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HOW DEVIANT CAN YOU BE? THE COMPLETE SOLUTION

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Abstract. We consider the problem of optimal deterministic lower and upper bounds on arbitrary linear combinations of order statistics centered about the sample mean in units generated by the sample central absolute moments of various orders. The signs of the evaluations depend merely on the coefficients of the linear combinations. Hitherto all the positive upper and negative lower bounds have been established as well as a few exceptional positive lower and negative upper ones. In the paper, we complete the solution by presenting all the positive lower bounds and negative upper bounds and respective samples attaining them. We also specify the general results by considering several important examples.

1. Introduction

In his famous paper, Samuelson (1968) stated the following problem: How a single observation can be deviant from the sample mean in the standard deviation units s_2, for a fixed deterministic sample $\tilde{x} = (x_1, \dots, x_n)$ of size n, with the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and positive sample variance $s_2^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$? Samuelson (1968) presented the following solution

$$\sup_{\tilde{x}} \max_{1 \leqslant i \leqslant n} \frac{|x_i - \bar{x}|}{s_2} = \sup_{\tilde{x}} \frac{x_{n:n} - \bar{x}}{s_2} = \sqrt{n-1}, \tag{1.1}$$

with the respective attainability conditions

$$x_{1:n} = \dots = x_{n-1:n} = \bar{x} - s_2/\sqrt{n-1} < x_{n:n} = \bar{x} + s_2\sqrt{n-1}.$$

Here and later on $x_{1:n} \leqslant \dots \leqslant x_{n:n}$ denote the order statistics based on the sample $x_1, \dots, x_n$. This immediately implies the following lower bound

$$\inf_{\tilde{x}} \frac{x_{1:n} - \bar{x}}{s_2} = -\sqrt{n-1},$$

which is attained by

$$x_{1:n} = \bar{x} - s_2\sqrt{n-1} < x_{2:n} = \dots = x_{n:n} = \bar{x} + s_2/\sqrt{n-1}.$$

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The paper stimulated extensive investigations on the subject. Some earlier results, including Thompson (1935), Scott (1936), Nair (1948), David et al. (1954), and Thomson (1955), were rediscovered, and bound (1.1) was found in Thompson (1935) and Scott (1936). Alternative proofs and various generalizations were presented in Mallows and Richter (1969), Boyd (1971), Hawkins (1971), Koop (1972), Beesack (1973, 1976), Arnold (1974, 1985), Arnold and Groeneveld (1974, 1978, 1979, 1981), Dwass (1975), O'Reilly (1975, 1976), Prescott (1977), Loynes (1979), Wolkowicz and Styan (1979), Smith (1980), Fahmy and Proschan (1981), Nagaraja (1981), Groeneveld (1982), Mărgĕritescu and Vŏda (1983), Gonzacenco and Mărgăritescu (1987), Mărgăritescu (1987), David (1988), Mărgăritescu and Nicolae (1990), Gonzacenco, Mărgăritescu and Vŏda (1992), Olkin (1992), and Rychlik (1992, 1993). Comprehensive reviews can be found in Arnold and Balakrishnan (1989) and Rychlik (1998). The most natural generalizations consist in extending the results to other linear combinations of order statistics, called L-statistics and replacing the standard deviation units s_2 by more general ones

$$s_p = \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}|^p \right)^{1/p}, \leqslant p < +\infty, \tag{1.2}$$

with the limiting value

$$s_\infty = \max_{1 \leqslant i \leqslant n} |x_i - \bar{x}| = \max\{\bar{x} - x_{1:n}, x_{n:n} - \bar{x}\}. \tag{1.3}$$

Except for (1.1), Scott (1936) established s_2-bounds for $x_{n-1:n} - \bar{x}$. Respective results for arbitrary order statistics were derived independently by Boyd (1971) and Hawkins (1971). The sample range and the difference between the second greatest order statistic and the sample minimum were examined by Nair (1948), and David et al. (1954), respectively. Fahmy and Proschan (1981) solved the problem for arbitrary differences of order statistics. Mallows and Richter (1969) studied selection differentials, and David (1988) considered L-statistics with nondecreasing coefficients $c_1 \leqslant \ldots \leqslant c_n$. Bounds for single order statistics in general dispersion units, including (1.2), were presented by Beesack (1973, 1976). Results for specific L-statistics, expressed in terms of scale units different from the standard deviation were derived by Arnold and Groeneveld (1981), Groeneveld (1982), Arnold (1985), and Mărgăritescu and Nicolae (1990).

The most general hitherto known deterministic bounds on L-statistics are due to Rychlik (1992). They are based on the projection of the coefficient vector $\tilde{c} = (c_1 \ldots, c_n) \in R^n$ onto the convex cone of nondecreasing vectors in the n-dimensional Euclidean space norm. Given $\tilde{c} \in R^n$, we define the subset of indices $\{j_1, \ldots, j_M\} \subset \{1, \ldots, n\}$ and the nondecreasing vector of projection $\underline{\tilde{c}} = (\underline{c}_1, \ldots, \underline{c}_n) \in R^n$ by means of the following algorithm

$$\begin{aligned} j_0 &= 0, \\ j_m &= \min\left\{ j_{m-1} < j \leqslant n : \frac{1}{j - j_{m-1}} \sum_{r=j_{m-1}+1}^{j} c_r = \min_{j_{m-1} < k \leqslant n} \frac{1}{k - j_{m-1}} \sum_{r=j_{m-1}+1}^{k} c_r \right\}, \end{aligned} \tag{1.4}$$

