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SOME MAXIMUM PRINCIPLES FOR A CLASS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. For a class of elliptic boundary value problems we construct, in this paper, some general elliptic inequalities from which we derive maximum principles and we also give some applications in physics and geometry of surfaces.

1. Introduction

The maximum principle is one of the most useful and best known tools employed in the study of partial differential equations to obtain informations for such topics as uniqueness, approximation and boundedness of solutions without any explicit knowledge of solutions themselves.

The goal of this paper is to employ Hopf's maximum principles [1], [2] to derive bounds for some quantities related to the following class of nonlinear boundary value problems:

$$\left(g\left(\mathbf{x},|\nabla u|^2\right)u_{,i}\right)_{,i}+h(\mathbf{x})f(u)=0, \mathbf{x}\in\Omega, \tag{1.1}$$

$$u=0, \mathbf{x}\in\partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N\geqslant 2,$ with smooth boundary $\partial\Omega\in C^{2,\varepsilon}$, and f, g and h are given functions assumed to satisfy the following conditions:

$$f,\ h\geqslant 0,\quad g>0, \tag{1.3}$$

$$f,\ h\in C^1,\quad g\in C^2. \tag{1.4}$$

Moreover, we assume that (1.1) is uniformly elliptic, i.e. we impose throughout the strong ellipticity condition

$$G(\mathbf{x},s):=g(\mathbf{x},s)+2s\frac{\partial g}{\partial s}>0,\ s>0, \mathbf{x}\in\Omega. \tag{1.5}$$

Under these assumptions, it then follows, from the maximum principle, that the solutions $u(\mathbf{x})$ of (1.1) assume their minima on $\partial\Omega$.

For the class of nonlinear boundary problems (1.1)-(1.2), sufficient conditions on f, g and h for the existence of classical solutions are known and have been well studied

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in the literature. For an account on these topics we refer, for instance, to Krylov [3] or to Ladyzenskaya and Uraltzeva [4]. Consequently, we shall tacitly assume the existence of classical solutions of the problems considered in this paper.

Some particular cases of our work have been considered and investigated in previous works, see for instance Payne [5], Payne and Phillipin [6], [7] Payne and Stakgold [8], Philippin [9], [10], Schaeffer [12], Schaeffer and Sperb [13], [14]. In order to handle our more general case, we make use of the techniques developed in these papers. The main tools of our investigations are Hopf's first and second maximum principles [1], [2]. We refer to the books of Protter and Weinberger [11] and of Sperb [17] for expository texts on these topics.

In this paper, we consider only two particular cases for the function g, namely $g = g(|\nabla u|^2)$ in Section 2, and $g = g(\mathbf{x})$ in Section 3. In both cases, we shall derive some maximum principles for appropriate combinations of u and $|\nabla u|^2$. These combinations will be of the following form

$$\Phi(\mathbf{x},a,b) := \int_0^{|\nabla u|^2} G(s)ds + 2a\int_0^u f(s)ds + bu^2, \tag{1.6}$$

in Section 2, with $G(s) := g(s) + 2sg'(s) > 0$, where a and b are some real positive parameters to be appropriately chosen, and

$$\Psi(\mathbf{x},\alpha,\beta) := |\nabla u|^2 + 2\alpha\int_0^u f(s)ds + \beta u^2, \tag{1.7}$$

in Section 3, where α and β are also real positive parameters to be appropriately chosen. Some possible applications in physics or geometry of surfaces will be given in Section 4.

The notations $u_{,i} := \dfrac{\partial u}{\partial x_i}$, $u_{,ij} := \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ will be used throughout the paper and summation from 1 to N is understood on repeated indices. Using these notations we have, for instance,

$$u_{,ij}u_{,i}u_{,j} = \sum_{i=1}^N\sum_{j=1}^N \frac{\partial^2 u}{\partial x_i\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}. \tag{1.8}$$

2. Derivation of maximum principles for Φ

Since the particular case $h \equiv const.$ has already been treated by Payne and Philippin in [7], we consider only the general case $h(\mathbf{x}) \neq const.$, $g = g\left(|\nabla u|^2\right)$. From (1.6), we compute succesively

$$\Phi_{,k} = 2Gu_{,ik}u_{,i} + 2afu_{,k} + 2buu_{,k}, \tag{2.1}$$

$$\begin{aligned}\Phi_{,kj} = {} & 4G'u_{,ik}u_{,i}u_{,lj}u_{,l} + 2G\left[u_{,ikj}u_{,i} + u_{,ik}u_{,ij}\right] \\ & + 2af'u_{,k}u_{,j} + 2afu_{,kj} + 2bu_{,j}u_{,k} + 2buu_{,kj},\end{aligned} \tag{2.2}$$

$$\Delta\Phi = 4G' u_{,ik}u_{,i}u_{,lk}u_{,l} + 2G\left[(\Delta u)_{,i}u_{,i} + u_{,ik}u_{,ik}\right] + 2af'|\nabla u|^2 + 2af\Delta u + 2b|\nabla u|^2 + 2bu\Delta u. \tag{2.3}$$

