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AN EXPLICIT REPRESENTATION AS QUASI–SUM OF SQUARES OF A POLYNOMIAL GENERATED BY THE AG INEQUALITY

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Abstract. An explicit representation of the difference $(x_1+\cdots+x_n)^n - n^n x_1\cdots x_n$ for all natural $n \geqslant 2$ is given as a sum of $p_{ij}(x_i - x_j)^2$ over all $1 \leqslant i < j \leqslant n$ where $p_{ij} = p_{ij}(x_1,\ldots,x_n)$ are homogeneous polynomials of degree $n-2$ whose coefficients at all possible monomials of degree $n-2$ are positive.

1. Introduction

In paper [2] by I. Gusić it was proved that the homogeneous symmetric polynomial

$$(x_1+\cdots+x_n)^n - n^n x_1\cdots x_n \tag{1}$$

of degree $n \geqslant 2$ is a quasi-sum of squares, i. e. it can be represented in the form

$$(x_1+\cdots+x_n)^n - n^n x_1\cdots x_n = \sum_{1\leqslant i<j\leqslant n} p_{ij}(x_1,\ldots,x_n)(x_i-x_j)^2\,, \tag{2}$$

where p_{ij} are homogeneous polynomials of degree $n-2$ with non-negative coefficients.

This representation clearly implies the inequality

$$(x_1+\cdots+x_n)^n \geqslant n^n x_1\cdots x_n$$

for all non-negative real numbers $x_1,\ldots,x_n$ with equality if and only if $x_1 = \cdots = x_n$, which can be viewed as one of the algebraic versions of the arithmetic-geometric (AG) inequality

$$\frac{x_1+\cdots+x_n}{n} \geqslant \sqrt[n]{x_1\cdots x_n}\,.$$

(Another one is $x_1^n+\cdots+x_n^n \geqslant n\,x_1\cdots x_n$.)

In particular, the following formulas were derived

$$\begin{aligned}&(x_1+x_2+x_3)^3 - 3^3x_1x_2x_3\\ &= \frac{1}{2}\big((x_1+x_2+7x_3)(x_1-x_2)^2+(x_1+7x_2+x_3)(x_1-x_3)^2+(7x_1+x_2+x_3)(x_2-x_3)^2\big),\end{aligned} \tag{3}$$

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$$
\begin{aligned}
&(x_1+x_2+x_3+x_4)^4 - 4^4 x_1x_2x_3x_4 \\
&= \frac{1}{3}\big((x_1^2+x_2^2+11(x_3^2+x_4^2)+14x_1x_2+58x_3x_4)(x_1-x_2)^2+\cdots\big),
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
&(x_1+x_2+x_3+x_4+x_5)^5 - 5^5 x_1x_2x_3x_4x_5 \\
&= \frac{1}{24}\big((6(x_1^3+x_2^3)+122(x_3^3+x_4^3+x_5^3)+132(x_1^2x_2+x_1x_2^2) \\
&+361(x_3^2x_4+x_3x_4^2+x_3^2x_5+x_3x_5^2+x_4^2x_5+x_4x_5^2) \\
&+362x_1x_2(x_3+x_4+x_5)+3606x_3x_4x_5)(x_1-x_2)^2+\cdots\big).
\end{aligned} \tag{5}
$$

Dots mean that the polynomial coefficient $p_{ij}(x_1,\dots,x_n)$ at $(x_i-x_j)^2$ is obtained from the polynomial coefficient $p_{12}(x_1,\dots,x_n)$ at $(x_1-x_2)^2$ by swapping x_1 and x_i, x_2 and x_j respectively.

Even more complicated formula for $n=6$ was also derived, however, no closed general formula for the difference (1) for an arbitrary natural $n \geqslant 2$ was given.

In this paper we derive a general representation of the difference (1) as a quasi-sum of squares for an arbitrary natural $n \geqslant 2$. For $n=3$ it coincides with (3), but for $n=4,5$ it differs from (4) and (5).

2. Non-uniqueness of the representation as quasi-sum of squares

We note that representations (3), (4) and (5) are representations of type (2) where the polynomial coefficients p_{ij} at $(x_i-x_j)^2$ satisfy the following conditions:

$(c1)$ p_{ij} is obtained from p_{12} by swapping x_1 and x_i and also x_2 and x_j, and

$(c2)$ p_{12} is symmetric in x_1,x_2 and also in $x_3,\dots,x_n$.

With these assumptions for $n=3$ representation (3) is defined uniquely: see Corollary 1 below.