$m = 1, \ldots, M$, for some $1 \leqslant M \leqslant n$ such that $j_M = n$, and

$$\underline{c}_i = \frac{1}{j_m - j_{m-1}} \sum_{r=j_{m-1}+1}^{j_m} c_r, i = j_{m-1}+1, \dots, j_m, \ m = 1, \dots, M. \tag{1.5}$$

In the first step, the algorithm determines the smallest partial mean of the first elements of $\tilde{c}$, and the smallest index j_1 which attains the minimum. It defines the first identical j_1 elements of $\tilde{\underline{c}}$ as the minimal partial mean. In the next step, the procedure is performed for the truncated sequence $(c_{j_1+1}, \dots, c_n)$. It is terminated in $1 \leqslant M \leqslant n$ steps if $j_M = n$. Alternative graphical construction of $\tilde{\underline{c}}$ consists in drawing the greatest convex minorant of the points $(0,0), (1, c_1), \dots, (j, \sum_{r=1}^{j} c_r), \dots, (n, \sum_{r=1}^{n} c_r)$. The minorant is a broken line with the breaking points at some integers contained in $\{j_1, \dots, j_{M-1}\}$, and $\underline{c}_i$ amounts to the slope of the line in the interval $(i-1, i)$ for $i = 1, \dots, n$. Algorithm (1.4), (1.5) is a particular form of the pool adjacent violators algorithm (PAVA) of isotonic regression, which has numerous applications in order restricted statistical inference (see, e.g., Barlow et al (1972), and Robertson et al (1988)). Rychlik (1992) proved that for arbitrary $\tilde{c}, \tilde{x} \in R^n$, we have

$$\sum_{i=1}^{n} c_i x_{i:n} \leqslant \sum_{i=1}^{n} \underline{c}_i x_{i:n} \tag{1.6}$$

and the equality holds in (1.6) iff

$$x_{j_{m-1}+1:n} = \dots = x_{j_m}, \ m = 1 \dots, M. \tag{1.7}$$

He applied (1.6) for establishing general bounds

$$\sum_{i=1}^{n} c_i (x_{i:n} - \bar{x}) \leqslant \|\tilde{\underline{c}} - \underline{c}_* \tilde{1}\|_* \|\tilde{x} - \bar{x}\tilde{1}\|, \tag{1.8}$$

where $\|\cdot\|$ and $\|\cdot\|_*$ denote arbitrary permutation invariant norm in R^n and its conjugate, respectively, $\tilde{x} - \bar{x}\tilde{1} = (x_1 - \bar{x}, \dots, x_n - \bar{x})$, $\tilde{\underline{c}} - \underline{c}_* \tilde{1} = (\underline{c}_1 - \underline{c}_*, \dots, \underline{c}_n - \underline{c}_*)$, where $\underline{c}_* \tilde{1}$ is the projection of $\tilde{\underline{c}}$ defined by (1.4) and (1.5) onto the subspace of constant vectors in the conjugate norm. In particular, combining (1.6) with the Hölder inequality (see, e.g., Mitrinović (1970, Theorem 2.8.1, pp. 50–51)), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} c_i (x_{i:n} - \bar{x}) &\leqslant \sum_{i=1}^{n} \underline{c}_i (x_{i:n} - \bar{x}) = \sum_{i=1}^{n} (\underline{c}_i - \underline{c}_q)(x_{i:n} - \bar{x}) \\
&\leqslant \left(\sum_{i=1}^{n} |\underline{c}_i - \underline{c}_q|^q \right)^{1/q} \left(\sum_{i=1}^{n} |x_{i:n} - \bar{x}|^p \right)^{1/p} \\
&= n^{1/p} \left(\sum_{i=1}^{n} |\underline{c}_i - \underline{c}_q|^q \right)^{1/q} s_p \\
&= U_p(\tilde{c}) s_p,
\end{aligned} \tag{1.9}$$

say, where $1<p<\infty$, $q=p/(p-1)$, and $\underline{c}_q$ minimizes $\|\underline{\tilde{c}}-c\tilde{1}\|_q^q=\sum_{i=1}^n|\underline{c}_i-c|^q$, $c\in R$. If

$$\min_{1\leqslant j\leqslant n-1}\frac{1}{j}\sum_{r=1}^{j}c_r<\frac{1}{n}\sum_{r=1}^{n}c_r, \tag{1.10}$$

then $M\geqslant 2$, and the projection vector $\underline{\tilde{c}}$ is nonconstant. Furthermore, $\underline{c}_q\in(\underline{c}_1,\underline{c}_n)$ is the unique solution to

$$\sum_{i=1}^n|\underline{c}_i-c|^{q-1}\mathrm{sgn}\{\underline{c}_i-c\}=0,$$

and the upper bound in (1.9) is positive. The equality in the latter inequality of (1.9) holds iff

$$x_{i:n}=\bar{x}+s_pn^{1/p}\frac{|\underline{c}_i-\underline{c}_q|^{q-1}\mathrm{sgn}\{\underline{c}_i-\underline{c}_q\}}{\left(\sum_{r=1}^n|\underline{c}_r-\underline{c}_q|^q\right)^{1/p}},\ i=1\ldots,n, \tag{1.11}$$

for arbitrary choices of $\bar{x}\in R$ and $s_p\geqslant 0$. Also, (1.11) satisfy conditions (1.7) providing the equality in the former inequality as well. In the case

