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NEIGHBORHOODS OF A NEW CLASS OF P–VALENTLY STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. A certain subclass $T^m(n,p,\alpha,\lambda)$ of p-valently starlike functions in the unit disk is introduced. By making use of the familiar concept of neigborhoods of p-valent functions, the author proves coefficient bounds and distortion inequalities, and associated inclusion relations for the $(n,\delta)-$ neighborhoods of functions belonging to the class $T^m(n,p,\alpha,\lambda)$, which is defined by means of a certain nonhomogeneous Cauchy-Euler differential equation.

1. Introduction

Let $T(n,p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geqslant 0; p,n \in \mathbb{N} := \{1,2,3,...\}), \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Following the earlier investigations by Goodman [7], Ruscheweyh [6] and Altintaş et al. [5], we define the $(n,\delta)-$ neighborhood of a function $f(z) \in T(n,p)$ by

$$N_{n,\delta}(f) := \left\{ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in T(n,p) : \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqslant \delta \right\} \tag{2}$$

so that, obviously,

$$N_{n,\delta}(h) := \left\{ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in T(n,p) : \sum_{k=n+p}^{\infty} k\,|b_k| \leqslant \delta \right\} \tag{3}$$

where, and in what follows,

$$h(z) = z^p (p \in \mathbb{N}). \tag{4}$$

We denote by $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ the classes of p-valently starlike functions of order α in U $(0 \leqslant \alpha < p)$ and p-valently convex functions of order α in U $(0 \leqslant \alpha < p)$, respectively.

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Thus, by definition, we have

$$S_n^*(p,\alpha) := \left\{ f \in T(n,p) : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \; (z \in U; 0 \leqslant \alpha < p) \right\} \tag{5}$$

and

$$C_n(p,\alpha) := \left\{ f \in T(n,p) : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \; (z \in U; 0 \leqslant \alpha < p) \right\}. \tag{6}$$

An unification of the function classes $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ is provided by the class $T_n(p,\alpha,\lambda)$ of functions $f(z) \in T(n,p)$, which also satisfy the following inequality:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right) > \alpha, \; (z \in U; 0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1). \tag{7}$$

The class $T_n(p,\alpha,\lambda)$ was investigated by Altintaş et al. [2] .

Using the Salagean operator [8]; we can write the following equalities for the functions $f(z)$ belong to the class $T(n,p)$

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = \frac{z}{p} f'(z) = z^p - \sum_{k=n+p}^{\infty} \frac{k}{p} a_k z^k,$$

$$D^2 f(z) = D(Df(z)) = z^p - \sum_{k=n+p}^{\infty} \frac{k^2}{p^2} a_k z^k,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D^m f(z) = D(D^{m-1} f(z)) = z^p - \sum_{k=n+p}^{\infty} \frac{k^m}{p^m} a_k z^k.$$

Here we also consider the new class $T^m(n,p,\alpha,\lambda)$. A function $f(z) \in T(n,p)$ is said to be in the class $T^m(n,p,\alpha,\lambda)$ if it satisfies the inequality:

$$\Re\left\{\frac{(1-\lambda)z(D^m f(z))' + \lambda z\left(D^{m+1} f(z)\right)'}{(1-\lambda)D^m f(z) + \lambda D^{m+1} f(z)}\right\} > \alpha \tag{8}$$

for some $\alpha(0 \leqslant \alpha < p)$, $\lambda(0 \leqslant \lambda \leqslant 1)$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and for all $z \in U$.

Let $A(n)$ be the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqslant 0, n \in \mathbb{N} = \{1,2,3,...\})$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $S_n^*(\alpha)$ denote the subclass of $A(n)$ consisting of functions which satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{9}$$

for some $\alpha\,(0 \leqslant \alpha < 1)$. A function $f(z)$ in $S_n^*(\alpha)$ is said to be starlike of order α in U.

