Mathematical
Inequalities
& Applications
Volume 9, Number 4 (2006), 707–716

THE STABILITY OF A WILSON TYPE AND A PEXIDER TYPE FUNCTIONAL EQUATION

YONG-SOO JUNG AND KIL-WOUNG JUN

(*communicated by Th. M. Rassias*)

Abstract. In this paper we study the stability of the Wilson type functional equation $f(x+y)-f(x-y)=2g(x)g(y)$ and the Pexider type functional equation $f(x+y-xy)=(1-x)^{\alpha}g(y)+(1-y)^{\alpha}h(x)$, respectively.

1. Introduction

Given an approximately homomorphism, is it possible to approximate it by a true homomorphism? This problem is called the stability of functional equations which was originally raised by S. M. Ulam [25]. For Banach spaces, the problem was first solved by D. H. Hyers [9] which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$||f(x+y)-f(x)-f(y)|| \leqslant \delta$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x)-T(x)|| \leqslant \delta$$

for all $x,\ y \in X$. In connection with these results, we say that the additive functional equation $f(x+y)=f(x)+f(y)$ is stable in the sense of Hyers and Ulam. This terminology is also applied to other functional equations.

Th. M. Rassias [16] generalized the theorem of Hyers by considering the case when the inequality is not bounded. The Rassias type results obtained by modifying the result can be found in [5, 7, 11, 13, 17-22]. Afterwards, the stability problem of functional equations has been extended in various directions (see, for example, [4, 8, 19, 20]). In particular, the stability of trigonometric and related maps was investigated by J. Baker [3], P. W. Cholewa [4], and R. Ger and P. Šemrl [8], etc.

We now introduce the following Wilson type functional equation

$$f(x+y)-f(x-y)=2g(x)g(y) \tag{1}$$

Mathematics subject classification (2000): 39B52, 39B72.

Key words and phrases: functional equation, stability.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2005-070-C00009).

Paper MIA-09-62

which is a special case of the Wilson's functional equation

$$f(x+y)+g(x-y)=h(x)k(y)$$

(see [1]). It is easy to see that the hyperbolic functions

$$f=cosh, \quad g=sinh$$

satisfy the functional equation (1). We here deal with the stability of the functional equation (1).

Recently Gy. Maksa and Zs. Páles [15] obtained the stability result for the following functional equation on the unit interval $(0,1]$:

$$f(xy)=x^{\alpha}f(y)+y^{\alpha}f(x), \tag{2}$$

where α is a fixed real number. The Hyers-Ulam stability of the equation (2) for the case $\alpha=1$ was proposed by Gy. Maksa [14] and was established by J. Tabor [24].

In this note we also examine the Hyers-Ulam stability of the following Pexider type functional equation on the unit interval $[0,1)$ inspired by the above equation:

$$f(x+y-xy)=(1-x)^{\alpha}g(y)+(1-y)^{\alpha}h(x). \tag{3}$$

2. Stability of the equation (1)

Throughout this section, G will represent an Abelian group with the group operation denoted by $+$ in which division by 2 is uniquely determined, and $\mathbb{C}$ will denote a complex field.

Our first result is

THEOREM 2.1. *Let $f,\ g: G\to\mathbb{C}$ be functions satisfying the inequality*

$$|f(x+y)-f(x-y)-2g(x)g(y)|\leqslant\delta \tag{4}$$

for all $x,\ y\in G$ and some $\delta\geqslant 0$. Then either g is bounded (equivalently, f is bounded) or g solves the sine functional equation

$$s(x+y)s(x-y)=s(x)^2-s(y)^2.$$

Proof. Let g be bounded. Putting $y=x$ in (4) yields

$$|f(2x)-f(0)-2g(x)^2|\leqslant\delta,$$

for all $x\in R$ and then setting $x=\frac{x}{2}$, we get

$$\left|f(x)-f(0)-2g\left(\frac{x}{2}\right)^2\right|\leqslant\delta$$

for all $x\in R$. Hence it follows that

$$\begin{aligned}|f(x)|&=\left|f(x)-f(0)-2g\left(\frac{x}{2}\right)^2+f(0)+2g\left(\frac{x}{2}\right)^2\right|\\&\leqslant\left|f(x)-f(0)-2g\left(\frac{x}{2}\right)^2\right|+|f(0)|+2\left|g\left(\frac{x}{2}\right)\right|^2\end{aligned}$$

for all $x\in R$ which implies that f is bounded.