$$\frac{1}{j}\sum_{r=1}^{j}c_r\geqslant\frac{1}{n}\sum_{r=1}^{n}c_r,\ j=1,\ldots,n-1, \tag{1.12}$$

opposite to (1.10), the projection is constant equal to $\underline{\tilde{c}}=\bar{c}\tilde{1}=(\bar{c},\ldots,\bar{c})$. Then clearly $\underline{c}_*=\underline{c}_q=\bar{c}$, and the projection bounds of (1.8) and (1.9) amount to 0. Moreover, (1.11) cannot be defined, because the denominator of the right-hand side vanishes then. Therefore the projection method provides the sharp strictly positive upper bounds (1.9) with the unique attainability conditions (1.11) up to arbitrary choice of location and scale parameters $\bar{x}\in R$ and $s_p\geqslant 0$. It can also be noticed that the method provides all the negative lower bounds $L_p(\tilde{c})$. Indeed, setting $y_i=2\bar{x}-x_i$, $c_i'=c_{n+1-i}$, $i=1,\ldots,n$, for arbitrary $\tilde{x},\tilde{c}\in R^n$, we obtain $\bar{y}=\bar{x}, s_p(\tilde{y})=s_p(\tilde{x})$, $y_{i:n}=2\bar{x}-x_{n+1-i:n}$, $i=1,\ldots,n$, so that

$$\sum_{i=1}^n c_i\frac{y_{i:n}-\bar{y}}{s_p(\tilde{y})}=\sum_{i=1}^n c_i\frac{\bar{x}-x_{n+1-i:n}}{s_p(\tilde{x})}=-\sum_{i=1}^n c_{n+1-i}\frac{x_{i:n}-\bar{x}}{s_p(\tilde{x})}$$

and

$$L_p(\tilde{c})=-U_p(\tilde{c}) \tag{1.13}$$

hold. Accordingly $L_p(\tilde{c})<0$ iff

$$\min_{1\leqslant j\leqslant n-1}\frac{1}{j}\sum_{r=n+1-j}^{n}c_r<\frac{1}{n}\sum_{r=1}^{n}c_r.$$

The purpose of this paper is to determine all the nonpositive upper bounds and nonnegative lower bounds for centered L-statistics in s_p units defined in (1.2) and (1.3). Since (1.13) holds for arbitrary vectors of coefficients, we restrict our theoretical investigations to the upper bounds. General results are presented in Section 2, and they are specified for some important examples in Section 3. We also mention that optimal negative upper and positive lower bounds for some particular L-statistics were already presented in the literature. Boyd (1971) and Hawkins (1971) determined negative upper bound on the sample minimum, and corresponding positive lower bound for the sample maximum in the standard deviation units. Analogous lower bound on the sample range is due to Thomson (1955). A generalization to the lower and upper selection differentials and their differences can be found Mallows and Richter (1969). Fahmy and Proschan (1981) noted that the zero lower bound for the differences of order statistics cannot be improved except for the sample range. All the particular inequalities can be concluded from the statements presented below. Finally, we point out that, independently of the statistical interpretations, our results provide sharp analytic evaluations for arbitrary linear combinations of elements of monotone sequences.

2. Main results

Throughout this section we assume that $\tilde{c} \in R^n$ is fixed so that (1.12) holds. Let

$$\mathscr{X} = \{\tilde{x} = (x_1, \ldots, x_n) : s_p > 0\} = \{\tilde{x} : x_{1:n} < x_{n:n}\} \subset R^n \tag{2.1}$$

denote the set of all nonconstant vectors in R^n. We aim at evaluating

$$U_p(\tilde{c}) = \sup_{\tilde{x} \in \mathscr{X}} \sum_{i=1}^{n} c_i \frac{x_{i:n} - \bar{x}}{s_p}, \ 1 \leqslant p \leqslant +\infty.$$

We show that the bounds are attained by some $\tilde{x}$ with arbitrarily fixed location $\bar{x} \in R$ and scale $s_p > 0$. We first consider $1 \leqslant p < \infty$.

THEOREM 1. *Set*

$$U_p^{(j)}(\tilde{c}) = \left[\frac{n^{p+1}}{j(n-j)^p + j^p(n-j)}\right]^{1/p} \sum_{i=1}^{j} (c_i - \bar{c}), \ j = 1, \ldots, n-1, \tag{2.2}$$

and take $\tilde{x}^{(j)} \in \mathscr{X}$, $j = 1, \ldots, n-1$, *satisfying*

$$x_{i:n}^{(j)} = \begin{cases} \bar{x} - s_p(n-j)\left[\frac{n}{j(n-j)^p + j^p(n-j)}\right]^{1/p}, & i = 1, \ldots, j, \\ \bar{x} + s_p j \left[\frac{n}{j(n-j)^p + j^p(n-j)}\right]^{1/p}, & i = j+1, \ldots, n, \end{cases} \tag{2.3}$$

for some fixed $\bar{x} \in R$ *and* $s_p > 0$. *Suppose that* $1 \leqslant j_* \leqslant n-1$ *minimizes* (2.2). *Then*

$$U_p(\tilde{c}) = \sup_{\tilde{x} \in \mathscr{X}} \sum_{i=1}^{n} c_i \frac{x_{i:n} - \bar{x}}{s_p} = \sum_{i=1}^{n} c_i \frac{x_{i:n}^{(j_*)} - \bar{x}}{s_p} = -U_p^{(j_*)}(\tilde{c}). \tag{2.4}$$