However there is no uniqueness for $n>3$. For example, for $n=4$ all representations (2) satisfying conditions $(c1)$ and $(c2)$ have the form

$$
\begin{aligned}
&(x_1+x_2+x_3+x_4)^4 - 4^4 x_1x_2x_3x_4 \\
&= \frac{1}{3}\big((x_1^2+x_2^2+11(x_3^2+x_4^2)+14x_1x_2+58x_3x_4 \\
&-\alpha(2x_1x_2-(x_1+x_2)(x_3+x_4)+x_3^2+x_4^2))(x_1-x_2)^2+\cdots\big) \\
&= \frac{1}{3}\big(x_1^2+x_2^2+(11-\alpha)(x_3^2+x_4^2)+(14-2\alpha)x_1x_2 \\
&+\alpha(x_1+x_2)(x_3+x_4)+58\,x_3x_4\big)(x_1-x_2)^2+\cdots), \text{ where}^{0}\ \alpha\in\mathbb{R}
\end{aligned} \tag{6}
$$

[0]Note that, for $n=3$, condition $(c1)$ implies condition $(c2)$, because by $(c1)$

$$
\begin{aligned}
&(x_1+x_2+x_3)^3 - 3^3x_1x_2x_3 \\
&= p_{12}(x_1,x_2,x_3)(x_1-x_2)^2 + p_{12}(x_1,x_3,x_2)(x_1-x_3)^2 + p_{12}(x_2,x_3,x_1)(x_2-x_3)^2
\end{aligned}
$$

and the polynomial $p_{12}(x_1,x_3,x_2)(x_1-x_3)^2+p_{12}(x_2,x_3,x_1)(x_2-x_3)^2$ is symmetric in x_1,x_2.

All coefficients of p_{12} are non-negative if and only if $0 \leqslant \alpha \leqslant 7$ and in this case this is a quasi-sum-of-squares representation. For $\alpha = 0$ (6) coincides with (4). In the other limit case $\alpha = 7$ it takes the form

$$\begin{aligned} &(x_1 + x_2 + x_3 + x_4)^4 - 4^4 x_1x_2x_3x_4 \\ &= \frac{1}{3}\big(x_1^2 + x_2^2 + 4(x_3^2 + x_4^2) + 7(x_1 + x_2)(x_3 + x_4) + 58\, x_3x_4\big)(x_1 - x_2)^2 + \cdots). \end{aligned} \tag{7}$$

In the limit cases the coefficients of the polynomial p_{12} at some monomials of degree 2 are equal to 0. In all non-limit cases, when $0 < \alpha < 7$, all coefficients of p_{12} at all monomials of degree 2 are positive. Representation (6) follows by equating the coefficients at all monomials of degree 4 in (2) (due to symmetry it suffices to consider the monomials $x_1^4, x_1^3x_2, x_1^2x_2^2, x_1^2x_2x_3, x_1x_2x_3x_4$) and solving the corresponding system of 5 linear equations in 4 unknowns.)

Moreover, if one omits conditions $(c1)$ and $(c2)$, then there is no uniqueness also for $n = 3$. For $n = 3$ all representations (2) have the form

$$\begin{aligned} &(x_1 + x_2 + x_3)^3 - 3^3 x_1x_2x_3 \\ &= \frac{1}{2}\big((\alpha x_1 + \beta x_2 + (9 - \alpha - \beta)x_3)(x_1 - x_2)^2 \\ &+(2 - \alpha)x_1 + (6 + 2\alpha - \beta)x_2 + (1 - \alpha + \beta)x_3)(x_1 - x_3)^2 \\ &+(6 - \alpha + 2\beta)x_1 + (2 - \beta)x_2 + (1 + \alpha - \beta)x_3)(x_2 - x_3)^2\big), \text{ where } \alpha \in \mathbb{R}. \end{aligned} \tag{8}$$

All coefficients are non-negative if and only if $0 \leqslant \alpha, \beta \leqslant 2$ and $|\alpha - \beta| \leqslant 1$ and in this case this is a quasi-sum-of-squares representation. If $\alpha = \beta = 1$, then (8) coincides with (3). If $\alpha = 1, \beta = 2$ it takes the form

$$\begin{aligned} &(x_1 + x_2 + x_3)^3 - 3^3 x_1x_2x_3 \\ &= \frac{1}{2}\big((x_1{+}2x_2{+}6x_3)(x_1{-}x_2)^2{+}(x_1{+}6x_2{+}2x_3)(x_1{-}x_3)^2{+}9x_1(x_2{-}x_3)^2\big). \end{aligned} \tag{9}$$

In all limit cases, when (α, β) belongs to the boundary of the hexagon defined by $0 \leqslant \alpha, \beta \leqslant 2$ and $|\alpha - \beta| \leqslant 1$ at least one of the coefficients at x_1, x_2 or x_3 of at least one of the polynomials p_{12}, p_{13} or p_{23} is equal to 0, as for example in (9). In all non-limit cases, when (α, β) is inside that hexagon, all coefficients at x_1, x_2 and x_3 of all polynomials p_{12}, p_{13} and p_{23} are positive. Representation (8) also follows by equating the coefficients at all monomials of degree 3 in (2) and solving the corresponding system now of 10 linear equations in 9 unknowns.