A function $f(z) \in A(n)$ is said to be convex of order α if it satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U) \tag{10}$$

for some $\alpha\,(0 \leqslant \alpha < 1)$. We denote by $C_n\,(\alpha)$ the subclass of $A(n)$ consisting of all such functions [3] .

Clearly, we have

$$T(n,1) = A(n), T^0(n,1,\alpha,\lambda) = T_n(1,\alpha,\lambda), T^0(n,p,\alpha,0) = S_n^*(p,\alpha)$$

$$T^0(n,p,\alpha,1) = C_n(p,\alpha), T^0(n,1,\alpha,0) = S_n^*(\alpha) \text{ and } T^0(n,1,\alpha,1) = C_n(\alpha)$$

in terms of the simpler classes $S_n^*(p,\alpha)$, $C_n(p,\alpha)$, $T_n(p,\alpha,\lambda)$, $T^m(n,p,\alpha,\lambda)$, $S_n^*(\alpha)$ and $C_n\,(\alpha)$ defined by (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10).

The main object of the present sequel to the aforementioned recent works is to derive several coefficient bounds and distortion inequalities, and associated inclusion relations for the $(n,\delta)-$ neighborhood of functions in the subclass $\kappa_n(p,\alpha,\lambda,\mu)$ of the class $T(n,p)$, which consists of functions $f \in T(n,p)$ satisfying the following nonhomogeneous Cauchy-Euler differential equation:

$$\begin{gathered} z^2\frac{d^2w}{dz^2} + 2(\mu+1)z\frac{dw}{dz} + \mu(\mu+1)w = (p+\mu)(p+\mu+1)g(z) \\ (w = f(z) \in T(n,p);\ g \in T^m(n,p,\alpha,\lambda);\ \mu > -p(\mu \in \mathbb{R})). \end{gathered} \tag{1.11}$$

2. Coefficient bounds and distortion inequalities for the class $T^m(n,p,\alpha,\lambda)$

In our present investigation of the class $T^m(n,p,\alpha,\lambda)$, we shall require Theorem 2.1 and Theorem 2.2 below.

THEOREM 2.1. *A function $f(z) \in T(n,p)$ is in the class $T^m(n,p,\alpha,\lambda)$ if and only if*

$$\begin{gathered} \sum_{k=n+p}^{\infty} k^m(k-\alpha)(p+\lambda k-\lambda p)a_k \leqslant p^{m+1}(p-\alpha) \\ (0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0). \end{gathered} \tag{2.1}$$

The result is sharp with the extremal function given by

$$f(z) = z^p - \frac{p^{m+1}(p-\alpha)}{(n+p)^m(n+p-\alpha)\,(p+\lambda n)} z^{n+p};\ (p,n \in \mathbb{N}; m \in \mathbb{N}_0). \tag{13}$$

Proof. Suppose that $f(z) \in T^m(n,p,\alpha,\lambda)$. Then from (1.8) we find that

$$\Re\left\{\frac{pz^p - \sum\limits_{k=n+p}^{\infty} \frac{k^{m+1}}{p^{m+1}}(p+\lambda k-\lambda p)a_k z^k}{z^p - \sum\limits_{k=n+p}^{\infty} \frac{k^m}{p^{m+1}}(p+\lambda k-\lambda p)a_k z^k}\right\} > \alpha$$

$$(0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0).$$

If we choose z to be real and let $z \to 1^-$; we get

$$\Re\left\{\frac{p-\sum\limits_{k=n+p}^{\infty}\frac{k^{m+1}}{p^{m+1}}(p+\lambda k-\lambda p)a_k}{1-\sum\limits_{k=n+p}^{\infty}\frac{k^{m}}{p^{m+1}}(p+\lambda k-\lambda p)a_k}\right\} \geqslant \alpha$$

$$(0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0)$$

or, equivalently,

$$\sum_{k=n+p}^{\infty}\frac{k^{m+1}}{p^{m+1}}(p+\lambda k-\lambda p)a_k-\alpha\sum_{k=n+p}^{\infty}\frac{k^{m}}{p^{m+1}}(p+\lambda k-\lambda p)a_k \leqslant (p-\alpha)$$

$$(0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0).$$

Thus we obtain

$$\sum_{k=n+p}^{\infty} k^m(k-\alpha)\left[p+\lambda k-\lambda p\right]a_k \leqslant p^{m+1}(p-\alpha)$$

$$(0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0).$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial U = \{z : z \in \mathbb{C}, |z| = 1\}.$$