Proof. We easily check that all $\tilde{x}^{(j)}$ defined in (2.3) have the sample mean $\bar{x}$ and pth absolute central moment s_p^p. We also have

$$\begin{aligned} \sum_{i=1}^{n} c_i \frac{x_{i:n}^{(j)} - \bar{x}}{s_p} &= \left[-(n-j)\sum_{i=1}^{j} c_i + j \sum_{i=j+1}^{n} c_i\right] \left[\frac{n}{j(n-j)^p + j^p(n-j)}\right]^{1/p} \\ &= -U_p^{(j)}(\tilde{c}) \leqslant -U_p^{(j_*)}(\tilde{c}) \leqslant 0, \end{aligned}$$

by definition and (1.12). Thus it suffices to show that

$$\sum_{i=1}^{n} c_i \frac{x_{i:n}-\bar{x}}{s_p} \leqslant -U_p^{(j_*)}(\tilde{c}), \ \tilde{x} \in \mathscr{X}. \tag{2.5}$$

If

$$\frac{1}{j_*}\sum_{i=1}^{j_*} c_i = \frac{1}{n}\sum_{i=1}^{n} c_i \tag{2.6}$$

for some $1 \leqslant j_* \leqslant n-1$, then $U_p^{(j_*)}(\tilde{c}) = 0$. On the other hand, (1.6) and (1.12) imply

$$\sum_{i=1}^{n} c_i \frac{x_{i:n}-\bar{x}}{s_p} \leqslant \sum_{i=1}^{n} (\underline{c}_i - \bar{c}) \frac{x_{i:n}-\bar{x}}{s_p} = 0, \ \tilde{x} \in \mathscr{X}, \tag{2.7}$$

because $\underline{c}_i = \bar{c}$, $i = 1, \ldots, n$.

More elaborate arguments are necessary in the case

$$\frac{1}{j}\sum_{i=1}^{j} c_i > \frac{1}{n}\sum_{i=1}^{n} c_i, \ j = 1, \ldots, n-1. \tag{2.8}$$

We obtain (2.7) as well, but here algorithm (1.4) and (1.5) terminates in one step, and, by (1.7), the equality is attained iff $x_{1:n} = \ldots = x_{n:n}$. Therefore we have

$$T_{\tilde{c}}(\tilde{x}) = \sum_{i=1}^{n} c_i \frac{x_{i:n}-\bar{x}}{s_p} = \sum_{i=1}^{n} d_i \frac{x_{i:n}-\bar{x}}{s_p} < 0, \ \tilde{x} \in \mathscr{X}, \tag{2.9}$$

with $d_i = c_i - \bar{c}$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{j} d_i > \sum_{i=1}^{n} d_i = 0, \ j = 1 \ldots, n-1. \tag{2.10}$$

Consider the closed and bounded subset

$$\mathscr{X}_0 = \left\{ \tilde{x} \in R^n : \sum_{i=1}^{n} x_i = 0, \sum_{i=1}^{n} |x_i|^p = n \right\} \tag{2.11}$$

of standardized elements of (2.1). Continuous function (2.9) restricted to (2.11) takes on the form

$$T_{\tilde{c}}(\tilde{x}) = \sum_{i=1}^{n} d_i x_{i:n}, \ \tilde{x} \in \mathscr{X}_0, \tag{2.12}$$

and attains its finite and strictly negative extremes there. Evidently, these extremes coincide with the general ones $-\infty < L_p(\tilde{c}) \leqslant U_p(\tilde{c}) < 0$, because $T_{\tilde{c}}(\tilde{x}) = T_{\tilde{c}}(\tilde{x}_0)$ for arbitrary $\tilde{x} \in \mathscr{X}$ with $\tilde{x}_0 = ((x_1-\bar{x})/s_p, \ldots, (x_n-\bar{x})/s_p) \in \mathscr{X}_0$. Accordingly, it suffices to maximize (2.12) over (2.11).

Take arbitrary $a \in [-U_p(\tilde{c}); -L_p(\tilde{c})] \subset (0, +\infty)$. For every $\tilde{x} \in \mathscr{X}_0$ satisfying $T_{\tilde{c}}(\tilde{x}) = -a$, define a nondecreasing sequence $\tilde{y} \in R^n$ by $y_i = y_i(\tilde{x}) = x_{i:n}/a$, $i = 1, \dots, n$. Then

$$\sum_{i=1}^{n} y_i = 0, \tag{2.13}$$

$$\sum_{i=1}^{n} d_i y_i = -1, \tag{2.14}$$

and

$$\sum_{i=1}^{n} |y_i|^p = \frac{n}{a^p} \in \left[\frac{n}{|L_p(\tilde{c})|^p}, \frac{n}{|U_p(\tilde{c})|^p}\right] \subset (0, +\infty).$$

Our original problem will be solved once we determine

$$\max \left\{ \sum_{i=1}^{n} |y_i|^p : \tilde{y} \in \mathscr{Y} \right\} \tag{2.15}$$

(which amounts to $n/|U_p(\tilde{c})|^p$), where

$$\mathscr{Y} = \left\{ \tilde{y} \in R^n : y_1 \leqslant \dots \leqslant y_n, \sum_{i=1}^{n} y_i = 0, \sum_{i=1}^{n} d_i y_i = -1 \right\}$$

for fixed $d_1, \dots, d_n$ satisfying (2.10). To this end, we change the variables

$$\begin{aligned} z_0 &= y_1 \in R, \\ z_i &= y_{i+1} - y_i \geqslant 0, \ i = 1, \dots, n-1. \end{aligned}$$