LEMMA 1. *Assume that for a polynomial $q_d(x_1, x_2, x_3)$ of degree d symmetric in x_1, x_2*

$$q_d(x_1,x_2,x_3)(x_1{-}x_2)^2{+}q_d(x_1,x_3,x_2)(x_1{-}x_3)^2{+}q_d(x_2,x_3,x_1)(x_2{-}x_3)^2 = 0 \tag{10}$$

for all $x_1, x_2, x_3 \in \mathbb{R}$.

If $d < 2$, then $q_d \equiv 0$. If $d = 2$, then q_2 is defined uniquely up to a constant multiple:

$$q_2(x_1, x_2, x_3) = c(x_1 - x_2)(x_2 - x_3)\,, \tag{11}$$

where[1] $c \in \mathbb{R}$. *If* $d > 2$, *then*

$$q_d(x_1,x_2,x_3) = (x_1 - x_2)(x_2 - x_3)q_{d-2}(x_1,x_2,x_3)\,, \tag{12}$$

where the polynomial q_{d-2}, *which is also symmetric in* x_1,x_2, *satisfies the equation*

$$q_{d-2}(x_1,x_2,x_3)(x_1-x_2)-q_{d-2}(x_1,x_3,x_2)(x_1-x_3)+q_{d-2}(x_2,x_3,x_1)(x_2-x_3)=0 \tag{13}$$

for all $x_1,x_2,x_3 \in \mathbb{R}$.

Proof. Equality (10) with $x_2 = x_3$ implies that $q_d(x_1,x_2,x_2) = 0$. Therefore, for some polynomial q_{d-1} of degree $d-1$, $q_d(x_1,x_2,x_2) = (x_2 - x_3)q_{d-1}(x_1,x_2,x_3)$. By (10)

$$\begin{aligned} q_{d-1}(x_1,x_2,x_3)(x_1 - x_2)^2 - q_{d-1}(x_1,x_3,x_2)(x_1 - x_3)^2 \\ -q_{d-1}(x_2,x_3,x_1)(x_1 - x_3)(x_2 - x_3) = 0 \end{aligned} \tag{14}$$

for all $x_1,x_2,x_3 \in \mathbb{R}$. If $x_1 = x_3$, this equality implies that, for some polynomial q_{d-2} of degree $d-2$, $q_{d-1}(x_1,x_2,x_2) = (x_1 - x_3)q_{d-2}(x_1,x_2,x_3)$ and equality (12) follows. Moreover equality (14) implies that

$$\begin{aligned} (x_1 - x_3)q_{d-2}(x_1,x_2,x_3)(x_1 - x_2)^2 - (x_1 - x_2)q_{d-2}(x_1,x_3,x_2)(x_1 - x_3)^2 \\ -(x_2 - x_1)q_{d-2}(x_2,x_3,x_1)(x_1 - x_3)(x_2 - x_3) = 0 \end{aligned}$$

and, hence, (13) for all $x_1,x_2,x_3 \in \mathbb{R}$. □

REMARK 1. Equation (13) is satisfied by any polynomial q_{d-2} of the form

$$q_{d-2}(x_1,x_2,x_3) = q^{(1)}_{d-2}(x_1,x_2,x_3) + x_3 q^{(2)}_{d-3}(x_1,x_3,x_2), \tag{15}$$

where $q^{(1)}_{d-2}, q^{(2)}_{d-3}$ are polynomials of degrees $d-2$, $d-3$ respectively, symmetric in x_1,x_2,x_3. This easily follows since $(x_1 - x_2) - (x_1 - x_3) + (x_2 - x_3) = 0$ and $x_3(x_1 - x_2) - x_2(x_1 - x_3) + x_1(x_2 - x_3) = 0$.

COROLLARY 1. *For* $n = 3$ *representation* (2) *where homogeneous polynomials* $p_{ij}, 1 \leqslant i < j \leqslant 3$, *of degree* 1 *satisfy condition* (c1) *is unique and has form* (3).