Conversely, suppose that f is bounded, i.e, there exists a $K > 0$ such that $|f(x)| \leqslant K$ for all $x \in G$. By the above inequality $|f(2x) - f(0) - 2g(x)^2| \leqslant \delta$, we have

$$\begin{aligned}|2g(x)^2| &\leqslant |2g(x)^2 - f(2x) + f(0)| + |f(2x)| + |f(0)| \\ &\leqslant \delta + K + |f(0)| := M\end{aligned}$$

for all $x \in G$ which reduces to

$$|g(x)| \leqslant \frac{\sqrt{2M}}{2}$$

for all $x \in G$, that is, g is bounded. Therefore, f is bounded if and only if g is bounded. On the other words, g is unbounded if and only if f is unbounded.

Assume that $g : G \to \mathbb{C}$ is an unbounded solution of inequality (4). Putting $y = 0$ in (4) and dividing the result by $2|g(z)|$, it follows that $g(0) = 0$ since g is unbounded. Since g is unbounded, we also choose an x_0 in G with $|g(x_0)| \geqslant 2$. Now let $h : G \to \mathbb{C}$ be a function defined by

$$h(x) = \frac{g(x + x_0) - g(x - x_0)}{2g(x_0)}$$

for all $x \in G$. For convenience, set

$$\varphi(x, y) := |g(x + y) + g(x - y) - 2g(x)h(y)| \tag{5}$$

for all $x \in G$. We claim that for all $x,\ y \in G$,

$$\varphi(x, y) \leqslant \delta. \tag{6}$$

Utilizing the definition of h and (4), we see that

$$\begin{aligned}2|g(x_0)|\varphi(x, y) =& |2g(x + y)g(x_0) + 2g(x - y)g(x_0) - 4g(x_0)g(x)h(y)| \\ \leqslant& |2g(x + y)g(x_0) - f(x + y + x_0) + f(x + y - x_0)| \\ &+ |2g(x - y)g(x_0) - f(x - y + x_0) + f(x - y - x_0)| \\ &+ |f(x + y + x_0) - f(x - y - x_0) - 2g(x)g(y + x_0)| \\ &+ |2g(x)g(y - x_0) - f(x + y - x_0) + f(x - y + x_0)| \\ &+ |2g(x)(g(y + x_0) - g(y - x_0)) - 4g(x_0)g(x)h(y)| \\ \leqslant& 4\delta,\end{aligned}$$

which gives the inequality $\varphi(x, y) \leqslant \delta$ for all $x,\ y \in G$ since $|g(x_0)| \geqslant 2$.

Making use of (4), (5) and (6), we have for all $x,\ y,\ z \in G$,

$$\begin{aligned}2|g(z)|\varphi(x, y) =& |2g(z)g(x + y) + 2g(z)g(x - y) - 4g(x)g(z)h(y)| \\ \leqslant& |2g(z)g(x + y) - f(z + x + y) + f(z - x - y)| \\ &+ |2g(z)g(x - y) - f(z + x - y) + f(z - x + y)| \\ &+ |f(z + x + y) - f(z - x + y) - 2g(z + y)g(x)| \\ &+ |f(z + x - y) - f(z - x - y) - 2g(z - y)g(x)| \\ &+ |2(g(z + y) + g(z - y))g(x) - 4g(z)h(y)g(x)| \\ \leqslant& 4\delta + 2\varphi(z, y)|g(x)| \\ \leqslant& 4\delta + 2\delta|g(x)|,\end{aligned}$$

that is,

$$|g(z)|\varphi(x,y) \leqslant 2\delta + \delta|g(x)| \tag{7}$$

for all $x,y,z \in G$. Let x and y be arbitrary but fixed while z ranges over G. Dividing two-sided of (7) by $|g(z)|$, it follows from the unboundedness of g that $\varphi(x,y) = 0$ for all $x,y \in G$ which yields the functional equation

$$g(x+y) + g(x-y) = 2g(x)h(y). \tag{8}$$

Letting $x = 0$ in (8), we obtain that $g(-y) = -g(y)$ for all $y \in G$. If we make the change of variables $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$ in (8), then we get