The inverse transformation is

$$y_i = \sum_{r=0}^{i-1} z_r, \ i = 1, \dots, n. \tag{2.16}$$

Combining (2.16) with (2.10), (2.13), and (2.14), yields

$$\sum_{i=1}^{n} y_i = \sum_{i=0}^{n-1} (n-i) z_i = 0 \tag{2.17}$$

$$\sum_{i=1}^{n} d_i y_i = \sum_{i=0}^{n-1} \left(\sum_{r=i+1}^{n} d_r \right) z_i = -\sum_{i=1}^{n} D_i z_i = -1,$$

where

$$D_i = -\sum_{r=i+1}^{n} d_r = \sum_{r=1}^{i} d_r = \sum_{r=1}^{i} (c_r - \bar{c}) > 0, \ i = 1, \dots, n-1. \tag{2.18}$$

By (2.16) and (2.17) we get

$$\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{1-n}{n} & \frac{2-n}{n} & \frac{3-n}{n} & \dots & \dots & \frac{j-n}{n} & \frac{j-n+1}{n} & \dots & -\frac{2}{n} & -\frac{1}{n} \\ \frac{1}{n} & \frac{2-n}{n} & \frac{3-n}{n} & \dots & \dots & \frac{j-n}{n} & \frac{j-n+1}{n} & \dots & -\frac{2}{n} & -\frac{1}{n} \\ \frac{1}{n} & \frac{2}{n} & \frac{3-n}{n} & \dots & \dots & \frac{j-n}{n} & \frac{j-n+1}{n} & \dots & -\frac{2}{n} & -\frac{1}{n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \frac{j-n}{n} & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \dots & \frac{j-1}{n} & \frac{j}{n} & \frac{j-n+1}{n} & \dots & -\frac{2}{n} & -\frac{1}{n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{n} & \frac{2}{n} & \frac{3}{n} & \dots & \dots & \frac{j}{n} & \frac{j-n+1}{n} & \dots & \frac{n-2}{n} & \frac{n-1}{n} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_{n-1} \end{bmatrix}$$

We denote the $n \times (n-1)$ matrix by A, and its columns by $A_1, \dots, A_{n-1}$, respectively. We restate problem (2.15) as

$$\max \left\{ \|A\tilde{z}\|_p^p = \sum_{i=1}^{n} |(Az)_i|^p : \tilde{z} \in \mathscr{X} \right\}$$

where

$$\mathscr{X} = \left\{ \tilde{z} \in R^{n-1} : z_i \geqslant 0,\ i = 1, \dots, n-1,\ \sum_{i=1}^{n} D_i z_i = 1 \right\}, \tag{2.19}$$

and $D_i, i = 1, \dots, n-1$, are fixed positive numbers defined in (2.18). Obviously $\tilde{z} \mapsto \|A\tilde{z}\|_p$ is a convex function, and (2.19) is a closed convex and bounded body in R^{n-1}. A convex function attains its maximum over a closed bounded and convex set at some extreme point of the set (see, e.g., Rockafellar (1970, p. 344, Corollary 32.3.1)). Here the function is nonnegative so that raising it to power p does not affect the maximizers. We also easily see that the extreme points of (2.19) amount to $\frac{1}{D_i}\tilde{e}_i, i = 1, \dots, n-1$, where $\tilde{e}_i$ denotes the ith standard versor in R^{n-1}. Therefore

$$\begin{aligned} \frac{n}{|U_p(\tilde{c})|^p} &= \max_{\tilde{z} \in \mathscr{X}} \|A\tilde{z}\|_p^p \\ &= \max_{1 \leqslant j \leqslant n-1} \|\frac{1}{D_j} A_j\|_p^p \\ &= \max_{1 \leqslant j \leqslant n-1} \left[\frac{j(n-j)^p + (n-j)j^p}{(nD_j)^p} \right]. \end{aligned}$$

This, together with (2.2) and (2.18), establish (2.5) and complete the proof. □

In contrast to evaluations (1.9), valid under (1.10), ordered samples attaining bounds (2.4) are not necessarily uniquely defined. Firstly, we observe that (2.2) can be minimized by different $1 \leqslant j \leqslant n-1$. It is easy to determine coefficients $c_1, \dots, c_n$ so that all the $U_p^{(j)}(\tilde{c})$, $j = 1, \ldots, n-1$, have arbitrarily chosen nonnegative values. We first use (2.2) to define respective $D_j = \sum_{i=1}^{j}(c_i - \bar{c})$, $j = 1, \ldots, n-1$, then calculate the increments $c_i - \bar{c}$, $i = 1, \ldots, n$, and possibly add an arbitrary constant $\bar{c}$ at the end. Maximizers different from (2.3) are possible under (2.6) provided that at least two extreme points (2.3) attain the zero bound. Then $M > 2$ and (2.2) amounts to zero for all $j_* = j_1, \ldots, j_{M-1}$. The bound is attained by any $\tilde{x} \in \mathscr{X}$ satisfying

$$x_{j_{m-1}+1:n} = \ldots = x_{j_m:n} = a_m, \; m = 1, \ldots, M, \tag{2.20}$$

with

$$a_1 \leqslant \ldots \leqslant a_m, \tag{2.21}$$

$$\sum_{i=1}^{M}(j_m - j_{m-1})a_m = n\bar{x}, \tag{2.22}$$

$$\sum_{i=1}^{M}(j_m - j_{m-1})|a_m - \bar{x}|^p = ns_p^p. \tag{2.23}$$

If $M = 2$, then (2.20) to (2.23) determine a unique solution.