Proof. If there are two such representations with polynomials $p^{(1)}_{ij}$ and $p^{(2)}_{ij}$, then by subtracting we obtain equation (10) with the polynomial $q_1 = p^{(1)}_{12} - p^{(2)}_{12}$. Hence $q_1(x_1,x_2,x_3) \equiv 0 \Longrightarrow p^{(1)}_{12} = p^{(2)}_{12} \Longrightarrow p^{(1)}_{ij} = p^{(2)}_{ij}$ for all $1 \leqslant i < j \leqslant 3$. □

COROLLARY 2. *Let* $n \geqslant 4$, q_{n-4}, q_{n-5} *be symmetric polynomials of orders* $n-4$, $n-5$ *respectively* (*if* $n = 4$ *it is assumed that* $q_{n-5} \equiv 0$), *and*

$$\begin{aligned} &q_{n-2}(x_1,\dots,x_n) \\ &= ((n-2)x_1x_2 - (x_1+x_2)(x_3+\cdots+x_n)+x_3^2+\cdots+x_n^2)q_{n-4}(x_1,\dots,x_n) \\ &+(x_1x_2(x_3+\cdots+x_n)-(x_1+x_2)(x_3^2+\cdots+x_n^2)+x_3^3+\cdots+x_n^3)q_{n-5}(x_1,\dots,x_n). \end{aligned} \tag{16}$$

[1] In this paper we only consider polynomials of real-valued variables with real-valued coefficients.

Then

$$\sum_{1\leqslant i<j\leqslant n} q_{n-2}(x_i,x_j\dots,x_{i-1},x_1,x_{i+1},\dots,x_{j-1},x_2,x_{j+1},\dots,x_n)(x_i-x_j)^2=0\,. \quad (17)$$

REMARK 2. If $n=4$, then

$$q_2(x_1,x_2,x_3,x_4)=c(2x_1x_2-(x_1+x_2)(x_3+x_4)+x_3^2+x_4^2)\,,$$

where $c\in\mathbb{R}$, which is the polynomial that enters formula (6).

Proof. By linearity it suffices to prove equality (17) with q_{n-2} replaced by $q^{(1)}_{n-2}$ and by $q^{(2)}_{n-2}$, where

$$q^{(1)}_{n-2}(x_1,\dots,x_n)=((n-2)x_1x_2-(x_1+x_2)(x_3+\dots+x_n)+x_3^2+\dots+x_n^2)q_{n-4}(x_1,\dots,x_n)$$

and

$$\begin{gathered} q^{(2)}_{n-2}(x_1,\dots,x_n)\\ =(x_1x_2(x_3+\dots+x_n)-(x_1+x_2)(x_3^2+\dots+x_n^2)+x_3^3+\dots+x_n^3)q_{n-5}(x_1,\dots,x_n)\,. \end{gathered}$$

By Lemma 1 and Remark 1

$$\begin{aligned}\sum_{1\leqslant i<j<k\leqslant n}\Big[&(x_i-x_k)(x_j-x_k)q_{n-4}(x_1,\dots,x_n)(x_i-x_j)^2\\ &+(x_i-x_j)(x_k-x_j)q_{n-4}(x_1,\dots,x_n)(x_i-x_k)^2\\ &+(x_j-x_i)(x_k-x_i)q_{n-4}(x_1,\dots,x_n)(x_j-x_k)^2\Big]=0\,.\end{aligned}$$

Splitting this sum into three sub-sums, swapping k and j in the second sub-sum and replacing i by k, j by i and k by j in the third sub-sum, we get

$$\begin{aligned}&\Big(\sum_{1\leqslant i<j<k\leqslant n}+\sum_{1\leqslant i<k<j\leqslant n}+\sum_{1\leqslant k<i<j\leqslant n}\Big)(x_i-x_k)(x_j-x_k)q_{n-4}(x_1,\dots,x_n)(x_i-x_j)^2\\ &=\sum_{1\leqslant i<j\leqslant n}\Big(\sum_{k\neq i,j}(x_i-x_k)(x_j-x_k)\Big)q_{n-4}(x_1,\dots,x_n)(x_i-x_j)^2\\ &=\sum_{1\leqslant i<j\leqslant n}((n-2)x_ix_j-(x_i+x_j)(x_3+\dots+x_{i-1}+x_{i+1}+\dots+x_{j-1}+x_{j+1}+\dots+x_n)\\ &\quad+x_3^2+\dots+x_{i-1}^2+x_{i+1}^2+\dots+x_{j-1}^2+x_{j+1}^2+\dots+x_n^2)q_{n-4}(x_1,\dots,x_n)(x_i-x_j)^2\\ &=\sum_{1\leqslant i<j\leqslant n} q^{(1)}_{n-2}(x_i,x_j\dots,x_{i-1},x_1,x_{i+1},\dots,x_{j-1},x_2,x_{j+1},\dots,x_n)(x_i-x_j)^2=0.\end{aligned}$$