Then, from the definition (1.1) we find that

$$\left|\frac{(1-\lambda)z(D^mf(z))'+\lambda z\left(D^{m+1}f(z)\right)'}{(1-\lambda)D^mf(z)+\lambda D^{m+1}f(z)}-p^{m+1}(p-\alpha)\right|$$

$$=\left|\frac{pz^p-\sum\limits_{k=n+p}^{\infty}\frac{k^{m+1}}{p^{m+1}}(p+\lambda k-\lambda p)a_kz^k}{z^p-\sum\limits_{k=n+p}^{\infty}\frac{k^{m}}{p^{m+1}}(p+\lambda k-\lambda p)a_kz^k}-p^{m+1}(p-\alpha)\right|$$

$$=\left|\frac{p^{m+1}\{p-p^{m+1}(p-\alpha)\}z^p-\sum\limits_{k=n+p}^{\infty}\{k^{m+1}(p+\lambda k-\lambda p)-p^{m+1}(p-\alpha)k^m(p+\lambda k-\lambda p)\}a_kz^k}{p^{m+1}z^p-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_kz^k}\right|$$

$$\leqslant\frac{\left|p^{m+1}\{p-p^{m+1}(p-\alpha)\}z^p\right|+\sum\limits_{k=n+p}^{\infty}\left|\left\{\left[k-p^{m+1}(p-\alpha)\right]k^m(p+\lambda k-\lambda p)\right\}a_kz^k\right|}{\left|p^{m+1}z^p\right|-\sum\limits_{k=n+p}^{\infty}\left|k^m(p+\lambda k-\lambda p)a_kz^k\right|}$$

$$=\frac{p^{m+1}\left|p-p^{m+1}(p-\alpha)\right|+\sum\limits_{k=n+p}^{\infty}\left|k-p^{m+1}(p-\alpha)\right|k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= \frac{p^{m+1}\left\{-\left[p-p^{m+1}(p-\alpha)\right]\right\}+\sum\limits_{k=n+p}^{\infty}\left\{k-p^{m+1}(p-\alpha)\right\}k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= \frac{p^{m+1}\left\{-p+p^{m+1}(p-\alpha)\right\}+\sum\limits_{k=n+p}^{\infty}\left\{p+(k-p)-p^{m+1}(p-\alpha)\right\}k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= \frac{p^{m+1}\left\{-p+p^{m+1}(p-\alpha)\right\}+\left\{p-p^{m+1}(p-\alpha)\right\}\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$+\frac{\sum\limits_{k=n+p}^{\infty}(k-p)\,k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= \frac{\left\{-p+p^{m+1}(p-\alpha)\right\}\left\{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k\right\}}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$+\frac{\sum\limits_{k=n+p}^{\infty}[k-\alpha+\alpha-p]\,k^m(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= -p+p^{m+1}(p-\alpha)+\frac{(\alpha-p)\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k+\sum\limits_{k=n+p}^{\infty}k^m(k-\alpha)(p+\lambda k-\lambda p)a_k}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$\leqslant -p+p^{m+1}(p-\alpha)+\frac{(\alpha-p)\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k+p^{m+1}(p-\alpha)}{p^{m+1}-\sum\limits_{k=n+p}^{\infty}k^m(p+\lambda k-\lambda p)a_k}$$

$$= p^{m+1}(p-\alpha)-p+p-\alpha$$

$$= p^{m+1}(p-\alpha)-\alpha$$

$$(0 \leqslant \alpha < p; 0 \leqslant \lambda \leqslant 1; 1 \leqslant p^m(p-\alpha); p \in \mathbb{N}; n \in \mathbb{N}; m \in \mathbb{N}_0)$$

provided that the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem, we have

$$f(z) \in T^m(n,p,\alpha,\lambda).$$

Finally, we note that the assertion (2.1) of Theorem 2.1 is sharp, the extremal function being

$$f(z) = z^p - \frac{p^{m+1}(p-\alpha)}{(n+p)^m(n+p-\alpha)(p+\lambda n)} z^{n+p}; \ (p,n \in \mathbb{N}; m \in \mathbb{N}_0).$$

