqslant \left(\frac{nc}{s}-1\right)\frac{c_{r-1}(c-x)}{c_{r-1}(x)} \\ &\leqslant \left(\frac{nc}{s}-1\right)^2\frac{c_{r-2}(c-x)}{c_{r-2}(x)} \\ &\leqslant \cdots \leqslant \left(\frac{nc}{s}-1\right)^r\frac{c_0(c-x)}{c_0(x)} = \left(\frac{nc}{s}-1\right)^r. \end{aligned} \tag{3.11}$$

REMARK 3. Let $c = 1$, i.e., $\sum_{i=1}^{n} x_i \leqslant 1$, we can establish the following converse Ky Fan type inequality

$$\frac{A_n(x)}{A_n(1-x)} \leqslant \left(\frac{c_r(x)}{c_r(1-x)}\right)^{\frac{1}{r}}.$$

COROLLARY 3.2. *Assume that* $x_i > 0$, $\sum_{i=1}^{n} x_i = s$, *and* $c \geqslant 0$. *Then*

$$\begin{aligned} \frac{c_r(c+x)}{c_r(x)} &\leqslant \left(\frac{nc}{s}+1\right)\frac{c_{r-1}(c+x)}{c_{r-1}(x)} \\ &\leqslant \left(\frac{nc}{s}+1\right)^2\frac{c_{r-2}(c+x)}{c_{r-2}(x)} \\ &\leqslant \cdots \leqslant \left(\frac{nc}{s}+1\right)^r\frac{c_0(c-x)}{c_0(x)} = \left(\frac{nc}{s}+1\right)^r. \end{aligned} \tag{3.12}$$

In particular, let $c = 1$, it follows from (3.12) that

$$\frac{A_n(x)}{A_n(1+x)} \leqslant \left(\frac{c_r(x)}{c_r(1+x)}\right)^{\frac{1}{r}}.$$

THEOREM 3.4. *Assume that* $0 < x_i \leqslant \frac{1}{2}, i = 1, 2, \cdots, n$. *Then*

$$\begin{aligned} \frac{c_n(1-x)}{c_n(x)} \geqslant \cdots \geqslant \frac{c_r(1-x)}{c_r(x)} \geqslant \frac{c_{r-1}(1-x)}{c_{r-1}(x)} \\ \geqslant \cdots \geqslant \frac{c_1(1-x)}{c_1(x)} = \frac{A_n(1-x)}{A_n(x)}. \end{aligned} \tag{3.13}$$

Proof. By Theorem 3.2, one see that $\phi_r(x) = \frac{c_r(x)}{c_{r-1}(x)}$ is an increasing function in $A = \{x = (x_1, x_2, \cdots, x_n) | 0 < x_i < 1\}$. This together with the relation $1 - x \geqslant x$ yields that

$$\phi_r(1-x) \geqslant \phi_r(x).$$

Which shows that

$$\frac{c_r(1-x)}{c_r(x)} \geqslant \frac{c_{r-1}(1-x)}{c_{r-1}(x)}.$$

Thus, (3.13) holds and so the proof is complete.

4. Note on the generalized $r-$ th order symmetric mean

In this section, we prove that the inequality (1.7) holds for $n > 2$, and furthermore generalizes (1.7).

THEOREM 4.1. *Assume that* $x_i > 0, i = 1, 2, \cdots, n,\ n \geqslant 2$, *then*

$$D_{r-2}(x)D_{r+2}(x) - D_{r-1}(x)D_{r+1}(x) \geqslant 0, \quad \forall\ r \geqslant 2. \tag{4.1}$$

Proof. Using Lemma 2.5, we can obtain

$$D_r^2(x) \leqslant D_{r-1}(x)D_{r+1}(x), D_{r-1}^2(x) \leqslant D_{r-2}(x)D_r(x), D_{r+1}^2(x) \leqslant D_{r+1}(x)D_{r+2}(x).$$

Simply calculation shows that

$$D_{r-2}(x)D_{r+2}(x) - D_{r-1}(x)D_{r+1}(x) \geqslant 0.$$

And so the proof is complete.

We can also establish the following more general result than Theorem 4.1.

THEOREM 4.2. *Assume that* $x_i > 0, i = 1, 2, \cdots, n,\ n \geqslant 2$, *then*

$$D_{r-s-1}(x)D_{r+s+1}(x) - D_{r-s}(x)D_{r+s}(x) \geqslant 0 \ \ (0 \leqslant s < r). \tag{4.2}$$

Proof. From Lemma 2.5, it follows that

$$\begin{gathered}
D^2_{r-s}(x) \leqslant D_{r-s-1}(x)D_{r-s+1}(x),\\
D^2_{r-s+1}(x) \leqslant D_{r-s}(x)D_{r-s+2}(x),\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
D^2_{r+s-1}(x) \leqslant D_{r+s-2}(x)D_{r+s}(x),\\
D^2_{r+s}(x) \leqslant D_{r+s-1}(x)D_{r+s+1}(x).
\end{gathered}$$

Multiplying the two hands of the above inequalities respectively, and simplifying arrives at (4.2). Thus, the proof is complete.

REMARK 4. In [1] the following problem was posed

$$D_{r-s}(x)D_{r+s}(x) - D_{r-s-1}(x)D_{r+s+1}(x) \geqslant 0 \quad (0 \leqslant s < r).$$

By Remark 1, we find that it is mistake.

Acknowledgment. The author is very grateful to the referee for her (his) valuable suggestions for the improvement of this paper.

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(Received February 15, 2004)

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