The proof for $q^{(2)}_{n-2}$ is similar. One should only note that

$$\sum_{k\neq i,j}(x_i - x_k)(x_j - x_k)x_k$$
$$= x_ix_j(x_3 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_{j-1} + x_{j+1} + \cdots + x_n)$$
$$-(x_i + x_j)(x_3^2 + \cdots + x_{i-1}^2 + x_{i+1}^2 + \cdots + x_{j-1}^2 + x_{j+1} + \cdots + x_n^2)$$
$$+x_3^3 + \cdots + x_{i-1}^3 + x_{i+1}^3 + \cdots + x_{j-1}^3 + x_{j+1}^3 + \cdots + x_n^3. \qquad \square$$

COROLLARY 3. *For $n > 3$ there exist infinitely many representations* (2) *where homogeneous polynomials $p_{ij}, 1 \leqslant i < j \leqslant n$, of degree $n-2$ satisfy conditions* $(c1)$ *and* $(c2)$ *and all coefficients of the polynomial p_{12} at all possible monomials of degree $n-2$ are positive.*

Proof. Existence of such polynomials will be proved in Section 3 (see Theorem 1 and Remark 4). If representation (2) holds with some polynomial satisfying the stated conditions, then by Corollary 2 it also holds if the polynomial p_{12} is replaced by the polynomial

$$p_{12}(x_1,\dots,x_n)+\varepsilon((n-2)x_1x_2-(x_1+x_2)(x_3+\cdots+x_n)+x_3^2+\cdots+x_n^2)(x_1+\cdots+x_n)^{n-4},$$

satisfying condition $(c2)$, and all other polynomials are defined via it by condition $(c1)$. It suffices to note that for sufficiently small $\varepsilon > 0$ all coefficients of this polynomials p_{ij} are positive. $\square$

3. Explicit representations as quasi-sum of squares

We start with two general observations.

LEMMA 2. *For all natural $n \geqslant 2$ and all representations* (2) *the sum of all coefficients of the polynomials p_{ij} is equal to $\frac{n^{n-1}}{2}$ for all $1 \leqslant i < j \leqslant n$.*

Proof. In (2) let $x_1 = 1 + \xi_i\varepsilon, i = 1,\dots,n,$ with arbitrary $\varepsilon, \xi_1,\dots,\xi_n \in \mathbb{R}$. Then

$$n^n\left[\left(1 + \frac{1}{n}(\xi_1 + \cdots + \xi_n)\varepsilon\right)^n - (1 + \xi_1\varepsilon)\cdots(1 + \xi_n\varepsilon)\right]$$
$$= \varepsilon^2 \sum_{1\leqslant i<j\leqslant n} p_{ij}(1 + \xi_1\varepsilon,\dots,1 + \xi_n\varepsilon)(\xi_i - \xi_j)^2.$$

Dividing by ε^2 and passing to the limit as $\varepsilon \to 0$, we get

$$n^n\left[\tfrac{n-1}{2n}(\xi_1 + \cdots + \xi_n)^2 - (\xi_1\xi_2 + \cdots + \xi_{n-1}\xi_n)\right] = \sum_{1\leqslant i<j\leqslant n} p_{ij}(1,\dots,1)(\xi_i - \xi_j)^2.$$

Since this equality holds for all $\xi_1,\dots,\xi_n \in \mathbb{R}$, equating coefficients at $\xi_i\xi_j$, we get $n^n(\frac{n-1}{n} - 1) = -2p_{ij}(1,\dots,1)$. Hence, the sum of all coefficients of the polynomial p_{ij} is equal to $p_{ij}(1,\dots,1) = \frac{n^{n-1}}{2}$. $\square$

LEMMA 3. *For all natural* $n \geqslant 2$ *and all representations* (2) *with polynomials* p_{ij} *satisfying conditions* $(c1)$ *and* $(c2)$*, in the polynomial* p_{12} *the coefficients at* x_1^{n-2} *and* x_2^{n-2} *are equal to* $\frac{1}{n-1}$ *and the coefficient at* $x_3 \cdots x_n$ *is equal to* $\frac{n^{n-1}-(n-1)!}{n-1}$.