$$g(u) + g(v) = 2g\Big(\frac{u+v}{2}\Big)h\Big(\frac{u-v}{2}\Big).$$

for all $u,v \in G$. Thus, for all $x,y \in G$, we obtain that

$$g(x+y) = g(x+y) + g(0) = 2g\Big(\frac{x+y}{2}\Big)h\Big(\frac{x+y}{2}\Big),$$

and that

$$g(x-y) = g(x-y) + g(0) = 2g\Big(\frac{x-y}{2}\Big)h\Big(\frac{x-y}{2}\Big),$$

Consequently, we get

$$g(x+y)g(x-y) = \Big[2g\Big(\frac{x+y}{2}\Big)h\Big(\frac{x-y}{2}\Big)\Big]\Big[2g\Big(\frac{x-y}{2}\Big)h\Big(\frac{x+y}{2}\Big)\Big].$$

for all $x,y \in G$. Hence, by (8), we have

$$\begin{aligned} g(x+y)g(x-y) &= [g(x)+g(y)][g(x)+g(-y)] \\ &= [g(x)+g(y)][g(x)-g(y)] \\ &= g(x)^2 - g(y)^2 \end{aligned}$$

for all $x,y \in G$, that is, g is a solution of the sine functional equation.

This completes the proof of the theorem. □

REMARK. Let f, $g : G \to \mathbb{C}$ be unbounded solutions of the equation (1). Take the functions $\overline{f}$, $\overline{g}$ defined on a group G with values in the algebra $M_2(\mathbb{C})$ of all complex 2×2-matrices given by

$$\overline{f}(x) = \begin{pmatrix} f(x) & 0 \\ 0 & c \end{pmatrix} \text{ and } \overline{g}(x) = \begin{pmatrix} g(x) & 0 \\ 0 & d \end{pmatrix}$$

for all $x \in G$, where c and d is the positive real constants. Then

$$\|\overline{f}(x+y) - \overline{f}(x-y) - 2\overline{g}(x)\overline{g}(y)\| = \text{constant} > 0$$

for all $x,\ y \in G$. Hence this difference is bounded but $\overline{g}$ is neither bounded nor satisfies the sine functional equation. This example shows that the stability of the equation (1) fails to hold in the case of vector-valued functions.

Nevertheless, we obtain a vector-valued analogue of Theorem 2.1.

THEOREM 2.2. *Let B be a semisimple commutative Banach algebra and let f, g : $G \to B$ be functions satisfying the inequality*

$$\|f(x+y)-f(x-y)-2g(x)g(y)\| \leqslant \delta \tag{9}$$

for all x, $y \in G$ and some $\delta \geqslant 0$. Then

$$g(x+y)g(x-y) = g(x)^2 - g(y)^2,$$

provided that for an arbitrary multiplicative linear functional $z^* \in B^*$ the superposition $z^* \circ g$ fails to be bounded (equivalently, the superposition $z^* \circ f$ fails to be bounded).

Proof. The arguments used in [6, p. 6, Theorem 3] carry over almost verbatim.

In fact, we assume that the inequality (9) is fulfilled for all $x, y \in G$ and that we are given an arbitrary fixed multiplicative linear functional $z^* \in B^*$.

Since $\|z^*\| = 1$, we have, for all $x, y \in G$,

$$\begin{aligned}\delta &\geqslant \|f(x+y)-f(x-y)-2g(x)g(y)\| \\ &= \sup_{\|z^*\|=1} |z^*(f(x+y)-f(x-y)-2g(x)g(y))| \\ &\geqslant |z^*(f(x+y)) - z^*(f(x-y)) - 2z^*(g(x))z^*(g(y))|,\end{aligned}$$

which shows that the superpositions $z^* \circ f$ and $z^* \circ g$ are solutions of the inequality (4). As in the proof of Theorem 2.1 with the above inequality, the superposition $z^* \circ g$ is unbounded if and only if the superposition $z^* \circ f$ is unbounded. Since, by hypothesis, the superposition $z^* \circ g$ is unbounded, $z^* \circ g$ satisfies the sine functional equation in view of Theorem 2.1. Since z^* was arbitrary in B^*, we see that

$$g(x+y)g(x-y) - g(x)^2 + g(y)^2 \in \bigcap_{z^* \in B^*} \ker z^*$$

for all $x, y \in G$. Since $\bigcap_{z^* \in B^*} \ker z^*$ is the Jacobson radical of B and B is semisimple, we conclude that

$$g(x+y)g(x-y) - g(x)^2 + g(y)^2 = 0$$

for all $x, y \in G$, as claim and the proof is complete. □

3. Solutions and stability of the equation (3)

Throughout this section, X will denote a real (or complex) vector space.