Negative bounds in case (2.8) cannot be achieved by points different from (2.3). This were possible for convex combinations of extreme maximizers under condition that

$$\left\| \frac{\alpha}{D_i}A_i + \frac{1-\alpha}{D_j}A_j \right\|_p = \frac{\alpha}{D_i}\,\|A_i\|_p + \frac{1-\alpha}{D_j}\,\|A_j\|_p \tag{2.24}$$

for some $i \neq j$ and $0 < \alpha < 1$. If $p > 1$, the equality conditions in the Minkowski inequality (cf, e.g., Mitrinović (1970, Theorem 2.9.1, p. 55)) force A_i and A_j to be proportional. This is impossible, because each A_j has the first j values equal to $\frac{j-n}{n} < 0$, and the other $n - j$ ones are $\frac{j}{n} > 0$. In the case $p = 1$, (2.24) holds if the signs of consecutive elements of A_i coincide with their counterparts for A_j, which is also impossible. Observe that (2.2) has much simpler representations in the most interesting cases $p = 1$ and $p = 2$

$$U_1^{(j)}(\tilde{c}) = \frac{n^2}{2j(n-j)}\sum_{i=1}^{j}(c_i - \bar{c}), \; j = 1, \ldots, n-1,$$

$$U_2^{(j)}(\tilde{c}) = \frac{n}{\sqrt{j(n-j)}}\sum_{i=1}^{j}(c_i - \bar{c}), \; j = 1, \ldots, n-1.$$

Slightly modifying arguments of the proof, we derive analogous bounds for $p = +\infty$. Verification of details is left to the reader.

THEOREM 2. *Put*

$$U_\infty^{(j)}(\tilde{c}) = \frac{n}{\max\{j, n-j\}}\sum_{i=1}^{j}(c_i - \bar{c}), \; j = 1, \ldots, n-1, \tag{2.25}$$

and choose $\tilde{x}^{(j)} \in \mathscr{X}$, $j = 1, \ldots, n-1$, *satisfying*

$$x_{i:n}^{(j)} = \begin{cases} \bar{x} - s_\infty, & i = 1, \ldots, j, \\ \bar{x} + s_\infty \frac{j}{n-j}, & i = j+1, \ldots, n, \end{cases} \quad 1 \leqslant j \leqslant \frac{n}{2}, \tag{2.26}$$

$$x_{i:n}^{(j)} = \begin{cases} \bar{x} - s_\infty \frac{n-j}{j}, & i = 1, \ldots, j, \\ \bar{x} + s_\infty, & i = j+1, \ldots, n, \end{cases} \quad \frac{n}{2} \leqslant j \leqslant n-1. \tag{2.27}$$

If $1 \leqslant j_* \leqslant n-1$ *minimizes* (2.25)*, then*

$$U_\infty(\tilde{c}) = \sup_{\tilde{x} \in \mathscr{X}} \sum_{i=1}^{n} \frac{x_{i:n} - \bar{x}}{s_\infty} = \sum_{i=1}^{n} c_i \frac{x_{i:n}^{(j_*)} - \bar{x}}{s_\infty} = -U_\infty^{(j_*)}(\tilde{c}). \tag{2.28}$$

If (2.6) holds, the attainability conditions of zero bound are (2.20) to (2.22) with (2.23) replaced by

$$\max\{\bar{x} - a_1, a_m - \bar{x}\} = s_\infty.$$

Under (2.8), we should check the requirements for (2.24) with $p = \infty$. The supremum norm of the sum of two vectors amounts to the sum of the norms iff both the vectors attain their maximal absolute value at a common coordinate, and the signs of the respective values coincide. In particular, this property holds for the pairs A_i, A_j if either $i, j \leqslant \frac{n}{2}$ or $i, j \geqslant \frac{n}{2}$. This means that if (2.25) is minimized at some $1 \leqslant j_1^* < \ldots < j_k^* \leqslant n-1$, then bound (2.28) is attained by convex combinations of (2.26) for some $1 \leqslant j_i^* \leqslant \frac{n}{2}$ and those of (2.27) with $\frac{n}{2} \leqslant j_i^* \leqslant n-1$.

3. Special cases

We specify the general results for some exemplary L-statistics of significant interest. We confine ourselves to presenting lower bounds, because positive L-statistics, especially ones used in estimating dispersion parameters, are more popular than negative ones.

EXAMPLE 1. Sample maximum $x_{n:n}$ with the coefficient vector $\tilde{c} = \tilde{e}_n = (0, \ldots, 0, 1)$.