REMARK 3. If $n = 2,3$, these properties, together with the property of Lemma 2, uniquely define representation (2) with polynomials p_{ij} satisfying conditions $(c1)$ and $(c2)$. If $n = 4$, then in the polynomial $p_{12}(x_1,x_2,x_3,x_4)$ the coefficients at x_1^2 and x_2^2 are always equal to $\frac{1}{3}$, the coefficient at x_3x_4 is always equal to $\frac{58}{3}$, and the sum of all coefficients is equal to 32 (as in formulas (4), (6), (7), and formula (20) below). If respectively $n = 5$, then in the polynomial $p_{12}(x_1,x_2,x_3,x_4,x_5)$ the coefficients at x_1^2 and x_2^2 are always equal to $\frac{1}{4}$, the coefficient at $x_3x_4x_5$ is always equal to $\frac{601}{4}$, and the sum of all coefficients is equal to $\frac{625}{2}$ (as in formula (5) and formula (21) below).

For $n = 3$ in representation (8) for any $\alpha, \beta \in \mathbb{R}$ the sums of all coefficients of the polynomials p_{12}, p_{13} and p_{23}, which do not satisfy conditions $(c1)$ and $(c2)$ if $(\alpha, \beta) \neq (1,1)$, are equal to $\frac{9}{2}$ which conforms with Lemma 2.

Proof. Let

$$p_{12}(x_1,\ldots,x_n) = \alpha_1 x_1^{n-2} + \alpha_2 x_2^{n-2} + \cdots + \beta x_3 \cdots x_n.$$

Then, for $1 \leqslant j \leqslant n$,

$$p_{1j}(x_1,\ldots,x_n) = \alpha_1 x_1^{n-2} + \cdots + \beta x_2 \cdots x_{j-1}x_{j+1} \cdots x_n.$$

and, for $1 < i < j \leqslant n$,

$$p_{ij}(x_1,\ldots,x_n) = \alpha_1 x_j^{n-2} \cdots + \beta x_2 \cdots x_{i-1}x_{i+1} \cdots x_{j-1}x_{j+1} \cdots x_n.$$

Next we equate in (2) the coefficients at x_1^n. In the right-hand side we should only consider the summands with $i = 1, 2 \leqslant j \leqslant n$, hence $1 = \alpha_1(n-1)$. Similarly $1 = \alpha_2(n-1)$. So $\alpha_1 = \alpha_2 = \frac{1}{n-1}$.

Equating coefficients at $x_1 \cdots x_n$, we get $n! - n^n = -2\beta\binom{n}{2}$, hence $\beta = \frac{n^{n-1}-(n-1)!}{n-1}$. □

Let $s_0 \equiv 1$, for $k = 1,\ldots,n$

$$s_k(x_1,\ldots,x_n) = x_1 \cdots x_{k-1}x_k + x_1 \cdots x_{k-1}x_{k+1} + \cdots + x_{n-k+1} \cdots x_{n-1}x_n$$

be the standard symmetric polynomial in variables $x_1,\ldots,x_n$ and, for $1 \leqslant i < j \leqslant n$,

$$s_k^{(ij)}(x_1,\ldots,x_n) = s_k(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n).$$

THEOREM 1. *For all natural* $n \geqslant 2$ *representation* (2) *holds with*

$$p_{ij} = \sum_{k=0}^{n-2} n^k \binom{n-1}{k+1}^{-1} s_1^{n-2-k} s_k^{(ij)}. \tag{18}$$

REMARK 4. Note that in the polynomials p_{ij} all coefficients at all possible monomials of degree $n-2$ are positive. Moreover, all of them are greater than or equal to $\frac{1}{n-1}$.

Proof. First we note that for any natural $m \leqslant n-1$

$$s_1(x_1,\dots,x_n)s_m(x_1,\dots,x_n)$$
$$= \alpha_m \sum_{1\leqslant i<j\leqslant n} s_{m-1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{j-1},x_{j+1},\dots,x_n)(x_i-x_j)^2 \tag{19}$$
$$+\beta_m s_{m+1}(x_1,\dots,x_n),$$

where

$$\alpha_m = \frac{1}{n-m}, \qquad \beta_m = \frac{n(m+1)}{n-m}.$$

This follows since the polynomials under consideration only contain monomials of the forms $x_{k_1}x_{k_2}\cdots x_{k_{m+1}}$ where all $k_1,k_2\dots,k_{m+1}$ are distinct and $x_{k_1}^2x_{k_2}\cdots x_{k_m}$ where all $k_1,k_2\dots,k_m$ are distinct. Due to symmetry equating coefficients at these monomials implies that equality (19) holds if and only if

$$m+1 = -2\binom{m+1}{2}\alpha_m + \beta_m, \quad 1 = (n-m)\alpha_m.$$

By (19) with $m=1$

$$s_1^2(x_1,\dots,x_n) = \alpha_1 \sum_{1\leqslant i<j\leqslant n}(x_i-x_j)^2 + \beta_1 s_2(x_1,\dots,x_n).$$