We will first find out the general solution of the functional equation (3).

THEOREM 3.1. *Let α be any fixed real number. The functions $f, g, h : [0,1) \to X$ satisfy the equation* (3) *for all x, $y \in [0,1)$ if and only if, for all x, $y \in [0,1)$,*

$$f(x) = (1-x)^{\alpha}[\ell(1-x) + f(0)] \tag{10}$$

$$g(x) = (1-x)^{\alpha}[\ell(1-x) + g(0)] \tag{11}$$

and

$$h(x) = (1-x)^{\alpha}[\ell(1-x) + h(0)], \tag{12}$$

where $\ell : (0,1] \to X$ is a logarithmic function and $f(0) = g(0) + h(0)$.

Proof. $(\Rightarrow)$ Define the functions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ on $(0,1]$ by

$$\tilde{f}(x) = x^{-\alpha}f(1-x),\ \ \tilde{g}(x) = x^{-\alpha}g(1-x)\ \text{ and }\ \tilde{h}(x) = x^{-\alpha}h(1-x),$$

respectively. Then the equation (3) yields the well-known Pexider equation

$$\tilde{f}(xy) = \tilde{h}(x) + \tilde{g}(y)$$

for all $x,y \in (0,1]$. By applying [2, p. 43, Theorem 9], we easily obtain (10), (11) and (12).

$(\Leftarrow)$ This is obvious. □

Let us now investigate the Hyers-Ulam stability of the equation (3).

DEFINITION. A function $g : [0,\infty) \to [0,\infty)$ is called exponentially increasing if it is increasing and there exist constants $\gamma > 1$ and $h \in [0,\infty)$ such that

$$g(x+h) \geqslant \gamma g(x)$$

for all $x \in [0,\infty)$.

First, we state a result of J. Tabor [24] concerning the stability of the additive functional equation $f(x+y) = f(x) + f(y)$ on the interval $[0,\infty)$.

THEOREM 3.2. *Assume that $g : [0,\infty) \to [0,\infty)$ is exponentially increasing with γ and h as in Definition and $g(0) > 0$. Let $K = 2\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma-1}$. Let X be a sequentially complete topological vector space and V be a closed convex, bounded and symmetric with respect to zero subset of X. If $f : [0,\infty) \to X$ is a function such that*

$$f(x+y) - f(x) - f(y) \in g(x+y)V$$

for all $x \in [0,\infty)$, then there exists a unique additive function $A : [0,\infty) \to X$ such that $A(h) = f(h)$ and that

$$f(x) - A(x) \in Kg(x)V$$

for all $x \in [0,\infty)$.

Our stability result concerning the equation (3) is

THEOREM 3.3. *Let X be a Banach space and α be any fixed real number. Let $f,g,h : [0,1) \to X$ be functions satisfying the inequality*

$$\|f(x+y-xy) - (1-x)^{\alpha}g(y) - (1-y)^{\alpha}h(x)\| \leqslant \varepsilon \tag{13}$$

for all $x,y \in [0,1)$ and some $\varepsilon \geqslant 0$. Then there exist functions $\Phi,\Psi,\Lambda : [0,1) \to X$ satisfying the functional equation (3) such that

$$\|f(x) - \Phi(x)\| \leqslant \kappa\varepsilon, \tag{14}$$

$$\|g(x) - \Psi(x)\| \leqslant (\kappa+1)\varepsilon \tag{15}$$

and

$$\|h(x) - \Lambda(x)\| \leqslant (\kappa+1)\varepsilon, \tag{16}$$

for all $x \in [0,1)$, where $\kappa := 3$ if $\alpha = 0$ and $\kappa := 9+6\sqrt{2}$ if $\alpha \neq 0$.

Proof. Let us define functions $p,q,r:[0,1)\to X$ by

$$p(x)=\frac{f(x)}{(1-x)^{\alpha}},\quad q(x)=\frac{g(x)}{(1-x)^{\alpha}}\ \text{ and }\ r(x)=\frac{h(x)}{(1-x)^{\alpha}}$$

for all $x\in[0,1)$, respectively. Then the inequality (13) yields the inequality

$$\|p(x+y-xy)-q(y)-r(x)\|\leqslant\frac{\varepsilon}{(1-x)^{\alpha}(1-y)^{\alpha}} \tag{17}$$

for all $x,\ y\in[0,1)$.