In order to get the lower bounds, we transform it into $\tilde{c}' = \tilde{e}_1 = (1, 0, \ldots, 0)$ and determine

$$L_p^{(j)}(\tilde{e}_n) = U_p^{(j)}(\tilde{e}_1) = \left[\frac{n(n-j)^p}{j(n-j)^p + j^p(n-j)} \right]^{1/p}, \quad 1 \leqslant p < \infty,$$

$$L_\infty^{(j)}(\tilde{e}_n) = U_\infty^{(j)}(\tilde{e}_1) = \min\left\{ \frac{n-j}{j}, 1 \right\}.$$

The former is decreasing in j, and so

$$\frac{x_{n:n} - \bar{x}}{s_p} \geqslant \left[\frac{n}{(n-1)^p + n - 1} \right]^{1/p}, \quad 1 \leqslant p < +\infty, \tag{3.1}$$

and the equality is achieved by (2.3) with $j^* = n-1$. The latter implies

$$\frac{x_{n:n} - \bar{x}}{s_\infty} \geqslant \frac{1}{n-1}, \tag{3.2}$$

and the equality holds for (2.27) with any $\frac{n}{2} \leqslant j \leqslant n-1$. Especially, for $p = 1$ and $p = 2$ we have

$$\frac{x_{n:n} - \bar{x}}{s_1} \geqslant \frac{n}{2(n-1)}, \tag{3.3}$$

$$\frac{x_{n:n} - \bar{x}}{s_2} \geqslant \frac{1}{\sqrt{n-1}}. \tag{3.4}$$

Negatives of (3.1) to (3.4) are the respective upper bounds for the sample minimum. Inequality (3.1) was derived by Beesack (1973), whereas (3.4) independently by Boyd (1971) and Hawkins (1971).

EXAMPLE 2. Differences of order statistics $x_{k:n} - x_{j:n}$, $1 \leqslant j < k \leqslant n$.

Here $\tilde{c} = \tilde{e}_k - \tilde{e}_j$, $\tilde{c}' = \tilde{e}_{n+1-k} - \tilde{e}_{n+1-j}$, $\bar{c} = 0$, and so

$$D_i' = \sum_{r=1}^{i} c_r' = \begin{cases} 0, & i = 1, \dots, n-k, \\ 1, & i = n+1-k, \dots, n-j, \\ 0, & i = n+1-j, \dots, n-1. \end{cases} \tag{3.5}$$

It follows that if either $n - k \geqslant 1$ or $n + 1 - j \leqslant n - 1$, i.e. for all pairs except for $(j,k) = (1,n)$ we have

$$\sup_{\tilde{x} \in \mathscr{X}} \frac{x_{k:n} - x_{j:n}}{s_p} = 0, \ 1 \leqslant p \leqslant +\infty,$$

and the zero bound is attained by (2.3) and (2.26), (2.27), respectively, with either $j_* < j$ or $j_* > k$. It is also easily deduced once we realize that the dual problem is to maximize $\|\tilde{x} - \bar{x}\tilde{1}\|_p$ under the constraint $x_{k:n} - x_{j:n} = 1$. If either $j > 1$ or $k < n$, we can remove one of the extremes arbitrarily far increasing the norm to infinity without affecting the restriction. It was firstly noted by Fahmy and Proschan (1981).

In the exceptional case of the sample range, (3.5) rewrites into $D_i = 1$, for $i = 1, \dots, n-1$. Therefore

$$L_p^{(j)}(\tilde{e}_n - \tilde{e}_1) = \left[\frac{n^{p+1}}{j(n-j)^p + j^p(n-j)}\right]^{1/p}, \ 1 \leqslant p < \infty, \tag{3.6}$$

$$L_\infty^{(j)}(\tilde{e}_n - \tilde{e}_1) = \frac{n}{\max\{j, n-j\}} \tag{3.7}$$

are to be minimized with respect to $1 \leqslant j \leqslant n-1$. Analysis of (3.6) and (3.7) is of special interest due to the fact that these are the common factors independent of the coefficient vector $\tilde{c}$ which appear in evaluations of all L-statistics. For solving the first problem, we maximize

$$f_p(x) = x(1-x)^p + x^p(1-x), \ 0 \leqslant x \leqslant 1, \tag{3.8}$$

for fixed $p \geqslant 1$. This is a positive function symmetric about $\frac{1}{2}$, and vanishing at the ends. It has an odd number of extremes located symmetrically about $\frac{1}{2}$. Simple calculus arguments show that $\frac{1}{2}$ is a local maximum point for $1 \leqslant p \leqslant 3$, and it is a minimum for $p > 3$. We claim that this is the global maximum in the first case, and otherwise there is a unique symmetric pair of maximizers $x_{1,p} = 1 - x_{2,p} \in (0, \frac{1}{2})$ determined by the equation

$$f_p'(x) = (1-x)^p - px(1-x)^{p-1} + px^{p-1}(1-x) - x^p = 0. \tag{3.9}$$

We shall have established the claim once we show that the derivative has less than five zeros in $(0,1)$. Dividing it by $(1-x)^p$ and introducing a new variable $y = \frac{x}{1-x} \in (0,+\infty)$, we derive function

$$g_p(y) = 1 - py + py^{p-1} - y^p, \ 0 < y < \infty,$$

with the same number of zeros. We easily check that this is strictly convex on $(0, p-2)$ and strictly concave on $(p-2, \infty)$. It can have two zeros at most in each interval, and there are no more than three altogether. If $1 \leqslant p \leqslant 3$, then $x = \frac{1}{2}$ is the maximum point of (3.8) and so there are no other extremes elsewhere. If $p > 3$, then $x = \frac{1}{2}$ is a minimum and it has to be a symmetric pair of (global) maxima uniquely determined by (3.9). For instance, $x_{1,4} = 1 - x_{2,4} = \frac{1}{2} - \frac{1}{\sqrt{12}}$. Since

$$L_p^{(j)}(\tilde{e}_n - \tilde{e}_1) = f_p^{-1/p}\left(\frac{j}{n}\right),$$

we arrive at the following conclusions. If $1 \leqslant p \leqslant 3$ and n is even, then

$$\frac{x_{n:n} - x_{1:n}}{s_p} \geqslant 2, \ 1 \leqslant p < \infty,$$

and the bound is attained for the symmetric sample (2.3) with $j^* = \frac{n}{2}$. If n is odd, then

$$\frac{x_{n:n} - x_{1:n}}{s_p} \geqslant \left\{ \frac{(2n)^{p+1}}{(n^2-1)[(n+1)^{p-1} + (n-1)^{p-1}]} \right\}^{1/p}.$$