Next, we replace Δu and $(\Delta u)_{,i}u_{,i}$ in (2.3) using the differential equation (1.1) in the equivalent form

$$\Delta u = -2\frac{g'}{g}u_{,lk}u_{,l}u_{,k} - \frac{fh}{g}. \tag{2.4}$$

Differentiating (2.4), we obtain

$$(\Delta u)_{,i}u_i = -4\left(\frac{g'}{g}\right)'(u_{,lk}u_{,l}u_{,k})^2 - 2\frac{g'}{g}\left[u_{,ikl}u_{,i}u_{,k}u_{,l} + 2u_{,lk}u_{,k}u_{,li}u_{,i}\right] - 2fh\left(\frac{1}{g}\right)'u_{,ij}u_{,i}u_{,j} - \frac{f'h}{g}|\nabla u|^2 - \frac{f}{g}h_{,i}u_{,i}. \tag{2.5}$$

Now, we would like to construct a second order elliptic differential inequality for Φ that contains no third order derivatives of u. This will be achieved if we consider the following operator

$$L\Phi := \Delta\Phi + 2\frac{g'}{g}\Phi_{,kj}u_{,k}u_{,j}, \tag{2.6}$$

for which we obtain after some reductions

$$L\Phi = 4\left[G' - \frac{Gg'}{g}\right]u_{,ik}u_{,i}u_{,lk}u_{,l} + 8\left[\frac{g'}{g}G' - G\left(\frac{g'}{g}\right)'\right](u_{,lk}u_{,l}u_{,k})^2 - 4Gfh\left(\frac{1}{g}\right)'u_{,ij}u_{,i}u_{,j} - 2\frac{Gf'h}{g}|\nabla u|^2 - 2\frac{Gf}{g}h_{,i}u_{,i} + 2Gu_{,ik}u_{,ik} + 2af'|\nabla u|^2 - 2a\frac{f^2h}{g} + 4b\frac{g'}{g}|\nabla u|^4 + 4a\frac{f'g'}{g}|\nabla u|^4 + 2b|\nabla u|^2 - 2b\frac{fh}{g}u. \tag{2.7}$$

Making use of the Cauchy-Schwarz inequality in the following form

$$|\nabla u|^2 u_{,ik}u_{,ik} \geqslant u_{,ik}u_{,k}u_{,ij}u_{,j}, \tag{2.8}$$

and of (2.1), we obtain

$$u_{,ik}u_{,ik} \geqslant \frac{(af+bu)^2}{G^2} + ..., \text{ in } \Omega\backslash\omega. \tag{2.9}$$

In (2.9), $\omega := \{\mathbf{x} \in \Omega : \nabla u(\mathbf{x}) = 0\}$ is the set of critical points of u and dots stand for terms containing $\Phi_{,k}$. We also make use of (2.1) to obtain the following equations

$$u_{,ik}u_{,i}u_{,k} = -\frac{(af+bu)}{G}|\nabla u|^2 + \cdots, \tag{2.10}$$

$$(u_{,ik}u_{,i}u_{,k})^2 = \frac{(af+bu)^2}{G^2}|\nabla u|^4 + \cdots, \tag{2.11}$$

$$u_{,ik}u_{,i}u_{,lk}u_{,l} = \frac{(af+bu)^2}{G^2}|\nabla u|^2 + \cdots, \tag{2.12}$$

where dots have the same meaning as above.

Next, the insertion of (2.9), (2.10), (2.11) and (2.12) in (2.8) gives, after some reductions,

$$L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \geqslant \frac{2G}{g}\left\{ [(a-h)f' + b]\, |\nabla u|^2 - f\, h_{,i} u_{,i} + \frac{1}{g}\left[(af + bu)^2 - f\, h\,(af + bu) \right] \right\}, \tag{2.13}$$

valid in $\Omega\backslash\omega$, where W_k is the k-th component of a vector field regular throughout Ω. Now, we consider the following two inequalities:

$$(af + bu)^2 - f\, h\,(af + bu) \geqslant \left[\left(a - \frac{h}{2} \right)^2 - \frac{h^2}{2} \right] f^2, \tag{2.14}$$

and

$$\left[b + (a-h)f' \right] |\nabla u|^2 - f\, h_{,i} u_{,i} \geqslant -\frac{|\nabla h|^2 f^2}{4\left[b + (a-h)f' \right]}, \tag{2.15}$$

valid if $b + (a-h)f' > 0$. Inserting (2.14) and (2.15) in (2.13) we obtain, in $\Omega\backslash\omega$, the following inequality

$$L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \geqslant \frac{2G}{g^2} f^2 \left\{ \left(a - \frac{h}{2} \right)^2 - \frac{h^2}{2} - \frac{|\nabla h|^2 g}{4\left[b + (a-h)f' \right]} \right\}. \tag{2.16}$$