THEOREM 2.2. *Let the function $f(z)$ given by (1.1) be in the class $T^m(n,p,\alpha,\lambda)$. Then*

$$\sum_{k=n+p}^{\infty} a_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^m(n+p-\alpha)(p+\lambda n)} \tag{14}$$

and

$$\sum_{k=n+p}^{\infty} k a_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)(p+\lambda n)}. \tag{15}$$

Proof. By using Theorem 2.1, we find from (2.1) that

$$\begin{aligned}(n+p)^m(n+p-\alpha)(p+\lambda n)\sum_{k=n+p}^{\infty} a_k &\leqslant \sum_{k=n+p}^{\infty} k^m(k-\alpha)(p+\lambda k-\lambda p)\, a_k \\ &\leqslant p^{m+1}(p-\alpha),\end{aligned}$$

which immediately yields the first assertion (2.3) of Theorem 2.2.

On the other hand, by appealing to (2.1), we also have

$$(p+\lambda n)(n+p)^m \sum_{k=n+p}^{\infty} (k-\alpha) a_k \leqslant p^{m+1}(p-\alpha),$$

that is,

$$(p+\lambda n)(n+p)^m \sum_{k=n+p}^{\infty} k a_k \leqslant p^{m+1}(p-\alpha) + (p+\lambda n)(n+p)^m \alpha \sum_{k=n+p}^{\infty} a_k$$

which, in view of the coefficient inequality (2.3), can be put in the form:

$$(p+\lambda n)(n+p)^m \sum_{k=n+p}^{\infty} k a_k \leqslant p^{m+1}(p-\alpha) + (p+\lambda n)(n+p)^m \alpha \frac{p^{m+1}(p-\alpha)}{(n+p)^m (n+p-\alpha)(p+\lambda n)}$$

or, equivalently,

$$\sum_{k=n+p}^{\infty} k a_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)(p+\lambda n)}.$$

THEOREM 2.3. *If $f \in T(n,p)$ is in the class $\kappa_n(p,\alpha,\lambda,\mu)$, then*

$$|f(z)| \leqslant |z|^p + \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^m(n+p-\alpha)(p+\lambda n)(n+p+\mu)} |z|^{n+p} \quad (z \in U) \tag{16}$$

and

$$|f(z)| \geqslant |z|^p - \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^m(n+p-\alpha)(p+\lambda n)(n+p+\mu)} |z|^{n+p} \quad (z \in U). \tag{17}$$

Proof. Suppose that $f \in T(n,p)$ is given by (1.1). Also let the function $g \in T^m(n,p,\alpha,\lambda)$, occurring in the nonhomogeneous Cauchy-Euler differential equation (1.11), be given as in the definitions (1.2) and (1.3) with, of course,

$$b_k \geqslant 0 \ \ (k = n+p, n+p+1, n+p+2, ...).$$

Then we readily find from (1.11) that

$$a_k = \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} b_k \ \ (k = n+p, n+p+1, n+p+2, ...) \tag{18}$$

so that

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k = z^p - \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} b_k z^k \tag{19}$$

and

$$|f(z)| \leqslant |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} b_k \ \ (z \in U). \tag{20}$$