Multiplying this equality by $s_1(x_1,\dots,x_n)$ and applying (19) with $m=2$ we get

$$s_1^3(x_1,\dots,x_n)$$
$$= \sum_{1\leqslant i<j\leqslant n}\big(\alpha_1 s_1(x_1,\dots,x_n) + \beta_1\alpha_2 s_1^{(ij)}(x_1,\dots,x_n)\big)(x_i-x_j)^2 + \beta_1\beta_2 s_3(x_1,\dots,x_n).$$

Similarly

$$s_1^4(x_1,\dots,x_n) = \sum_{1\leqslant i<j\leqslant n}\big(\alpha_1 s_1^2(x_1,\dots,x_n) + \beta_1\alpha_2 s_1(x_1,\dots,x_n)s_1^{(ij)}(x_1,\dots,x_n)$$
$$+\beta_1\beta_2\alpha_3 s_2^{(ij)}(x_1,\dots,x_n)\big)(x_i-x_j)^2 + \beta_1\beta_2\beta_3 s_4(x_1,\dots,x_n),$$

and so on. Thus we get

$$s_1^n(x_1,\cdots,x_n) = \sum_{1\leqslant i<j\leqslant n}\Big(\sum_{k=0}^{n-2}\beta_1\cdots\beta_k\alpha_{k+1}s_1^{n-2-k}(x_1,\dots,x_n)s_k^{(ij)}(x_1,\dots,x_n)\Big)(x_i-x_j)^2$$
$$+\beta_1\cdots\beta_{n-1}s_n(x_1,\dots,x_n).$$

If $k=0$, we assume that $\beta_1\cdots\beta_k\alpha_{k+1}\Big|_{k=0} = \alpha_1$.

Finally we note that

$$\beta_1 \cdots \beta_{n-1} = \frac{2n}{n-1} \cdot \frac{3n}{n-2} \cdots \frac{(n-1)n}{2} \cdot \frac{n \cdot n}{1} = n^n$$

and, for $k = 0, 1, \dots, n-2$,

$$\begin{aligned} \beta_1 \cdots \beta_k \alpha_{k+1} &= \frac{2n}{n-1} \cdot \frac{3n}{n-2} \cdots \frac{(k+1)n}{n-k} \cdot \frac{1}{n-k-1} \\ &= \frac{n^k (k+1)!}{(n-1)\cdots(n-k+1)} = n^k \binom{n-1}{k+1}^{-1}, \end{aligned}$$

hence the statement follows. □

Let us consider several particular cases of (18). If $n = 3$, then formula (18) coincides with (3) (which should be by Corollary 1).

If $n = 4$, then

$$\begin{aligned} &(x_1+x_2+x_3+x_4)^4 - 4^4 x_1x_2x_3x_4 \\ &= \frac{1}{3}\big(((x_1+x_2+x_3+x_4)^2 + 4(x_1+x_2+x_3+x_4)(x_3+x_4) + 48x_3x_4)(x_1-x_2)^2 + \cdots\big) \\ &= \frac{1}{3}\big((x_1^2+x_2^2+5(x_3^2+x_4^2)+2x_1x_2+6(x_1+x_2)(x_3+x_4)+58x_3x_4)(x_1-x_2)^2+\cdots\big), \end{aligned} \tag{20}$$

which coincides with (6) if there $\alpha = 6$.

If $n = 5$, then

$$\begin{aligned} &(x_1+x_2+x_3+x_4+x_5)^5 - 5^5 x_1x_2x_3x_4x_5 \\ &= \frac{1}{24}\big(6(x_1+x_2+x_3+x_4+x_5)^3 + 20(x_1+x_2+x_3+x_4+x_5)^2(x_3+x_4+x_5) \\ &\quad +150(x_1+x_2+x_3+x_4+x_5)(x_3x_4+x_3x_5+x_4x_5)+3000x_3x_4x_5)(x_1-x_2)^2+\cdots\big) \\ &= \frac{1}{24}\big(6(x_1^3+x_2^3)+26(x_3^3+x_4^3+x_5^3)+18(x_1^2x_2+x_1x_2^2) \\ &\quad +228(x_3^2x_4+x_3x_4^2+x_3^2x_5+x_3x_5^2+x_4^2x_5+x_4x_5^2) \\ &\quad +76x_1x_2(x_3+x_4+x_5)+38(x_1^2+x_2^2)(x_3+x_4+x_5)+58(x_1+x_2)(x_3^2+x_4^2+x_5^2) \\ &\quad +266(x_1+x_2)(x_3x_4+x_3x_5+x_4x_5)+3606x_3x_4x_5)(x_1-x_2)^2+\cdots\big). \end{aligned} \tag{21}$$

We conclude by deriving a corollary of Theorem 1 which may be of independent interest.