Consequently, we obtain

$$L\Phi + |\nabla u|^{-2} W_k \Phi_{,k} \geqslant 0, \text{ in } \Omega\backslash\omega, \tag{2.17}$$

if the positive constants a and b are chosen to satisfy the following two conditions:

$$b + (a-h)f' > 0, \tag{2.18}$$

$$a \geqslant \max_{\Omega} \left(\frac{h(\mathbf{x})}{2} + \sqrt{\frac{h^2(\mathbf{x})}{2} + \frac{|\nabla h|^2 g}{4\left[b + (a-h)f' \right]}} \right) := a_1. \tag{2.19}$$

The following result is now a direct consequence of Hopf's first maximum principle [1], [11]:

THEOREM 2.1. *Let* $u(\mathbf{x})$ *be a classical solution of* (1.1), *with* $g = g\left(|\nabla u|^2 \right)$, *in a bounded domain* $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, *and let* $\Phi(\mathbf{x}, a, b)$ *be the function defined in* (1.6). *If the positive parameters* a *and* b *are chosen to satisfy* (2.18) – (2.19), *then the function* $\Phi(\mathbf{x}, a, b)$ *takes its maximum value either on* $\partial\Omega$ *or at a critical point of* u *(i.e. a point in* Ω *where* $\nabla u = 0$).

REMARKS.

(i) In the case $N = 2$, we may replace the inequality (2.9) by the following identity

$$u_{,ik} u_{,ik} |\nabla u|^2 = |\nabla u|^2 (\Delta u)^2 + 2 u_{,i} u_{,ij} u_{,k} u_{,kj} - 2 \Delta u\, u_{,ij} u_{,i} u_{,j}. \tag{2.20}$$

This identity leads to the same result if we replace the condition (2.19) by the following one:

$$a \geqslant \max_{\Omega}\left(\frac{3h(\mathbf{x})}{4}+\sqrt{\frac{10h^2(\mathbf{x})}{16}+\frac{|\nabla h|^2 g}{8\left[b+(a-h)f'\right]}}\right):=a_2. \tag{2.21}$$

(ii) a_1 or a_2 may be difficult to compute if g is not bounded.

(iii) Theorem 2.1 holds independently of the boundary conditions for $u(\mathbf{x})$. However, in what follows, we will show that $\Phi(\mathbf{x},a,b)$ cannot take its maximum value on $\partial\Omega$, if Ω is a convex domain in $\mathbb{R}^N$.

Suppose that $\Phi(\mathbf{x},a,b)$ takes its maximum value at **P** on $\partial\Omega$. Then, by Hopf's second maximum principle [2], [11] we must have $\Phi \equiv cte$ in Ω or $\frac{\partial\Phi}{\partial n} > 0$ at **P**. We now compute the outward normal derivative $\frac{\partial\Phi}{\partial n}$ at an arbitrary point of $\partial\Omega$. Since $u=0$ on $\partial\Omega$, we obtain

$$\frac{\partial\Phi}{\partial n} = 2Gu_n u_{nn} + 2af\,u_n. \tag{2.22}$$

From the differential equation (1.1), evaluated on $\partial\Omega \in C^{2,\varepsilon}$, we have

$$Gu_{nn} + g(N-1)Ku_n + f\,h = 0. \tag{2.23}$$

In (2.22) and (2.23), u_n and u_{nn} are the first and second outward normal derivatives of u on $\partial\Omega$, and K is the average curvature of $\partial\Omega$. The insertion of (2.23) in (2.22) leads to

$$\frac{\partial\Phi}{\partial n} = -2g(N-1)Ku_n^2 + 2(a-h)f\,u_n, \text{on } \partial\Omega. \tag{2.24}$$

Clearly, if a satisfies (2.18) or (2.21), we have $\frac{\partial\Phi}{\partial n} \leqslant 0$ on $\partial\Omega$, so that Φ cannot take its maximum on $\partial\Omega$. Note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Hopf's second principle [2], [11]. We formulate these results in the following theorem:

