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ON ALMOST INCREASING SEQUENCES FOR GENERALIZED ABSOLUTE SUMMABILITY

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Abstract. In this paper, we establish a summability factor theorem for summability $|A,\delta|_k$ as defined in (2) where A is a lower triangular matrix with non-negative entries satisfying certain conditions.

This paper is an extension of the main result of [3] using definition (2) below.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{v=0}^{n} a_{nv} s_v.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geqslant 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty. \tag{1}$$

and it is said to be summable $|A,\delta|_k, k \geqslant 1$ and $\delta \geqslant 0$ if (see,[1])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1} |A_n - A_{n-1}|^k < \infty. \tag{2}$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \qquad n, v = 0, 1, 2, \ldots,$$

and

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \qquad n = 1, 2, 3, \ldots.$$

We may write

$$T_n = \sum_{i=0}^{n} a_{ni} \sum_{v=0}^{i} a_v \lambda_v = \sum_{v=0}^{n} a_v \lambda_v \sum_{i=v}^{n} a_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} a_v \lambda_v.$$

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Thus

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{v=0}^{n} \bar{a}_{nv} a_v \lambda_v - \sum_{v=0}^{n-1} \bar{a}_{n-1,v} a_v \lambda_v \\
&= \sum_{v=0}^{n} \bar{a}_{nv} a_v \lambda_v - \sum_{v=0}^{n} \bar{a}_{n-1,v} a_v \lambda_v \\
&= \sum_{v=0}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) a_v \lambda_v \\
&= \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v (s_v - s_{v-1}) \\
&= \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v s_v - \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v s_{v-1} \\
&= \sum_{v=1}^{n-1} \hat{a}_{nv} \lambda_v s_v + \hat{a}_{nn} \lambda_n s_n - \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v s_{v-1} \\
&= \sum_{v=1}^{n-1} \hat{a}_{nv} \lambda_v s_v + a_{nn} \lambda_n s_n - \sum_{v=0}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} s_v \\
&= \sum_{v=1}^{n} (\hat{a}_{nv} \lambda_v - \hat{a}_{n,v+1} \lambda_{v+1}) s_v + a_{nn} \lambda_n s_n.
\end{aligned}$$

We may write

$$\begin{aligned}
(\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1}) &= \hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v \\
&= (\hat{a}_{nv} - \hat{a}_{n,v+1})\lambda_v + \hat{a}_{n,v+1}(\lambda_v - \lambda_{v+1}) \\
&= \lambda_v \Delta_v \hat{a}_{nv} + \hat{a}_{n,v+1} \Delta\lambda_v.
\end{aligned}$$

Therefore

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{v=0}^{n-1} \Delta_v \hat{a}_{nv} \lambda_v s_v + \sum_{v=0}^{n-1} \hat{a}_{n,v+1} \Delta\lambda_v s_v + a_{nn} \lambda_n s_n \\
&= T_{n1} + T_{n2} + T_{n3}, \quad \text{say.}
\end{aligned}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \leqslant b_n \leqslant Bc_n$. Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = e^{(-1)^n} n$.

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry a_{nv}, $\Delta_v a_{nv} := a_{nv} - a_{nv+1}$.

THEOREM 1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that*

(i) $|\Delta\lambda_n| \leqslant \beta_n$,

(ii) $\lim_n \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$,

and

(iv) $|\lambda_n|X_n = O(1)$.

Let A be a lower triangular matrix with non-negative entries satisfying

(v) $na_{nn} \asymp O(1)$,

(vi) $a_{n-1,\nu} \geqslant a_{n\nu}$ *for* $n \geqslant \nu+1$,

(vii) $\bar{a}_{n0} = 1$ *for all* n,

$(viii)$ $\sum_{\nu=1}^{n-1} a_{\nu\nu}\hat{a}_{n\nu+1} = O\Big(a_{nn}\Big)$,

(ix) $\sum_{n=\nu+1}^{m+1} n^{\delta k}|\Delta_\nu \hat{a}_{n\nu}| = O\Big(\nu^{\delta k}a_{\nu\nu}\Big)$

and

(x) $\sum_{n=\nu+1}^{m+1} n^{\delta k}\hat{a}_{n\nu+1} = O\Big(\nu^{\delta k}\Big)$.

If

(xi) $\sum_{n=1}^{m} n^{\delta k-1}|s_n|^k = O(X_m)$.

then the series $\sum a_n\lambda_n$ *is summable* $|A,\delta|_k, k \geqslant 1, 0 \leqslant \delta < 1/k$.

The following lemma is pertinent for the proof of Theorem 1.