THEOREM 2. *For all natural* $n \geqslant 2$

$$(x_1+\cdots+x_n)^n - n^n x_1\cdots x_n \geqslant \tfrac{1}{n-1}(x_1+\cdots+x_n)^{n-2} \sum_{1\leqslant i<j\leqslant n} (x_i-x_j)^2 \tag{22}$$

for all non-negative $x_1, \dots, x_n$.

If $n \geqslant 3$, *then, for non-negative* $x_1, \dots, x_n$, *equality holds if and only if all* $x_1, \dots, x_n$ *are equal or all of them but one are equal to* 0.

Proof. For non-negative $x_1, \ldots, x_n$ inequality (22) follows by equality (2) where p_{ij} are defined by (18) because all summands in the right-hand side of (2) are non-negative and the summands corresponding to $k = 0$ have the form

$$\sum_{1 \leqslant i < j \leqslant n} \tfrac{1}{n-1}(x_1 + \cdots + x_n)^{n-2}(x_i - x_j)^2 .$$

Next assume that $n \geqslant 3$, all $x_1, \ldots, x_n$ are non-negative and equality holds in (22). By (2) and (18) this can happen if and only if

$$\left(\sum_{k=1}^{n-2} n^k \binom{n-1}{k+1}^{-1} (x_1 + \cdots + x_n)^{n-2-k} s_k^{(ij)}(x_1, \ldots, x_n) \right) (x_i - x_j)^2 = 0 \qquad (23)$$

for all $1 \leqslant i < j \leqslant n$. This is clearly satisfied if all $x_1, \ldots, x_n$ are equal. Let not all of them be equal, say $x_1 \neq x_2$. Then, by (23) with $i = 1, j = 2$ and $k = 1$, $x_3 + \cdots + x_n = 0$, hence $x_3 = \cdots = x_n = 0$. Next, by (23) with $i = 2, j = 3$ and $k = 1$, $(x_1 + x_4 + \cdots + x_n)x_2^2 = 0$, hence $x_1 x_2 = 0$. So either $x_1 = 0$ or $x_2 = 0$, in which case equalities (23) are satisfied for all $1 \leqslant i < j \leqslant n$. Indeed, if say $x_2 = \cdots = x_n = 0$, then for $i \neq 1$ $(x_i - x_j)^2 = 0$, and for $i = 1$ $s_k^{1j}(x_1, \ldots, x_n) = 0$ for all $1 \leqslant k \leqslant n-2$. □

REMARK 5. Recall that an explicit representation as quasi-sum of squares of the polynomial $x_1^n + \cdots + x_n^n - n\, x_1 \cdots x_n$ related to the second algebraic version of the AG inequality, mentioned in the Introduction, was obtained long ago by A. Hurwitz [3]. In [3] it was proved that

$$x_1^n + \cdots + x_n^n - n\, x_1 \cdots x_n = \sum_{1 \leqslant i < j \leqslant n} q_{ij}(x_1, \ldots, x_n)(x_i - x_j)^2, \qquad (24)$$

where

$$q_{ij}(x_1, \ldots, x_n)$$
$$= \tfrac{1}{(n-1)!} \sum_{k=1}^{n-1} \alpha_{k,n} (x_i^{n-k-1} + x_i^{n-k-2} x_j + \cdots + x_j^{n-k-1}) x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_{k+1}$$

and

$$\alpha_{k,n} = (k-1)!\ (n-k-1)! .$$

In particular,

$$x_1^3 + x_2^3 + x_3^3 - 3\, x_1 x_2 x_3 = \frac{1}{2}(x_1 + x_2 + x_3)\big((x_1 - x_2)^2 + (x_1 - x_3)^2 + 9x_2 - x_3)^2\big),$$

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\, x_1 x_2 x_3 x_4$$
$$= \frac{1}{6}\left((2(x_1^2 + x_1 x_2 + x_2^2) + (x_1 + x_2)x_3 + 2x_3 x_4)(x_1 - x_2)^2 + \cdots\right),$$

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\, x_1 x_2 x_3 x_4 x_5$$
$$= \frac{1}{12}\big((3(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) + (x_1^2 + x_1 x_2 + x_2^2)x_3$$
$$+ (x_1 + x_2)x_3 x_4 + 2x_3 x_4 x_5)(x_1 - x_2)^2 + \cdots\big).$$

See also book [1] by P. S. Bullen where on page 87 the proof of inequality (24) is reproduced amongst 74 proofs of the AG inequality.

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