THEOREM 2.2. *Let $u(\mathbf{x})$ be a classical solution of* (1.1) - (1.2)*, with $g = g(|\nabla u|^2)$ in a bounded convex domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, and let $\Phi(\mathbf{x},a,b)$ be the function defined in* (1.6) *with a and b as in Theorem* 2.1*. Then the function $\Phi(\mathbf{x},a,b)$ takes its maximum value at a critical point of u.*

3. Derivation of maximum principles for Ψ

In this section we will investigate the problem (1.1) - (1.2) with $g = g(\mathbf{x})$. Starting from (1.7), we compute succesively

$$\Psi_{,k} = 2u_{,ik}u_{,i} + 2\alpha f\,u_{,k} + 2\beta u u_{,k}, \tag{3.1}$$

$$\nabla\left(g(\mathbf{x})\nabla\Psi\right) = 2\left\{gu_{,ik}u_{,ik} + \left(gu_{,ik}\right)_{,k}u_{,i} - (\alpha f + \beta u)f\,h + \beta g\,|\nabla u|^2 + \alpha f' g\,|\nabla u|^2\right\}. \tag{3.2}$$

Next, we differentiate (1.1) to obtain

$$\left(g_{,i}u_{,k} + gu_{,ki}\right)_{,k} = \left(gu_{,k}\right)_{,ki} = -h_{,i}f - hf'u_{,i}, \tag{3.3}$$

from which we compute

$$(gu_{,ik})_{,k}\,u_{,i} = -f\,\nabla h \nabla u - hf'\,|\nabla u|^2 - g_{,ik}u_{,k}u_{,i} - \Delta u \nabla g \nabla u. \tag{3.4}$$

Inserting (3.4) in (3.2) and making use of (1.1), we obtain

$$\begin{aligned}\frac{1}{2}\nabla\left(g(\mathbf{x})\nabla\Psi\right) =& gu_{,ik}u_{,ik} - f\,\nabla h\nabla u - g_{,ik}u_{,k}u_{,i} - \Delta u\nabla g\nabla u \\ & - (\alpha f + \beta u) f\,h + \beta g\,|\nabla u|^2 + f'\,|\nabla u|^2\,(\alpha g - h)\,.\end{aligned} \tag{3.5}$$

Now, we consider the following two inequalities:

$$u_{,ik}u_{,ik} \geqslant \frac{(\Delta u)^2}{N},\ \text{in } \Omega, \tag{3.6}$$

and

$$u_{,ik}u_{,ik} \geqslant |\nabla u|^{-2}\, u_{,ik}u_{,k}u_{,ij}u_{,j} = (\alpha f + \beta u)^2 + ...,\ \text{in } \Omega\backslash\omega. \tag{3.7}$$

In (3.7), ω is the set of critical points of u and dots stand for terms containing $\Psi_{,k}$. Combining (3.6) and (3.7) we obtain.

$$u_{,ik}u_{,ik} \geqslant \frac{1}{2}\left(\alpha f + \beta u\right)^2 + \frac{1}{2N}\left(\Delta u\right)^2 + |\nabla u|^{-2}\, W_k\Psi_{,k},\ \text{in } \Omega\backslash\omega, \tag{3.8}$$

where $\mathbf{W} := (W_1, ..., W_N)$ is a vector field regular throughout Ω. Insertion of (3.8) in (3.5) gives

$$\begin{aligned} L\Psi :=& \frac{1}{2}\nabla\left(g(\mathbf{x})\nabla\Psi\right) - |\nabla u|^{-2}\, W_k\Psi_{,k} \\ \geqslant& \frac{1}{2}g\left(\alpha f + \beta u\right)^2 + \frac{g}{2N}\left(\Delta u\right)^2 - f\,\nabla h\nabla u - g_{,ik}u_{,k}u_{,i} \\ & - \Delta u\nabla g\nabla u - (\alpha f + \beta u) f\,h + \beta g\,|\nabla u|^2 + f'\,|\nabla u|^2\,(\alpha g - h)\,,\end{aligned} \tag{3.9}$$

valid in $\Omega\backslash\omega$. Our computation makes use of the following relations

$$\alpha\beta g f\, u \geqslant 0, \tag{3.10}$$

$$\frac{1}{2}g\alpha^2 f^2 - \alpha f^2 h = \frac{1}{2}gf^2\left[\alpha - \frac{h}{g}\right]^2 - \frac{h^2 f^2}{2g}, \tag{3.11}$$

$$\frac{1}{2}g\beta^2 u^2 - \beta huf = \frac{1}{2}g\beta^2\left[u - \frac{hf}{\beta g}\right]^2 - \frac{h^2 f^2}{2g} \geqslant -\frac{h^2 f^2}{2g}, \tag{3.12}$$