Next, since $g \in T^m(n,p,\alpha,\lambda)$, the first assertion (2.3) of Theorem 2.2 yields the coefficient inequality:

$$b_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^m(n+p-\alpha)\,(p+\lambda n)} \ \ (k = n+p, n+p+1, n+p+2, ...) \tag{21}$$

which, in conjunction with (2.9), yields

$$|f(z)| \leqslant |z|^p + \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^m(n+p-\alpha)\,(p+\lambda n)} |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} \ \ (z \in U). \tag{22}$$

Finally, in view of the telescopic sum:

$$\begin{aligned} \sum_{k=n+p}^{\infty} \frac{1}{(k+\mu)(k+\mu+1)} &= \sum_{k=n+p}^{\infty} \left(\frac{1}{(k+\mu)} - \frac{1}{(k+\mu+1)}\right) \\ &= \lim_{q\to\infty} \sum_{k=n+p}^{q} \left(\frac{1}{(k+\mu)} - \frac{1}{(k+\mu+1)}\right) \\ &= \lim_{q\to\infty} \left(\frac{1}{n+p+\mu} - \frac{1}{q+\mu+1}\right) \\ &= \frac{1}{n+p+\mu} (\mu \in \mathbb{R} - \{-p-n, -p-n-1, ...\}), \end{aligned} \tag{2.12}$$

the first assertion (2.5) of Theorem 2.3 follows at once from (2.11).

Similarly, we can write

$$|f(z)| \geqslant |z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} b_k \ \ (z \in U). \tag{24}$$

Since $g \in T^m(n,p,\alpha,\lambda)$, the first assertion (2.3) of Theorem 2.2 yields the coefficients inequality:

$$b_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^m(n+p-\alpha)\,(p+\lambda n)} \quad (k=n+p,n+p+1,n+p+2,...) \qquad (25)$$

which, in conjunction with (2.13), yields

$$|f(z)| \geqslant |z|^p - \frac{p^{m+1}(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)^m(n+p-\alpha)\,(p+\lambda n)}\,|z|^{n+p}\sum_{k=n+p}^{\infty}\frac{1}{(k+\mu)(k+\mu+1)} \quad (z\in U). \qquad (26)$$

Finally, in view of the telescopic sum:

$$\begin{aligned}\sum_{k=n+p}^{\infty}\frac{1}{(k+\mu)(k+\mu+1)} &= \sum_{k=n+p}^{\infty}\left(\frac{1}{(k+\mu)}-\frac{1}{(k+\mu+1)}\right)\\ &= \lim_{q\to\infty}\sum_{k=n+p}^{q}\left(\frac{1}{(k+\mu)}-\frac{1}{(k+\mu+1)}\right)\\ &= \frac{1}{n+p+\mu} \quad (\mu\in\mathbb{R}-\{-p-n,-p-n-1,...\})\end{aligned} \qquad (2.16)$$

the second assertion (2.6) of Theorem 2.3 follows at once from (2.15).

By setting $m=0, \lambda=0$ in Theorem 2.3, we arrive to Corollary 2.1.

COROLLARY 2.1. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation (1.11) with $g \in S_n^*(p,\alpha)$, then*

$$|z|^p - \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}\,|z|^{n+p} \leqslant |f(z)| \leqslant |z|^p + \frac{(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p-\alpha)(n+p+\mu)}\,|z|^{n+p}.$$

By setting $m=0,\ \lambda=1$ in Theorem 2.3, we arrive to Corollary 2.2.

COROLLARY 2.2. *If the functions f and g satisfy the nonhomogeneous Cauchy-Euler differential equation (1.11) with $g \in C_n(p,\alpha)$, then*

$$|z|^p - \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}\,|z|^{n+p} \leqslant |f(z)| \leqslant |z|^p + \frac{p(p-\alpha)(p+\mu)(p+\mu+1)}{(n+p)(n+p-\alpha)(n+p+\mu)}\,|z|^{n+p}.$$

3. Neighborhoods for the classes $T^m(n,p,\alpha,\lambda)$ and $\kappa_n(p,\alpha,\lambda,\mu)$

In this section, we determine inclusion relations for the classes $T^m(n,p,\alpha,\lambda)$ and $\kappa_n(p,\alpha,\lambda,\mu)$ involving the $(n,\delta)-$ neighborhoods defined by (1.2) and (1.3).