Assume that $\alpha\geqslant 0$. Then the functions $P,Q,R:[0,\infty)\to X$ defined by

$$P(u)=p(1-e^{-u}),\quad Q(u)=q(1-e^{-u})\ \text{ and }\ R(u)=r(1-e^{-u})$$

for all $u\in[0,\infty)$ satisfy the inequality

$$\begin{aligned}\|P(u+v)-Q(u)-R(v)\|&=\|p(1-e^{-(u+v)})-q(1-e^{-u})-r(1-e^{-v})\|\\&=\|p((1-e^{-u})+(1-e^{-v})-(1-e^{-u})(1-e^{-v}))\\&\quad-q(1-e^{-u})-r(1-e^{-v})\|\\&\leqslant\varepsilon e^{\alpha(u+v)}\end{aligned}$$

which is

$$\|P(u+v)-Q(u)-R(v)\|\leqslant\varepsilon e^{\alpha(u+v)} \tag{18}$$

for all $u,\ v\in[0,\infty)$. Putting $v=0$ in (18), we get

$$\|P(u)-Q(u)-R(0)\|\leqslant\varepsilon e^{\alpha u} \tag{19}$$

for all $u\in[0,\infty)$. Letting $u=0$ in (18), we have

$$\|P(v)-Q(0)-R(v)\|\leqslant\varepsilon e^{\alpha v} \tag{20}$$

for all $v\in[0,\infty)$. We now define a function $F:[0,\infty)\to X$ by

$$F(u)=P(u)-Q(0)-R(0) \tag{21}$$

for all $u\in[0,\infty)$. We assert that

$$\|F(u+v)-F(u)-F(v)\|\leqslant 3\varepsilon e^{\alpha(u+v)}$$

for all $u,\ v\in[0,\infty)$.

Indeed, it follows from (18), (19), (20) and (21) that

$$\begin{aligned}\|F(u+v)-F(u)-F(v)\|&=\|P(u+v)-P(u)-P(v)+G(0)+R(0)\|\\&\leqslant\|P(u+v)-Q(u)-R(v)\|+\|Q(u)-P(u)+R(0)\|\\&\quad+\|R(v)-P(v)+Q(0)\|\\&\leqslant\varepsilon e^{\alpha(u+v)}+\varepsilon e^{\alpha u}+\varepsilon e^{\alpha v}.\end{aligned}$$

Hence we have

$$\|F(u+v)-F(u)-F(v)\|\leqslant 3\varepsilon e^{\alpha(u+v)} \tag{22}$$

for all $u,\ v\in[0,\infty)$.

In the case $\alpha = 0$ in (22), by employing Skof's result [23], we see that there exists a unique additive function $A_1 : \mathbb{R} \to X$ such that

$$\|F(u) - A_1(u)\| \leqslant 3\varepsilon \tag{23}$$

for all $u \in [0, \infty)$.

Let $\alpha > 0$ in (22). Note that the exponential function $e^{(\cdot)}$ is exponentially increasing with $h := -ln(1-z)$, $\gamma := e^{\alpha h} = \frac{1}{(1-z)^{\alpha}}$, where $z \in (0,1)$ in Theorem 3.2. Therefore, by applying (22) to Theorem 3.2, we see that there exists an additive function $A_2 : [0,\infty) \to X$ such that

$$\|F(u) - A_2(u)\| \leqslant (9 + 6\sqrt{2})\varepsilon e^{\alpha u} \tag{24}$$

for all $u \in [0,\infty)$ since the minimal value of $K = \frac{2}{(1-z)^{\alpha}} + \frac{1}{1-(1-z)^{\alpha}}$ in $(0,1)$, in view of Theorem 3.2, is equal to $3 + \sqrt{2}$ when $z = 1 - (2 - \sqrt{2})^{1/\alpha}$.