$$\begin{aligned}\frac{g}{2N}\left(\Delta u\right)^2 - \Delta u\nabla g\nabla u =& \frac{g}{2N}\left[\Delta u - \frac{N\nabla g\nabla u}{g}\right]^2 - \frac{N\left(\nabla g\nabla u\right)^2}{2g} \\ & \geqslant -\frac{N\left(\nabla g\nabla u\right)^2}{2g} \geqslant -\frac{N\,|\nabla g|^2\,|\nabla u|^2}{2g},\end{aligned} \tag{3.13}$$

and

$$g_{,ik}u_{,i}u_{,k} \leqslant \sqrt{g_{,ik}g_{,ik}}\,|\nabla u|^2\,, \tag{3.14}$$

the last two inequalities being the consequence of the Cauchy-Schwarz inequality.

From (3.13) and (3.14) we obtain

$$\begin{aligned}\frac{g}{2N}(\Delta u)^2 - \Delta u \nabla g \nabla u - g_{,ik}u_{,i}u_{,k} - f\nabla h \nabla u + \beta g |\nabla u|^2 &\geqslant A|\nabla u|^2 - f\nabla h \nabla u \\ &\geqslant -\frac{f^2|\nabla h|^2}{4A},\end{aligned} \tag{3.15}$$

with

$$A := \beta g - \frac{N|\nabla g|^2}{2g} - \sqrt{g_{,ik}g_{,ik}} > 0. \tag{3.16}$$

Inserting (3.10) - (3.12) and (3.15) in (3.9), we obtain the inequality

$$L\Psi \geqslant f^2\left\{\frac{1}{2}g\left[\alpha - \frac{h}{g}\right]^2 - \frac{h^2}{g} - \frac{|\nabla h|^2}{4A}\right\} + f'(\alpha g - h)|\nabla u|^2, \tag{3.17}$$

in $\Omega\backslash\omega$. Consequently, we obtain

$$L\Psi \geqslant 0, \text{ in } \Omega\backslash\omega, \tag{3.18}$$

if the constants α and β are chosen to satisfy the following two conditions:

$$\beta > \max_{\Omega}\left\{\frac{N|\nabla g|^2}{2g^2} + \frac{\sqrt{g_{,ik}g_{,ik}}}{g}\right\}, \tag{3.19}$$

$$\alpha \geqslant \max_{\Omega}\left\{\frac{h}{g} + \sqrt{\frac{2h^2}{g^2} + \frac{|\nabla h|^2}{2gA}}\right\}, \tag{3.20}$$

and the function f satisfies

$$f' \geqslant 0. \tag{3.21}$$

The following result is now a direct consequence of Hopf's first maximum principle [1], [11]:

THEOREM 3.1. *Let $u(\mathbf{x})$ be a classical solution of (1.1), with $g = g(\mathbf{x})$, in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, and let $\Psi(\mathbf{x},\alpha,\beta)$ be the function defined in (1.7). If the positive parameters α and β are chosen to satisfy (3.19) - (3.20) and f satisf ies (3.21), then the function $\Psi(\mathbf{x},\alpha,\beta)$ takes its maximum value either on $\partial\Omega$ or at a critical point of u (i.e. a point in Ω where $\nabla u = 0$).*

Note that if $g(\mathbf{x})$ is concave, we may omit the term $-g_{,ik}u_{,i}u_{,k}$ in (3.9) so that the condition (3.16) may be replaced by

$$A := \beta g - \frac{N|\nabla g|^2}{2g} > 0. \tag{3.22}$$

As in Theorem 2.1, we note that the conclusion of Theorem 3.1 holds independently of the boundary conditions for $u(\mathbf{x})$. We shall now establish a condition on the average curvature K and on g which implies that the maximum of $\Psi(\mathbf{x},\alpha,\beta)$ cannot occur on $\partial\Omega$.

Suppose that $\Psi(\mathbf{x},\alpha,\beta)$ takes its maximum value at $\mathbf{P}$ on $\partial\Omega$. Then, by Hopf's second maximum principle [2], [11], we must have $\Psi \equiv cte.$ in Ω or $\dfrac{\partial\Psi}{\partial n} > 0$ at $\mathbf{P}$.