LEMMA 1. *([2]) Under the conditions on* $\{X_n\}, \{\beta_n\}$ *and* $\{\lambda_n\}$ *as taken in the statement of the theorem, the following conditions hold, when* (iii) *is satisfied*

(1) $n\beta_n X_n = O(1)$.

(2) $\sum_{n=1}^{\infty} \beta_n X_n < \infty$.

Proof. To complete the proof it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|T_{nr}|^k < \infty, \quad \text{for} \quad r = 1,2,3.$$

From the definition of $\hat{A}$ and using (vi) and (vii);

$$\begin{aligned}
\hat{a}_{n,\nu+1} &= \bar{a}_{n,\nu+1} - \bar{a}_{n-1,\nu+1} \\
&= \sum_{i=\nu+1}^{n} a_{ni} - \sum_{i=\nu+1}^{n-1} a_{n-1,i} \\
&= 1 - \sum_{i=0}^{\nu} a_{ni} - 1 + \sum_{i=0}^{\nu} a_{n-1,i} \\
&= \sum_{i=0}^{\nu}\Big(a_{n-1,i} - a_{n,i}\Big) \geqslant 0. \qquad (3)
\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned} I_1 := \sum_{n=1}^{m} n^{\delta k+k-1}|T_{n1}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\Big|\sum_{v=1}^{n-1}\Delta_v\hat{a}_{nv}\lambda_v s_v\Big|^k \\ &= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v||s_v|\Big)^k \\ &= O(1)\sum_{n=1}^{m+1} n^{\delta k+k-1}\Big(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v|^k|s_v|^k\Big)\Big(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\Big)^{k-1} \end{aligned}$$

$$\begin{aligned} \Delta_v\hat{a}_{nv} &= \hat{a}_{nv} - \hat{a}_{n,v+1} \\ &= \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} \\ &= a_{nv} - a_{n-1,v} \leqslant 0. \end{aligned}$$

Thus, using (vii),

$$\sum_{v=0}^{n-1}|\Delta_v\hat{a}_{nv}| = \sum_{v=0}^{n-1}(a_{n-1,v} - a_{nv}) = 1 - 1 + a_{nn} = a_{nn}.$$

Since $\{X_n\}$ is an almost increasing sequence, condition (iv) implies that $\{\lambda_n\}$ is bounded.Then,using $(v), (ix), (xi)$, and (i) and condition (2) of Lemma 1,

$$\begin{aligned} I_1 &= O(1)\sum_{n=1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|\lambda_v|^k|s_v|^k|\Delta_v\hat{a}_{nv}| \\ &= O(1)\sum_{n=1}^{m+1} n^{\delta k}\Big(\sum_{v=1}^{n-1}|\lambda_v|^{k-1}|\lambda_v||\Delta_v\hat{a}_{nv}||s_v|^k\Big) \\ &= O(1)\sum_{v=1}^{m}|\lambda_v||s_v|^k\sum_{n=v+1}^{m+1} n^{\delta k}|\Delta_v\hat{a}_{nv}| \\ &= O(1)\sum_{v=1}^{m} v^{\delta k}|\lambda_v|a_{vv}|s_v|^k \\ &= O(1)\sum_{v=1}^{m}|\lambda_v|\Big[\sum_{r=1}^{v}a_{rr}|s_r|^k r^{\delta k} - \sum_{r=1}^{v-1}a_{rr}|s_r|^k r^{\delta k}\Big] \\ &= O(1)\Big[\sum_{v=1}^{m}|\lambda_v|\sum_{r=1}^{v}a_{rr}|s_r|^k r^{\delta k} - \sum_{v=0}^{m-1}|\lambda_{v+1}|\sum_{r=1}^{v}a_{rr}|s_r|^k r^{\delta k}\Big] \\ &= O(1)\Big[\sum_{v=1}^{m-1}\Delta(|\lambda_v|)\sum_{r=1}^{v}a_{rr}|s_r|^k r^{\delta k} + |\lambda_m|\sum_{r=1}^{m}a_{rr}|s_r|^k r^{\delta k}\Big] \end{aligned}$$

$$= O(1)\Big[\sum_{v=1}^{m-1}\Delta(|\lambda_v|)\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k + |\lambda_m|\sum_{r=1}^{m} r^{\delta k-1}|s_r|^k\Big]$$