Now we suppose that $\alpha < 0$. Defining the functions $P, Q, R : [0, \infty) \to X$ by

$$P(u) = p(1 - e^u), \ \ Q(u) = q(1 - e^u) \ \text{ and } \ R(u) = r(1 - e^u)$$

for all $u \in [0, \infty)$, the inequality (17) gives the inequality

$$\|P(u+v) - Q(u) - R(v)\| \leqslant \varepsilon e^{-\alpha(u+v)}$$

for all $u,\ v \in [0,\infty)$. If we pass through the same process as in (19) $\sim$ (22), then we also get the inequality

$$\|F(u+v) - F(u) - F(v)\| \leqslant 3\varepsilon e^{-\alpha(u+v)} \tag{25}$$

for all $u,\ v \in [0,\infty)$, and so it follows from (25) with Theorem 3.2 that there exists an additive function $A_3 : [0,\infty) \to X$ such that

$$\|F(u) - A_3(u)\| \leqslant (9 + 6\sqrt{2})\varepsilon e^{-\alpha u} \tag{26}$$

for all $u \in [0,\infty)$ as in the case $\alpha \geqslant 0$.

For the sake of convenience, let us $\kappa := 3$ if $\alpha = 0$ and $\kappa := 9 + 6\sqrt{2}$ if $\alpha \neq 0$. Now the inequalities (23), (24) and (26) can be rewritten to the form

$$\|F(u) - A(u)\| \leqslant \kappa\varepsilon e^{\beta u} \tag{27}$$

for all $u \in [0,\infty)$, where $\beta := \alpha$, $A := A_1$(or A_2) if $\alpha \geqslant 0$ and $\beta := -\alpha$, $A := A_3$ if $\alpha < 0$.

Let $\vartheta := -1$ if $\alpha \geqslant 0$ and $\vartheta := 1$ if $\alpha \leqslant 0$. With the definition of F and P, the inequality (27) implies that

$$\|p(x) - A(\vartheta ln(1-x)) - q(0) - r(0)\| \leqslant \frac{\kappa\varepsilon}{(1-x)^{\alpha}},$$

that is,

$$\left\|\frac{f(x)}{(1-x)^{\alpha}} - A(\vartheta ln(1-x)) - g(0) - h(0)\right\| \leqslant \frac{\kappa\varepsilon}{(1-x)^{\alpha}}$$

which gives

$$\|f(x) - (1-x)^{\alpha}[A(\vartheta ln(1-x)) + g(0) + h(0)]\| \leqslant \kappa\varepsilon \tag{28}$$

for all $x \in [0,1)$.

On the other hand, the inequalities (19), (21) and (27) yield that

$$\begin{aligned}\|Q(v)-A(v)-Q(0)\| &= \|Q(v)-A(v)+F(v)-P(v)+R(0)\| \\ &\leqslant \|F(v)-A(v)\|+\|P(v)-Q(v)-R(0)\| \\ &\leqslant (\kappa+1)\varepsilon e^{\alpha v}\end{aligned}$$

for all $v \in [0,\infty)$, and so this and the definition of Q, q imply that

$$\left\|\frac{g(x)}{(1-x)^{\alpha}}-A(\vartheta ln(1-x))-g(0)\right\| \leqslant \frac{(\kappa+1)\varepsilon}{(1-x)^{\alpha}},$$

that is,

$$\|g(x)-(1-x)^{\alpha}[A(\vartheta ln(1-x))+g(0)]\| \leqslant (\kappa+1)\varepsilon \tag{29}$$

for all $x \in [0,1)$. We use the inequalities (20), (21) and (27) to obtain the inequality

$$\begin{aligned}\|R(v)-A(v)-R(0)\| &= \|R(v)-A(v)+F(v)-P(v)+Q(0)\| \\ &\leqslant \|F(v)-A(v)\|+\|P(v)-R(v)-Q(0)\| \\ &\leqslant (\kappa+1)\varepsilon e^{\alpha v}\end{aligned}$$

for all $v \in [0,\infty)$, and hence from this and the definition of Q, q, it follows that

$$\left\|\frac{h(x)}{(1-x)^{\alpha}}-A(\vartheta ln(1-x))-h(0)\right\| \leqslant \frac{(\kappa+1)\varepsilon}{(1-x)^{\alpha}},$$

that is,

$$\|h(x)-(1-x)^{\alpha}[A(\vartheta ln(1-x))+h(0)]\| \leqslant (\kappa+1)\varepsilon \tag{30}$$

for all $x \in [0,1)$.