We now compute the outward normal derivative $\dfrac{\partial\Psi}{\partial n}$ at an arbitrary point of $\partial\Omega$. Since $u=0$ on $\partial\Omega$, we obtain

$$\frac{\partial\Psi}{\partial n} = 2u_n u_{nn} + 2\alpha f\, u_n. \tag{3.23}$$

From the differential equation (1.1), evaluated on $\partial\Omega \in C^{2,\varepsilon}$, we have

$$g\,\{u_{nn} + (N-1)Ku_n\} + \nabla g \nabla u = -f\,h. \tag{3.24}$$

In (3.23) and (3.24), u_n and u_{nn} are the first and second outward normal derivatives of u on $\partial\Omega$ and K is the average curvature of $\partial\Omega$. The insertion of (3.24) in (3.23) leads to

$$\frac{\partial\Psi}{\partial n} = -\frac{2}{g}\left[(N-1)Kg + \frac{\partial g}{\partial n}\right]|\nabla u|^2 + 2u_n f\left[\alpha - \frac{h}{g}\right], \text{ on } \partial\Omega. \tag{3.25}$$

Clearly, if α and β satisfy (3.19) and (3.20), and if the condition

$$(N-1)Kg + \frac{\partial g}{\partial n} \geqslant 0, \text{ on } \partial\Omega, \tag{3.26}$$

is satisfied, then we have $\dfrac{\partial\Psi}{\partial n} \leqslant 0$ on $\partial\Omega$. It then follows that Ψ cannot take its maximum on $\partial\Omega$. Note that $\nabla u \neq 0$ on $\partial\Omega$ in view of Hopf's second maximum principle [2], [11]. Moreover, if $\dfrac{\partial g}{\partial n} \leqslant 0$ on $\partial\Omega$, condition (3.26) is satisfied for convex domains. We formulate these results in the following theorem:

THEOREM 3.2. *Let $u(\mathbf{x})$ be a classical solution of (1.1)-(1.2), with $g = g(\mathbf{x})$, in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, and let $\Psi(\mathbf{x},\alpha,\beta)$ be the function defined in (1.7) with α, β and f as in Theorem 3.1. If the condition (3.26) is satisfied, then the function $\Psi(\mathbf{x},\alpha,\beta)$ takes its maximum value at a critical point of u.*

4. Applications

4.1. The Poisson equation

Let $u(\mathbf{x})$ be the solution of the Poisson equation

$$\Delta u = -h(\mathbf{x}),\ \mathbf{x} \in \Omega, \tag{4.1}$$

subject to the Dirichlet boundary condition

$$u = 0,\ \mathbf{x} \in \partial\Omega, \tag{4.2}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$, and $h \in C^1(\Omega)$ is a nonnegative function.

Here, we have $g = 1$, $f = 1$, and the corresponding auxiliary function $\Phi(\mathbf{x},a,b)$ defined in (1.6) is

$$\Phi(\mathbf{x},a,b) = |\nabla u|^2 + 2au + bu^2. \tag{4.3}$$

Theorem 2.2 implies that $\Phi(\mathbf{x},a,b)$ takes its maximum value in a critical point of u if we choose

$$a \geqslant \max_{\Omega}\left(\frac{h(\mathbf{x})}{2} + \sqrt{\frac{h^2(\mathbf{x})}{2} + \frac{|\nabla h|^2}{4b}}\right), \tag{4.4}$$

where b is an arbitrary positive constant. This leads to the following inequality

$$|\nabla u|^2 \leqslant 2a\left(u_m - u\right) + b\left(u_m^2 - u^2\right) = b\left\{\left(u_m + \frac{a}{b}\right)^2 - \left(u + \frac{a}{b}\right)^2\right\}, \tag{4.5}$$

where $u_m := \max_{\overline{\Omega}} u(\mathbf{x})$. Inequality (4.5) may be used to derive an upper bound for u_m. To this end, let $\mathbf{P}$ be a point where $u = u_m$ and $\mathbf{Q}$ a point on $\partial\Omega$ nearest to $\mathbf{P}$. Let r measure the distance from $\mathbf{P}$ along the ray connecting $\mathbf{P}$ and $\mathbf{Q}$. Clearly, we have

$$-\frac{du}{dr} \leqslant |\nabla u|. \tag{4.6}$$

Integrating (4.6) from $\mathbf{Q}$ to $\mathbf{P}$ and making use of (4.5), we obtain

$$\int_0^{u_m} \frac{du}{\sqrt{\left(u_m + \frac{a}{b}\right)^2 - \left(u + \frac{a}{b}\right)^2}} \leqslant \sqrt{b}\int_{\mathbf{P}}^{\mathbf{Q}} dr = \sqrt{b}\delta \leqslant \sqrt{b}d, \tag{4.7}$$

where $\delta = d(\mathbf{P},\mathbf{Q})$ and d is the radius of the largest ball inscribed in Ω.