$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m$$

$$= O(1)\sum_{v=1}^{m}\beta_v X_v + O(1)|\lambda_m|X_m$$

$$= O(1).$$

Using (i) and Hölder's inequality

$$I_2 := \sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\Big|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}(\Delta_v\lambda_v)s_v\Big|^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\Big[\sum_{v=1}^{n-1}|\Delta_v\lambda_v|\hat{a}_{n,v+1}|s_v|\frac{a_{vv}}{a_{vv}}\Big]^k$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v\lambda_v|^k|s_v|^k\hat{a}_{n,v+1}a_{vv}^{-k}a_{vv}\Big]\Big[\sum_{v=1}^{n-1}a_{vv}\hat{a}_{n,v+1}\Big]^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}\beta_v^k|s_v|^k\hat{a}_{n,v+1}a_{vv}^{-k}a_{vv}\Big]$$

We have, using (v),

$$I_2 = O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{v=1}^{n-1}\hat{a}_{n,v+1}(v\beta_v)^k|s_v|^k a_{vv}$$

$$= O(1)\sum_{v=1}^{m}(v\beta_v)^k|s_v|^k a_{vv}\sum_{n=v+1}^{m+1} n^{\delta k}\hat{a}_{n,v+1}.$$

From the condition (1) of Lemma 1, $(v\beta_v)$ is bounded and therefore, from (x),

$$I_2 = O1)\sum_{v=1}^{m} v^{\delta k}(v\beta_v)^k|s_v|^k a_{vv}$$

$$= O1)\sum_{v=1}^{m} v^{\delta k}(v\beta_v)(v\beta_v)^{k-1}|s_v|^k a_{vv}$$

$$= O1)\sum_{v=1}^{m} v^{\delta k}(v\beta_v)|s_v|^k a_{vv}$$

$$= O(1)\sum_{v=1}^{m} v\beta_v|s_v|^k v^{\delta k-1}.$$

Using summation by parts, (xi), conditions (1) and (2) of Lemma 1 and (iii),

$$
\begin{aligned}
I_2 :&= O(1)\sum_{v=1}^{m} v\beta_v\Big[\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k - \sum_{r=1}^{v-1} r^{\delta k-1}|s_r|^k\Big]\\
&= O(1)\Big[\sum_{v=1}^{m}(v\beta_v)\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k - \sum_{r=1}^{m-1}(v+1\beta_{v+1})\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k\Big]\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v} r^{\delta k-1}|s_r|^k + O(1)m\beta_m\sum_{r=1}^{m} r^{\delta k-1}|s_r|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1} v|\Delta(\beta_v)|X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m\\
&= O(1).
\end{aligned}
$$

Finally, using (iv) and (v) we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n3}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\Big|a_{nn}\lambda_n s_n\Big|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k+k-1}|a_{nn}|^k|\lambda_n|^k|s_n|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||s_n|^k\\
&= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n||s_n|^k\\
&= O(1),
\end{aligned}
$$

as in the proof of I_1

Setting $\delta = 0$ in the theorem yields the following corollary:

COROLLARY 1. *Let $\{X_n\}$ be an almost increasing sequence and $\{\beta_n\}$ and $\{\lambda_n\}$ sequences satisfying conditions $(i)-(iv)$ of Theorem 1. Let A be a triangle satisfying conditions $(v)-(viii)$ of Theorem 1.*
If
(ix) $\sum_{n=1}^{m}\frac{1}{n}|s_n|^k = O(X_m)$,
then the series $\sum a_n\lambda_n$ is summable $|A|_k, k \geqslant 1$.

COROLLARY 2. *Let $\{p_n\}$ be a positive sequence such that*
$P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfies
(v) $np_n \asymp O(P_n)$.

(vi) $\displaystyle\sum_{n=v+1}^{m+1} n^{\delta k}\left|\frac{p_n}{P_n P_{n-1}}\right| = O\Big(\frac{v^{\delta k}}{P\ \ _v}\Big).$

Let $\{X_n\}$ *is an almost increasing sequence,* $\{\beta_n\}$ *and* $\{\lambda_n\}$ *sequences satisfying conditions* $(i)-(iv)$ *of Theorem* 1.

If

(vii) $\sum_{n=1}^{m} n^{\delta k-1}|s_n|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ *is summable* $|\bar{N},p,\delta|_k, k \geqslant 1$ *for* $0 \leqslant \delta < 1/k$.

Proof. Conditions $(i)-(iv)$ and (vii) of Corollary 2 are, respectively, conditions $(i)-(iv)$ and (xi) of Theorem 1.

Conditions (vi), (vii) and $(viii)$ of Theorem 1 are automatically satisfied for any weighted mean method. Condition (v) of Theorem 1 becomes condition (v) of Corollary 2, and conditions (ix) and (x) of Theorem 1 become condition (vi) of Corollary 2.

It should be noted that, in [3], an incorrect definition of absolute summability was used. Corollary 2 gives the correct version of Ozarslan's theorem.

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