For $p > 3$, (3.6) is minimized at either of integer neighbors $j = \lfloor nx_{i,p} \rfloor, \lceil nx_{i,p} \rceil$ of $nx_{i,p}$, $i = 1,2$. We exclude $j = 0$ and $j = n$ in the case $x_{1,p} < \frac{1}{n}$. We can see that $x_{i,p}$ approach the borders of the domain, as p increases. For $p = \infty$, (3.7) implies

$$\frac{x_{n:n} - x_{1:n}}{s_\infty} \geqslant \frac{n}{n-1}$$

with j_* equal either to 1 or $n-1$.

EXAMPLE 3. Gini mean differences

$$\frac{1}{n(n-1)} \sum_{i,j=1}^{n} |x_i - x_j| = \frac{2}{n(n-1)} \sum_{i=1}^{n} (2i-n-1)x_{i:n}.$$

Here we have

$$c_i' = -c_i = \frac{2(n+1-2i)}{n(n-1)},\ i = 1, \dots, n,$$

$\bar{c} = 0$, and

$$D_j' = \sum_{i=1}^{j} c_i' = \frac{2j(n-j)}{n(n-1)},\ j = 1, \dots, n-1.$$

Out objective is to minimize

$$L_p^{(j)}(\tilde{c}) = \frac{2n^{1/p} j(n-j)}{(n-1)[j(n-j)^p + j^p(n-j)]^{1/p}},\ 1 \leqslant p < \infty,$$

and

$$L_\infty^{(j)}(\tilde{c}) = \frac{2j(n-j)}{(n-1)\max\{j, n-j\}}$$

for $j = 1, \dots, n-1$. Both sequences are symmetric about $\frac{n}{2}$, and first increasing and then decreasing. Therefore

$$\frac{1}{n(n-1)} \sum_{i,j=1}^{n} \frac{|x_i - x_j|}{s_p} \ \geqslant\ L_p^{(1)}(\tilde{c}) = L_p^{(n-1)}(\tilde{c}) = 2\left[\frac{n}{(n-1)^p + n - 1}\right]^{1/p},\ 1 \leqslant p < +\infty,$$

$$\frac{1}{n(n-1)} \sum_{i,j=1}^{n} \frac{|x_i - x_j|}{s_\infty} \ \geqslant\ L_\infty^{(1)}(\tilde{c}) = L_\infty^{(n-1)}(\tilde{c}) = \frac{2}{n-1}.$$

EXAMPLE 4. Mean absolute median deviation

$$\frac{1}{n} \sum_{i=1}^{n} |x_i - \mathrm{med}(\tilde{x})| = \frac{1}{n} \sum_{i \geqslant \frac{n}{2}+1} x_{i:n} - \frac{1}{n} \sum_{i \leqslant \frac{n}{2}} x_{i:n}.$$

For $1 \leqslant p < \infty$, we have

$$L_p^{(j)}(\tilde{c}) = \frac{n^{1/p} \min\{j, n-j\}}{[j(n-j)^p + j^p(n-j)^p]^{1/p}} \text{ if } j \leqslant \frac{n}{2} \text{ and } j \geqslant \frac{n}{2} + 1, \tag{3.10}$$

and

$$L_p^{((n+1)/2)}(\tilde{c}) = \left\{\frac{2n(n-1)^{p-1}}{(n+1)[(n-1)^{p-1} + (n+1)^{p-1}]}\right\}^{1/p} \tag{3.11}$$

for the sample median in the odd sized samples. Sequence (3.10) is symmetric about $\frac{n}{2}$ and has the minimal value

$$L_p^{(1)}(\tilde{c}) = L_p^{(n-1)} = \left[\frac{n}{(n-1)^p + n - 1}\right]^{1/p}.$$

This is equal to (3.11) for $n=3$ (which is obvious since $\frac{n+1}{2}=n-1$ then), and strictly less for all odd $n>3$. Therefore

$$\frac{1}{n}\sum_{i=1}^{n}\frac{|x_i-\mathrm{med}(\tilde{x})|}{s_p}\geqslant\left[\frac{n}{(n-1)^p+n-1}\right]^{1/p},\ 1\leqslant p<\infty. \tag{3.12}$$

We can check in a similar manner that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{|x_i-\mathrm{med}(\tilde{x})|}{s_\infty}\geqslant\frac{1}{n-1}. \tag{3.13}$$

Both (3.12) and (3.13) are attained when all the elements of the sample except for one are identical.

Note that all these bounds are identical with the respective ones for the sample maximum and twice less than the bounds for the Gini mean differences. All they vanish at the rate $\mathcal{O}(n^{-(1-1/p)})$, as $n\to\infty$. For $p=+\infty$, the convergence rate is $\mathcal{O}(n^{-1})$, and the limits are strictly positive for $p=1$. For the sample range, the limits of the lower bounds range between 1 and 2 for all $1\leqslant p\leqslant+\infty$.

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