Choosing now b (> 0) such that $\sqrt{b}d < \frac{\pi}{2}$, we obtain the inequality

$$\frac{a/b}{u_m + a/b} \geqslant \cos\left(d\sqrt{b}\right), \tag{4.8}$$

which leads to the following upper bound for u_m

$$u_m \leqslant \frac{a}{b}\left(\frac{1}{\cos\left(d\sqrt{b}\right)} - 1\right). \tag{4.9}$$

For instance, taking

$$b := \frac{\pi^2}{16d^2}, \quad a = \frac{h_m}{2} + \sqrt{\frac{h_m^2}{2} + \frac{4d^2}{\pi^2}}, \tag{4.10}$$

with $h_m := \max_{\overline{\Omega}} h(\mathbf{x})$, we obtain

$$u_m \leqslant \frac{a}{b}\left(\sqrt{2} - 1\right) = \frac{16d^2}{\pi^2}\left(\sqrt{2} - 1\right)\left(\frac{h_m}{2} + \sqrt{\frac{h_m^2}{2} + \frac{4d^2}{\pi^2}}\right). \tag{4.11}$$

4.2. The Saint-Venant equation

Let $u(\mathbf{x})$ be the solution of the Saint-Venant equation

$$\nabla\left(g(\mathbf{x})\nabla u\right) = -1, \ \mathbf{x} \in \Omega, \tag{4.12}$$

subject to the Dirichlet boundary condition

$$u = 0, \ \mathbf{x} \in \partial\Omega, \tag{4.13}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$, and $g \in C^2(\Omega)$ is a positive function satisfying the condition (3.26) and

$$\max_{\Omega}\left\{\frac{N|\nabla g|^2}{2g^2} + \frac{\sqrt{g_{,ik}g_{,ik}}}{g}\right\} < \frac{\pi^2}{4d^2}, \tag{4.14}$$

where d is the radius of the largest ball inscribed in Ω. With $f = 1,\ h = 1,$ the auxiliary function $\Psi(\mathbf{x},\alpha,\beta)$ defined in (1.7) is

$$\Psi(\mathbf{x},\alpha,\beta) = |\nabla u|^2 + 2\alpha u + \beta u^2. \tag{4.15}$$

Choosing the parameters α and β to satisfy the following conditions

$$\alpha \geqslant \left(1+\sqrt{2}\right) \max_{\Omega} \left\{ \frac{1}{g(\mathbf{x})} \right\}, \tag{4.16}$$

$$\frac{\pi^2}{4d^2} > \beta > \max_{\Omega} \left\{ \frac{N|\nabla g|^2}{2g^2} + \frac{\sqrt{g_{,ik}g_{,ik}}}{g} \right\}, \tag{4.17}$$

it then follows from Theorem 3.2 that $\Psi(\mathbf{x},\alpha,\beta)$ takes its maximum value at a critical point of u. This leads to the inequality

$$|\nabla u|^2 \leqslant 2\alpha(u_m - u) + \beta\left(u_m^2 - u^2\right) = \beta\left\{ \left(u_m + \frac{\alpha}{\beta}\right)^2 - \left(u + \frac{\alpha}{\beta}\right)^2 \right\}, \tag{4.18}$$

where $u_m := \max_{\overline{\Omega}} u(\mathbf{x})$. In the same way as in Section 4.1, we obtain

$$u_m \leqslant \frac{\alpha}{\beta}\left(\frac{1}{\cos\left(d\sqrt{\beta}\right)} - 1 \right). \tag{4.19}$$

For instance, with $\Omega := \left\{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}| \leqslant R\right\}$ and $g(\mathbf{x}) := 1 + |\mathbf{x}|^2$, we have

$$\max_{\Omega} \left\{ \frac{N|\nabla g|^2}{2g^2} + \frac{\sqrt{g_{,ik}g_{,ik}}}{g} \right\} \leqslant 2\left(NR^2 + \sqrt{N}\right), \tag{4.20}$$

and we may choose $\alpha := 1+\sqrt{2},\ \beta := 2\left(NR^2 + \sqrt{N}\right)$. Then, the first inequality in (4.17) will be satisfied if $R \leqslant \left\{ \sqrt{\frac{\pi^2}{8} - \frac{1}{4N}} - \frac{1}{2\sqrt{N}} \right\}^{1/2}$.