Finally, for all $x \in [0,1)$, setting

$$\begin{aligned}\Phi(x) &= (1-x)^{\alpha}[A(\vartheta ln(1-x))+g(0)+h(0)], \\ \Psi(x) &= (1-x)^{\alpha}[A(\vartheta ln(1-x))+g(0)]\end{aligned}$$

and

$$\Lambda(x) = (1-x)^{\alpha}[A(\vartheta ln(1-x))+h(0)],$$

we see that the functions Φ, Ψ and Λ satisfy the functional equation (3), that is,

$$\Phi(x+y-xy) = (1-x)^{\alpha}\Psi(y)+(1-y)^{\alpha}\Lambda(x)$$

for all $x,y \in [0,1)$, and the inequalities (28), (29) and (30) give the inequalities (14), (15) and (16), respectively. The proof of the theorem is complete. □

Acknowledgments The authors wish to thank Professor Th. M. Rassias and the anonymous referees for their valuable comments.

REFERENCES

[1] J. ACZÉL, *Lectures on Functional Equations and Their Applications*, Academic Press, New York/London, 1966.
[2] J. ACZÉL, J. DHOMBRES, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.
[3] J. BAKER, *The stability of the cosine equation*, Proc. Amer. Math. Soc., **80** (1980), 411–416.
[4] P. W. CHOLEWA, *The stability of the sine equation*, Proc. Amer. Math. Soc., **88** (1983), 631–634.
[5] S. CZERWIK, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Hamburg, **62** (1992), 59–64.

[6] Z. DARÓCZY, ZS. PÁLES,, *Functional Equations-Results and Advances*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.
[7] P. GĂVRUŢA, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436.
[8] R. GER, P. ŠEMRL, *The stability of the exponential equation*, Proc. Amer. Math. Soc., **124** (1996), 117–125.
[9] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A., **27** (1941), 222–224.
[10] G. ISAC, TH. M. RASSIAS, *Functional inequalities for approximately additive mappings*, in Stability of Mappings of Hyers-Ulam Type (Th. M. Rassias and J. Tabor, Eds.), (1994), 117–125, Hadronic Press, Palm Harbour, FL.
[11] S.-M. JUNG, *Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl., **232** (1999), 384–392.
[12] S.-M. JUNG, *On the superstability of the functional equation* $f(x^y) = yf(x)$, Abh. Math. Sem. Univ. Hamburg, **67** (1997), 315–322.
[13] Y.-H. LEE, K.-W. JUN, *A generalization of the Hyers-Ulam-Rassias stability of Pexider equation*, J. Math. Anal. Appl., **246** (2000), 627–638.
[14] GY. MAKSA, *Problems 18, In Report on the 34th ISFE*, Aequationes Math., **53** (1997), 194.
[15] GY. MAKSA, ZS. PÁLES, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paed. Nyhazi., **17** (2001), 107–112.
[16] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300.
[17] TH. M. RASSIAS, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., **62** (2000), 23–130.
[18] TH. M. RASSIAS, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), 264–284.
[19] TH. M. RASSIAS (ED.), *Functional Equations and inequalities*, Kluwer Academic, Dordrecht/ Boston/ London, 2000.
[20] TH. M. RASSIAS, J. TABOR, *Stability of mappings of Hyers-Ulam type*, Hadronic Press, Inc., Florida, 1994.
[21] TH. M. RASSIAS, *On the the behavior of mappings which does not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., **114** (1992), 989–993.
[22] TH. M. RASSIAS, J. TABOR, *What is left of Hyers-Ulam stability?*, Journal of Natural Geometry, **1** (1992), 65–69.
[23] F. SKOF, *Sull'approssimazione delle appliazioni localmente* δ*-additive*, Atti Accad. Sc. Torino, **117** (1983), 377–389.
[24] J. TABOR, *Stability of the Cauchy functional equation with variable bound*, Publ. Math. Debrecen, **51** (1997), 165–173.
[25] S. M. ULAM, *Problems in Modern Mathematics,*, Chap. VI, Wiley, New York, 1964.

(Received March 19, 2005)

Yong-Soo Jung
Department of Mathematics
Chungnam National University
Taejon 305-764
Korea
e-mail: ysjung@math.cnu.ac.kr

Kil-Woung Jun
Department of Mathematics
Chungnam National University
Taejon 305-764
Korea
e-mail: kwjun@math.cnu.ac.kr