THEOREM 3.1. *If $f \in T(n,p)$ is in the class $T^m(n,p,\alpha,\lambda)$, then*

$$T^m(n,p,\alpha,\lambda) \subset N_{n,\delta}(h), \qquad (28)$$

where $h(z)$ is given by (1.4) and

$$\delta := \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)}.$$

Proof. The assertion (3.1) would follow easily from the definition (1.3) of $N_{n,\delta}(h)$ and the second assertion (2.4) of Theorem 2.2.

THEOREM 3.2. *If $f \in T(n,p)$ is in the class $\kappa_n(p,\alpha,\lambda,\mu)$, then*

$$\kappa_n(p,\alpha,\lambda,\mu) \subset N_{n,\delta}(g), \tag{29}$$

where

$$\delta := \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)} \left\{ \frac{n+(p+\mu)(p+\mu+2)}{(n+p+\mu)} \right\}. \tag{30}$$

Proof. Suppose that $f \in \kappa_n(p,\alpha,\lambda,\mu)$. Then upon substituting from (2.7) into the coefficient inequality:

$$\sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqslant \sum_{k=n+p}^{\infty} k(|a_k| + |b_k|) \leqslant \sum_{k=n+p}^{\infty} ka_k + \sum_{k=n+p}^{\infty} kb_k\,(a_k \geqslant 0; b_k \geqslant 0),$$

we obtain

$$\sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqslant \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)} kb_k + \sum_{k=n+p}^{\infty} kb_k. \tag{31}$$

Next, since $g \in T^m(n,p,\alpha,\lambda)$, the second assertion (2.4) of Theorem 2.2 yields

$$kb_k \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)} \quad (k = n+p, n+p+1, ...). \tag{32}$$

Finally, by making use of (2.4) as well as (3.5) on the right-hand side of (3.4), we find that

$$\sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)} (1 + \sum_{k=n+p}^{\infty} \frac{(p+\mu)(p+\mu+1)}{(k+\mu)(k+\mu+1)}), \tag{33}$$

which, by virtue of the telescopic sum (2.12), immediately yields

$$\begin{aligned}\sum_{k=n+p}^{\infty} k\,|a_k - b_k| &\leqslant \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)} \left\{ 1 + \frac{(p+\mu)(p+\mu+1)}{(n+p+\mu)} \right\} \\ &= \frac{p^{m+1}(p-\alpha)}{(n+p)^{m-1}(n+p-\alpha)\,(p+\lambda n)} \left\{ \frac{n+(p+\mu)(p+\mu+2)}{(n+p+\mu)} \right\} \\ &= \delta.\end{aligned}$$

Thus, by the definition (1.2), $f \in N_{n,\delta}(g)$. This evidently completes the proof of Theorem 3.2.

By setting $m = 0, p = 1, \lambda = 0$ in Theorem 3.1, we arrive to Corollary 3.1 [3].

COROLLARY 3.1. *If $f \in T(n,1) = A(n)$ is in the class $T^0(n,1,\alpha,0) = S_n^*(\alpha)$, then*

$$S_n^*(\alpha) \subset N_{n,\delta}(h),$$

where $h(z)$ *is given by* $h(z) = z$ *and*

$$\delta := \frac{(n+1)(1-\alpha)}{(n+1-\alpha)}.$$

By setting $m = 0, p = 1, \lambda = 1$ in Theorem 3.1, we arrive to Corollary 3.2 [3].

COROLLARY 3.2. *If* $f \in T(n,1) = A(n)$ *is in the class* $T^0(n,1,\alpha,1) = C_n(\alpha)$, *then*

$$C_n(\alpha) \subset N_{n,\delta}(h),$$

where $h(z)$ *is given by* $h(z) = z$ *and*

$$\delta := \frac{(1-\alpha)}{(n+1-\alpha)}.$$

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