4.3. The first eigenfunction of a semilinear operator

Let $u(\mathbf{x})$ be the solution of the following semi-linear problem

$$\nabla\left(g(\mathbf{x})\nabla u\right) + \lambda_1 u(\mathbf{x}) = 0,\ u > 0,\ \mathbf{x} \in \Omega, \tag{4.21}$$

with Dirichlet boundary condition

$$u = 0,\ \mathbf{x} \in \partial\Omega. \tag{4.22}$$

Ω is a bounded convex domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$, and $g \in C^2(\Omega)$ is a positive function satisfying the condition (3.26). Here we have $f(s) = s,\ g = g(\mathbf{x})$ and $h = \lambda_1$, so that the auxiliary function $\Psi(\mathbf{x},\alpha,\beta)$ defined in (1.7) is

$$\Psi(\mathbf{x},\alpha,\beta) = |\nabla u|^2 + (\alpha+\beta)\, u^2. \tag{4.23}$$

Thus, if the parameters α and β are chosen to satisfy the following conditions:

$$\alpha \geqslant \left(1+\sqrt{2}\right) \lambda_1 \max_{\Omega} \left\{ \frac{1}{g\left(\mathbf{x}\right)} \right\}, \tag{4.24}$$

$$\beta > \max_{\Omega} \left\{ \frac{N \left|\nabla g\right|^2}{2g^2} + \frac{\sqrt{g_{,ik} g_{,ik}}}{g} \right\}, \tag{4.25}$$

then Theorem 3.2 implies that $\Psi(\mathbf{x}, \alpha, \beta)$ takes its maximum value in a critical point of u. This leads to the inequality

$$\left|\nabla u\right|^2 \leqslant (\alpha+\beta) \left(u_m^2 - u^2\right), \tag{4.26}$$

where $u_m := \max_{\overline{\Omega}} u(\mathbf{x})$. Let $\mathbf{P}$ be a point where $u = u_m$ and $\mathbf{Q}$ a point on $\partial\Omega$ nearest to $\mathbf{P}$. Inequality (4.26) may be used, as in the previous applications, to find the inequality

$$\frac{\pi}{2} = \int_0^{u_m} \frac{du}{\sqrt{u_m^2 - u^2}} \leqslant \sqrt{\alpha+\beta}\delta, \tag{4.27}$$

where $\delta = d(\mathbf{P}, \mathbf{Q})$. This shows that the critical points of $u(\mathbf{x})$ are at distance $\delta \geqslant \dfrac{\pi}{2\sqrt{\alpha+\beta}}$ from the boundary.

4.4. The equation of a surface of prescribed mean curvature $\Lambda(\mathbf{x})$

Let Λ be the mean curvature of a surface S given in nonparametric form by:

$$x_{N+1} = u(x_1, x_2, ..., x_N), \quad (x_1, x_2, ..., x_N) \in \Omega, \tag{4.28}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega \in C^{2,\varepsilon}$.

The differential equation of this surface is given by

$$\left(\frac{u_{,i}}{\sqrt{1+\left|\nabla u\right|^2}} \right)_{,i} = -N\Lambda, \ \mathbf{x} \in \Omega. \tag{4.29}$$

We are concerned with the homogeneous Dirichlet problem, i.e.

$$u = 0, \text{ on } \partial\Omega, \tag{4.30}$$

and we consider the particular case $N = 2$ for simplicity. In this case Serrin's existence criterion [16] asserts that the curvature K of $\partial\Omega$ should satisfy

$$K(s) \geqslant 2\Lambda, \tag{4.31}$$

at all points of ∂D.

Here $g(s) = \dfrac{1}{\sqrt{1+s}}$, $f = 1$ and $h = 2\Lambda$ so that the corresponding auxiliary function $\Phi(\mathbf{x}, a, b)$ is

$$\Phi(\mathbf{x}, a, b) = 2\left(1 - \frac{1}{\sqrt{1+\left|\nabla u\right|^2}} + au\right) + bu^2. \tag{4.32}$$

It follows from Theorem 2.2 that $\Phi(\mathbf{x}, a, b)$ takes its maximum value in a critical point of u if we choose

$$a \geqslant \max_{\Omega} \left(\frac{\Lambda(\mathbf{x})}{2} + \sqrt{\frac{\Lambda^2(\mathbf{x})}{2} + \frac{|\nabla\Lambda|^2}{4b}} \right), \tag{4.33}$$

where b is an arbitrary positive constant. We are then led to the following inequality

$$\frac{1}{\sqrt{1+|\nabla u|^2}} \geqslant 1-a\,(u_m-u) - \frac{b}{2}\left(u_m^2-u^2\right) = 1-\frac{b}{2}\left\{ \left(u_m+\frac{a}{b}\right)^2 - \left(u+\frac{a}{b}\right)^2 \right\}, \text{ in } \Omega, \tag{4.34}$$

from which a lower bound may be obtained for $|\nabla u|^2$.

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